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SOME REMARKS ON REGULAR SUBGROUPS OF THE AFFINE GROUP

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Abstract. Let V be a vector space over a field $\mathbb{F}$ of characteristic $p \geqslant 0$ and let T be a regular subgroup of the affine group $\mathrm{AGL}(V)$. In the finite dimensional case we show that, if T is abelian or $p > 0$, then T is unipotent. For T abelian, pushing forward some ideas used in [A. Caranti, F. Dalla Volta and M. Sala, Abelian regular subgroups of the affine group and radical rings, Publ. Math. Debrecen **69** (2006), 297–308.], we show that the set $\{t - I \mid t \in T\}$ is a subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathbb{F} \oplus V)$, which is nilpotent when V has finite dimension. This allows a rather systematic construction of abelian regular subgroups.

1. Introduction

Let $\mathbb{F}$ be a field of characteristic $p \geqslant 0$ and $\mathbb{F}^{n+1}$ be the space of row vectors, with canonical basis $e_0, e_1, \ldots, e_n$. Considering $\mathrm{GL}_{n+1}(\mathbb{F})$ acting on the left, in the natural way, on the transpose of $\mathbb{F}^{n+1}$, we may identify the affine group $\mathrm{AGL}_n(\mathbb{F})$ with the stabilizer of the transpose of e_0. Thus $\mathrm{AGL}_n(\mathbb{F})$ acts on the right on the set of affine vectors

$$\Omega := (1,\, v) \mid v \in \mathbb{F}^n\}.$$

Moreover the subspace $\langle e_1, \ldots, e_n\rangle$ is invariant under $\mathrm{AGL}_n(\mathbb{F})$. We are concerned here with regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$. They are the subgroups which act transitively on Ω and whose non-trivial elements fix no point. Equivalently, a subgroup T of $\mathrm{AGL}_n(\mathbb{F})$ is regular if, for every $v \in \mathbb{F}^n$, there exists a unique element of T having $(1, v)$ as first row. Thus T can be identified with a function

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$T : \mathbb{F}^n \to \mathrm{GL}_n(\mathbb{F})$ and its elements can be written as:

$$\left\{ \begin{pmatrix} 1 & v \\ 0 & T(v) \end{pmatrix} \mid v \in \mathbb{F}^n \right\}. \tag{1.1}$$

The most natural example of a regular subgroup is the translation group, corresponding to the constant function $T(v) = I$ for all v. Another abelian example is given by the centralizer $C_U(J)$, where $U = U_{n+1}(\mathbb{F})$ denotes the group of all upper unitriangular matrices, and $J \in U$ is the $(n+1) \times (n+1)$ Jordan block with minimal polynomial $(x-1)^{n+1}$. Indeed, by a formula of Frobenius, the centralizer of J in $\mathrm{Mat}_{n+1}(\mathbb{F})$ has dimension $n+1$. Thus it coincides with the subalgebra $\mathbb{F}J^0 + \cdots + \mathbb{F}J^n$ generated by J. Note that, in general, $C_U(J)$ is not isomorphic to the translation group. For example, J does not have order p when $p > 0$ and $n \geqslant p+1$. Other examples can be found in [1]. In [2], Hegedus constructed some interesting examples of regular subgroups of $\mathrm{AGL}_n(\mathbb{F}_p)$ containing no non-trivial translations. His construction leads to non-abelian examples for $n \geq 4$. The observation after Lemma 5.1 shows how to construct non-abelian examples already for $n = 2$. For our purposes it is relevant to consider the regular subgroup of $\mathrm{AGL}_2(\mathbb{R})$ whose elements are:

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & e^y & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}. \tag{1.2}$$

Similarly, for $n \geq 3$, and $i \neq j$, we have the non-abelian regular subgroup:

$$\left\{ \begin{pmatrix} 1 & v & x \\ 0 & I + xE_{ij} & \mathbf{0}^{\mathbf{t}} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \mid v \in \mathbb{F}^{n-1},\ x \in \mathbb{F} \right\}.$$

We recall that a subgroup of $\mathrm{GL}_{n+1}(\mathbb{F})$ is called unipotent if all its elements have 1 as the unique eigenvalue. It is well known (see for example [3, 17.5, Corollary, p. 113]) that every unipotent subgroup of $\mathrm{GL}_{n+1}(\mathbb{F})$ is conjugate to a subgroup of $U_{n+1}(\mathbb{F})$.

Let T be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$. When $\mathbb{F}$ is finite, T is a p-group, hence it is unipotent. In Section 3 we show that T is unipotent whenever either T is abelian or $p > 0$. On the other hand, when $p = 0$, this may not be true, as example (1.2) shows. Note that, when $p > 0$, the exponential map cannot be defined over $\mathbb{F}$.

In [1], among other things, it is shown axiomatically that any regular abelian subgroup T of $\mathrm{AGL}(V)$, where V is a vector space over $\mathbb{F}$, allows to define on $(V, +)$ the structure of a radical ring. But the relation of this ring with the algebra $\mathrm{End}_{\mathbb{F}}(\mathbb{F} \oplus V)$, of which it is an additive subgroup, is not made explicit. In Section 4, pushing forward some of the ideas of [1], we show that the set $\{t - I \mid t \in T\}$ is a subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathbb{F} \oplus V)$. Moreover this subalgebra is nilpotent, whenever V has finite dimension.

As a byproduct we sketch a rather systematic construction of unipotent regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$, illustrating the cases $n \leq 3$.

We are grateful to the authors of [1] for having attracted our attention to this problem.

2. Preliminary facts

As above $\mathbb{F}^{n+1}$ has canonical basis $e_0, \dots, e_n$. We recall that $\langle e_1, \dots, e_n \rangle$ is invariant under $\mathrm{AGL}_n(\mathbb{F})$.

Lemma 2.1. *Let T be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$. Assume that $\langle e_1, \dots, e_n \rangle = U \oplus W$ where U and W are T-invariant subspaces, of dimensions m and $n-m$. Then T is the semidirect product of two subgroups $\widehat{T}_1 \sim T_1$ and $\widehat{T}_2 \sim T_2$, where T_1 and T_2 are regular subgroups of $\mathrm{AGL}_m(\mathbb{F})$ and $\mathrm{AGL}_{n-m}(\mathbb{F})$ respectively. Moreover, if $\widehat{T}_1$ and $\widehat{T}_2$ centralize each other, then*

$$T = \left\{ \begin{pmatrix} 1 & u & w \\ 0 & T_1(u) & 0 \\ 0 & 0 & T_2(w) \end{pmatrix} \mid u \in U, w \in W \right\}.$$

Proof. For every $u \in U$, there exists a unique matrix t_u in T having $(1, u, 0)$ as first row. Similarly, for every $w \in W$, there exists a unique matrix t_w in T having $(1, 0, w)$ as first row. Hence, by the assumption that U and W are T-invariant, up to conjugation, they have respective shapes

$$t_u = \begin{pmatrix} 1 & u & 0 \\ 0 & T_1(u) & 0 \\ 0 & 0 & A(u) \end{pmatrix}, \quad t_w = \begin{pmatrix} 1 & 0 & w \\ 0 & B(w) & 0 \\ 0 & 0 & T_2(w) \end{pmatrix}$$

with $T_1(u), A(u) \in \mathrm{GL}_m(\mathbb{F})$, $B(w), T_2(w) \in \mathrm{GL}_{n-m}(\mathbb{F})$.

Thus T contains the subgroups:

$$\widehat{T}_1 := \{t_u \mid u \in U\}, \quad \widehat{T}_2 := \{t_w \mid w \in W\}.$$

The maps

$$t_u \mapsto \begin{pmatrix} 1 & u \\ 0 & T_1(u) \end{pmatrix}, \quad t_w \mapsto \begin{pmatrix} 1 & w \\ 0 & T_2(w) \end{pmatrix}$$

are isomorphisms from $\widehat{T}_1$ and $\widehat{T}_2$ to regular subgroup T_1 and T_2 of $\mathrm{AGL}_m(\mathbb{F})$ and $\mathrm{AGL}_{n-m}(\mathbb{F})$. Since $UB(w) = U$, for all $w \in W$, we conclude that $T = \widehat{T}_1\widehat{T}_2$. Clearly $\widehat{T}_1 \cap \widehat{T}_2 = I$. Finally $\widehat{T}_1$ and $\widehat{T}_2$ centralize each other if and only if $wA(u) = w$ for all $w \in W$ and $uB(w) = u$ for all $u \in U$, if and only if $A(u) = I$ for all $u \in$ and $B(w) = I$ for all $w \in W$. Thus also our final claim follows. □

Lemma 2.2. *If $\begin{pmatrix} 1 & v \\ 0 & g \end{pmatrix}$ fixes no affine vector, then v does not belong to the image of $I - g$. In particular 1 is an eigenvalue of g.*

Proof. If $z(I - g) = v$ for some $z \in \mathbb{F}^n$ then the block matrix in the statement fixes the affine vector $(1, z)$. Thus $I - g$ is singular, whence our final claim. □

We will need the following fact, based on elementary linear algebra.

Lemma 2.3. *Let $f_1(x)$, $f_2(x)$ be coprime polynomials and let $f(x) = f_1(x)f_2(x)$ have degree n. Then every element of $\mathrm{GL}_n(\mathbb{F})$ with characteristic polynomial $f(x)$ is conjugate in $\mathrm{GL}_n(\mathbb{F})$ to a matrix*

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where g_1, g_2 have respective characteristic polynomials $f_1(x)$ and $f_2(x)$. Moreover the centralizer $C(g)$ of g in $\mathrm{GL}_n(\mathbb{F})$ consists of block-diagonal matrices of the same form.

3. **Unipotency**

Let T be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$.

Theorem 3.1. *If T is abelian, then it is unipotent.*

Proof. By contradiction assume that $\begin{pmatrix} 1 & v_0 \\ 0 & T(v_0) \end{pmatrix}$ is an element of T that is not unipotent. By Lemma 2.2 we have $n \geq 2$ and the characteristic polynomial of $T(v_0)$ factorizes as $(x-1)^m f(x)$, where $m \geqslant 1$ and $f(x)$ is coprime to $x - 1$. By Lemma 2.3, the space $V = \langle e_1, \dots, e_n \rangle$ is the direct sum of two subspaces U, W which are invariant under $C := C_{\mathrm{GL}_n(\mathbb{F})}(T(v_0))$. By the commutativity of T, we have $T(v) \leq C$ for all $v \in V$. Hence U, W are also T-invariant. Thus we can apply Theorem 2.1 to the abelian group T. Let $\widehat{T_1}$ and $\widehat{T_2}$ be defined as in that theorem. By induction on n, they are unipotent. It follows that T is unipotent, a contradiction. □

Theorem 3.2. *If $\operatorname{char} \mathbb{F} = p > 0$, then T is unipotent.*

Proof. Again suppose that T has a non-unipotent element t. By Lemmas 2.2 and 2.3, we may assume that, for some $v_1 \in \mathbb{F}^{n-m}$, $v_2 \in \mathbb{F}^m$,

$$t = \begin{pmatrix} 1 & v_1 & v_2 \\ 0 & g_1 & 0 \\ 0 & 0 & g_2 \end{pmatrix}$$

where g_1 does not have 1 as an eigenvalue and g_2 is unipotent.

If $z := v_1(I - g_1)^{-1}$, then $(1, z) = (1, z)\begin{pmatrix} 1 & v_1 \\ 0 & g_1 \end{pmatrix}$. Thus, for every $k \geqslant 1$:

$$(1, z, 0)t^k = \left(1,\ z,\ v_2 \sum_{j=0}^{k-1} g_2^j\right).$$

Since $p > 0$ and g_2 is an $m \times m$ unipotent matrix, $g_2^{p^{m-1}} = I$. From

$$\sum_{j=0}^{p^m-1} g_2^j \;=\; p \sum_{j=0}^{p^{m-1}-1} g_2^j$$

we have that t^{p^m} fixes $(1, z, 0)$. By the regularity assumption it follows that $t^{p^m} = I$, a contradiction. □

4. **Abelian regular subgroups as linear maps**

In this section, following [1], we consider a vector space V over $\mathbb{F}$, whose dimension may be infinite. The results of the previous sections apply to the case $V = \mathbb{F}^n$, the space of row vectors. We recall that a regular subgroup T of $\mathrm{AGL}(V)$ may be identified with a function $T : V \to \mathrm{GL}(V)$. We may also consider the constant function $I : V \to \mathrm{End}_{\mathbb{F}}(V)$ such that $I(v) =$ identity for all $v \in V$.

Theorem 4.1. *If T is abelian, then the map $T - I : V \to \mathrm{End}_{\mathbb{F}}(V)$ is linear. Moreover the set $\{t - I \mid t \in T\}$ is a subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathbb{F} \oplus V)$. When V has finite dimension, this subalgebra is nilpotent.*

Proof. Set $\delta = T - I$. By abuse of notation, which should create no confusion, we may represent the elements of T as "matrices" of shape

$$\left\{ \begin{pmatrix} 1 & v \\ 0 & I + \delta(v) \end{pmatrix} \mid v \in V \right\}.$$

As observed in [1], the commutativity of T gives:

$$v\delta(w) = w\delta(v), \quad \forall\, v, w \in V \tag{4.1}$$

and this implies that δ is linear. For the reader's convenience we give a proof of this fact. For all $v \in V$, we have:

$$v\delta(u + w) = (u + w)\delta(v) = u\delta(v) + w\delta(v) = v\delta(u) + v\delta(w) = v\left(\delta(u) + \delta(w)\right)$$

$$v\delta(\lambda u) = \lambda u\delta(v) = \lambda\left(u\delta(v)\right) = \lambda v\delta(u) = v\left(\lambda\delta(u)\right)$$

where $u, w \in V$ and $\lambda \in \mathbb{F}$. Thus $\delta(V)$ is a subspace of $\mathrm{End}_{\mathbb{F}}(V)$. By the shape of the matrices in T it follows that also the set $\{t - I \mid t \in T\}$ is a subspace of $\mathrm{End}_{\mathbb{F}}(\mathbb{F} \oplus V)$. In particular it is an additive subgroup. It follows that it is also closed under multiplication, hence a subalgebra. Indeed, for all $t_1, t_2 \in T$:

$$(t_1 - I)(t_2 - I) = (t_1 t_2 - I) - (t_1 - I) - (t_2 - I).$$

When V is finite dimensional, by Theorem 3.1 the group T is unipotent. Thus the corresponding subalgebra is nilpotent, hence a radical ring. □

5. **Low dimensional cases**

Clearly Theorem 4.1 allows to construct abelian regular subgroups starting from appropriate definitions of $T(v_i)$, where $v_1, \ldots, v_n$ is a basis of $\mathbb{F}^n$, and applying the linearity of $T - I$. So, in this Section, we exploit this observation to construct regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$, at least for n small.

Lemma 5.1. *The only regular subgroup of* $\mathrm{AGL}_1(\mathbb{F})$ *is the translation group. Every unipotent regular subgroup* T *of* $\mathrm{AGL}_2(\mathbb{F})$ *is conjugate to one of the form*

$$T = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & \tau(a) \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F} \right\}, \tag{5.1}$$

where τ *is any additive endomorphism of* $\mathbb{F}$.

Proof. The case $n = 1$ follows immediately from Lemma 2.2. If $n = 2$, up to conjugation, $T \leqslant U_3(\mathbb{F})$. Further use of the same Lemma gives that, for all $a, b \in \mathbb{F}$, the group T contains matrices of the form

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & \tau(a) \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Noting that the matrices of the second type form a central subgroup Z of T and considering the quotient T/Z, one gets $\tau(a_1 + a_1) = \tau(a_1) + \tau(a_2)$ for all a_1 and a_2. □

This group is abelian precisely when $a\tau(b) = b\tau(a)$ for all $a, b \in \mathbb{F}$, i.e., when τ is linear. This gives rise to non-abelian examples. E.g. take $\mathbb{F} = \mathbb{K}(\alpha)$ a quadratic extension of the field $\mathbb{K}$. Then any non-scalar matrix of $A \in \mathrm{GL}_2(\mathbb{K})$ gives rise to the endomorphism $k_0 + k_1\alpha \mapsto h_0 + h_1\alpha$, where $(h_0, k_0) = (k_0, h_0)A$. Since A is non-scalar, this group is not abelian.

Lemma 5.2. *Let* $T \leq U_{n+1}(\mathbb{F})$ *be abelian. Denote by* $\{e_1, \ldots, e_n\}$ *the canonical basis of* $\mathbb{F}^n$. *Then* $T(e_n) = I$.

Proof. By the commutativity, for all $v, w \in \mathbb{F}^n$ we have:

$$v - w = vT(w) - wT(v). \tag{5.2}$$

Since $T(v)$ fixes e_n for all v, taking $w = e_n$ in relation (5.2) we get $v = v\,T(e_n)$ for all v, i.e. $T(e_n) = I$, whence our claim. □

Inductive use of this Lemma gives that the matrix $T\,(e_{n-1})$ has the first $n-1$ columns equal to the (transposes of the) vectors of the canonical basis, etc.... So, for example, we have the following.

Corollary 5.3. *Every abelian regular subgroup* T *of* $\mathrm{AGL}_3(\mathbb{F})$ *is conjugate to one of the following, for some* $a, b, c \in \mathbb{F}$:

$$T_1 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & x_1 a + x_2 b \\ 0 & 0 & 1 & x_1 b + x_2 c \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{F} \right\}$$

or

$$T_2 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_1 a & x_1 b + x_2 c \\ 0 & 0 & 1 & x_1 c \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{F} \right\}.$$

Proof. Up to conjugation $T \leq U_4(\mathbb{F})$. By the previous lemma

$$T((0,0,1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T((0,1,0)) = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}$$

for some $d, e \in \mathbb{F}$. Assume that

$$T((1,0,0)) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

for some $a, b, c \in \mathbb{F}$. Imposing commutativity we get $d = c$ and $ae = 0$. Taking $a = 0$ we obtain T_1. Taking $e = 0$ we obtain T_2 (up to the names of parameters). In general, T_1 and T_2 are not isomorphic. For example, for $p = 3$, the group T_1 is elementary abelian for all a, b, c, whereas T_2 may have elements of order 9. This happens, for example, when $a = 2$, $b = 1$, $c = 1$. □

One might also try to characterize unipotent regular subgroups T of $\mathrm{AGL}_n(\mathbb{F})$ via the Jordan form of an element in the centre. For example, if the Jordan form of an element of the center of T consists of a unique block J of size $n+1$, then T is conjugate to $C_U(J)$.

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