

ON THE RELATION BETWEEN THE NON-COMMUTING GRAPH AND THE PRIME GRAPH

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Communicated by Ali Reza Ashrafi

ABSTRACT. Given a non-abelian finite group G , let $\pi(G)$ denote the set of prime divisors of the order of G and denote by $Z(G)$ the center of G . The *prime graph* of G is the graph with vertex set $\pi(G)$ where two distinct primes p and q are joined by an edge if and only if G contains an element of order pq and the *non-commuting graph* of G is the graph with the vertex set $G - Z(G)$ where two non-central elements x and y are joined by an edge if and only if $xy \neq yx$. Let G and H be non-abelian finite groups with isomorphic non-commuting graphs. In this article, we show that if $|Z(G)| = |Z(H)|$, then G and H have the same prime graphs and also, the set of orders of the maximal abelian subgroups of G and H are the same.

1. Introduction

For an integer $z > 1$, we denote by $\pi(z)$ the set of all prime divisors of z . If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *prime graph* $GK(G)$ of a finite group G is the graph with vertex set $\pi(G)$ where two distinct primes p and q are joined by an edge (we write $(p, q) \in GK(G)$) if and only if G contains an element of order pq . Let G be a non-abelian group and $Z(G)$ be its center. We will associate a graph $\Gamma(G)$ to G which is called the *non-commuting graph* of G . The vertex set $V(\Gamma(G))$ is $G - Z(G)$ and the edge set $E(\Gamma(G))$ consists of (x, y) , where x and y are distinct non-central elements of G such that $xy \neq yx$. The commuting graph associated to a non-abelian group G is the complement of $\Gamma(G)$, i.e., a graph with vertex set $G - Z(G)$ where distinct non-central elements x and y of G are joined by an edge if and only if $xy = yx$. Obviously, we are considering simple graphs, i.e., graphs with no loops or directed or repeated edges. The non-commuting graph of a non-abelian finite group has received some attention in existing literature. In [1], the authors

MSC(2010): Primary: 20D60; Secondary: 20D05

Keywords: Non-commuting graph; Prime graph, Maximal abelian subgroups, Maximal independent set of the graph.

Received: 9 July 2011, Accepted: 07 November 2011.

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studied some properties of non-commuting graph, especially, they showed that two non-abelian finite groups with isomorphic non-commuting graphs have the same set of lengths of conjugacy classes. In [5], it has been proved that the commuting graph of a non-abelian finite group G is disconnected if and only if $GK(G)$ is disconnected, where $Z(G) = 1$ and in [4], it has been proved that if S is a finite non-abelian simple group and G is a finite group such that $\Gamma(S) \cong \Gamma(G)$, then $|G| = |S|$. But until now, there doesn't exist enough information about the non-commuting graph of a non-abelian finite simple group and also, the structure of a finite group with given non-commuting graph. On the other hand, in [9], the authors pointed out the prime graph of a given non-abelian finite simple group and in [8], the structure of a finite group with given prime graph has been studied. Thus it is interesting to find a link between the non-commuting graph and the prime graph of a non-abelian finite group. The aim of this article is finding this link.

2. The Main Results

In the following, let G and H be non-abelian finite groups and let $M(G)$ denote the set of orders of maximal abelian subgroups of G . A subset of the vertices of a graph is called an independent set if its elements are pairwise non-adjacent and the independent set of maximal size is named a maximal independent set.

Lemma 2.1. [3] *If $M(G) = M(H)$, then G and H have the same prime graphs.*

Lemma 2.2. [4] *Let S be a non-abelian finite simple group. If G is a finite group such that $\Gamma(G) \cong \Gamma(S)$, then $Z(G) = 1$.*

Lemma 2.3. *If T is a maximal independent set of $\Gamma(G)$, then $C_G(T) - Z(G) = T$.*

Proof. Obviously, for $x, y \in G - Z(G)$, x and y are non-adjacent in $\Gamma(G)$ if and only if $xy = yx$, so we can see at once that for every $x, y \in T$, $xy = yx$. This shows that $T \subseteq C_G(T) - Z(G)$. If $z \in C_G(T) - Z(G)$, then for every $x \in T$, $zx = xz$, so $T \cup \{z\}$ is an independent set of $\Gamma(G)$. But T is a maximal independent set of $\Gamma(G)$ and hence, $T \cup \{z\} = T$. This implies that $z \in T$. Thus $C_G(T) - Z(G) \subseteq T$, so $C_G(T) - Z(G) = T$. \square

Lemma 2.4. *If T is a maximal independent set of $\Gamma(G)$ and M is a maximal abelian subgroup of G , then*

- (i) $T \cup Z(G)$ is a maximal abelian subgroup of G ;
- (ii) $M - Z(G)$ is a maximal independent set of $\Gamma(G)$.

Proof. It is easy to see that $C_G(T)$ is a subgroup of G . Thus Lemma 2.3 forces $T \cup Z(G)$ to be an abelian subgroup of G . If M is a maximal abelian subgroup of G containing $T \cup Z(G)$, then we can see that by Lemma 2.3, $M \subseteq C_G(T) = T \cup Z(G)$. This shows that $M = T \cup Z(G)$. Therefore, (i) follows. It remains to prove (ii). Of course for every $x, y \in M - Z(G)$, $xy = yx$, so $M - Z(G)$ is an independent subset of $V(\Gamma(G))$. Thus there exists a maximal independent set U of $\Gamma(G)$ containing $M - Z(G)$.

Since by (i), $U \cup Z(G)$ is an abelian subgroup of G and $M = (M - Z(G)) \cup Z(G) \subseteq U \cup Z(G)$, we conclude that $M = U \cup Z(G)$, because by our assumption, M is a maximal abelian subgroup of G . This shows that $M - Z(G) = U$, so the lemma follows. \square

Theorem 2.5. *If $|Z(G)| = |Z(H)|$ and $\Gamma(G) \cong \Gamma(H)$, then*

- (i) $M(G) = M(H)$;
- (ii) $GK(G) = GK(H)$.

Proof. Since $\Gamma(G) \cong \Gamma(H)$, $|V(\Gamma(G))| = |V(\Gamma(H))|$ and hence, $|G| - |Z(G)| = |H| - |Z(H)|$. But by assumption $|Z(G)| = |Z(H)|$, so we conclude that $|G| = |H|$. Thus $\pi(G) = \pi(H)$. Also, there exists a bijection $\varphi : V(\Gamma(H)) \rightarrow V(\Gamma(G))$ such that for every vertices $a, b \in V(\Gamma(H))$, $(a, b) \in \Gamma(H)$ if and only if $(\varphi(a), \varphi(b)) \in \Gamma(G)$. If M is an arbitrary maximal abelian subgroup of H , then by Lemma 2.4(ii), $M - Z(H)$ is a maximal independent set of $\Gamma(H)$ and hence $\varphi(M - Z(H))$ is a maximal independent set of $\Gamma(G)$. Thus by Lemma 2.4(i), $\varphi(M - Z(H)) \cup Z(G)$ is a maximal abelian subgroup of G . Also, $|M - Z(H)| = |\varphi(M - Z(H))|$ and $\varphi(M - Z(H)) \subseteq G - Z(G)$. Therefore, $|M| = |M - Z(H)| + |Z(H)| = |\varphi(M - Z(H))| + |Z(G)| = |\varphi(M - Z(H)) \cup Z(G)| \in M(G)$ and hence, $M(H) \subseteq M(G)$. Similarly, we can see that $M(G) \subseteq M(H)$, so the proof of (i) is complete. Also, (i) and Lemma 2.1 complete the proof of (ii). \square

The following corollary follows immediately from Lemma 2.2 and Theorem 2.5:

Corollary 2.6. *Let S be a non-abelian finite simple group. If $\Gamma(S) \cong \Gamma(G)$, then*

- (i) $M(G) = M(S)$;
- (ii) $GK(G) = GK(S)$.

From Theorem 2.5, we also obtain:

Corollary 2.7. *If Γ_1 is a maximal complete subgraph of the commuting graph of G , then the vertex set of Γ_1 is $M - Z(G)$ for some maximal abelian subgroups M of G .*

Problem 16.1 in the Kourovka notebook [6] contains the following conjecture:

Conjecture. (AAM's conjecture) Let S be a non-abelian finite simple group and let G be a group such that $\Gamma(S) \cong \Gamma(G)$. Then $S \cong G$.

In several papers, it has been proved that AAM's conjecture is true for the finite simple groups with disconnected commuting graphs, but one can see easily that those methods used for the groups with disconnected commuting graphs fail for finite simple groups with connected commuting graphs. According to [5], [7] and [10], if $n \geq 4$ is even, then the commuting graph of $B_n(q)$ is connected if and only if $n \neq 2^m$, for every natural number m . Here, as a main consequence of our result, we see that AAM's conjecture holds for an infinite series of finite simple groups of Lie type with connected commuting graphs.

Corollary 2.8. *Let $n \geq 4$ be even. If G is a finite group with $\Gamma(G) \cong \Gamma(B_n(q))$, then $G \cong B_n(q)$.*

Proof. Since $\Gamma(G) \cong \Gamma(B_n(q))$, by Theorem 2.5, we deduce that $M(G) = M(B_n(q))$, so [2, Main Theorem] shows that $G \cong B_n(q)$, as claimed. \square

After considering Theorem 2.5, the questions below may engage the reader's mind:

Problem 1. Let G and H be finite groups with isomorphic non-commuting graphs. If $\pi(Z(G)) = \pi(Z(H))$ and $|Z(G)| \neq |Z(H)|$, then what can we say about $GK(G)$ and $GK(H)$?

Let G be an infinite group such that $T(G) = \{x \in G : O(x) < \infty\} \neq \{1\}$. Then the prime graph of G is a graph with $\cup_{x \in T(G)} \pi(O(x))$ as its vertex set and in this graph primes p and q are joined by an edge if and only if G contains an element of order pq .

Problem 2. Let G and H be non-abelian infinite groups with isomorphic non-commuting graphs. If $T(G), T(H) \neq \{1\}$, then what can we say about the prime graphs of G and H ?

Acknowledgments

The authors would like to thank the referee for invaluable comments. The first author wishes to thank Shahrekord University for financial support.

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