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SOME SPECIAL CLASSES OF n-ABELIAN GROUPS

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Abstract. Given an integer n, we denote by $\mathfrak{B}_n$ and $\mathfrak{C}_n$ the classes of all groups G for which the map $\phi_n : g \mapsto g^n$ is a monomorphism and an epimorphism of G, respectively. In this paper we give a characterization for groups in $\mathfrak{B}_n$ and for groups in $\mathfrak{C}_n$. We also obtain an arithmetic description of the set of all integers n such that a group G is in $\mathfrak{B}_n \cap \mathfrak{C}_n$.

1. Introduction

Let n be an integer. A group G is said to be *n-abelian* if the map $\phi_n : g \mapsto g^n$ is an endomorphism of G. Then $(xy)^n = x^n y^n$ for all $x, y \in G$, from which it follows $[x^n, y] = [x, y]^n = [x, y^n]$. It is also easy to see that a group G is n-abelian if and only if it is $(1-n)$-abelian. The structure of n-abelian groups has been described in [2] and [1]. If $n \neq 0, 1$ and G is an n-abelian group, then the quotient group $G/Z(G)$ has finite exponent dividing $n(n-1)$. This implies that every torsion-free n-abelian group is abelian.

In this paper we denote by $\mathfrak{B}_n$ and $\mathfrak{C}_n$ the classes of all groups G for which ϕ_n is a monomorphism and an epimorphism of G, respectively. Then $\mathfrak{B}_0 = \mathfrak{C}_0$ contains only the trivial group, $\mathfrak{B}_1 = \mathfrak{C}_1$ is the class of all groups, and $\mathfrak{B}_{-1} = \mathfrak{C}_{-1}$ is the class of all abelian groups. Furthermore, with $|n| > 1$, $G \in \mathfrak{B}_n$ if and only if G is an n-abelian group having no elements of order dividing $|n|$. Similarly, $G \in \mathfrak{C}_n$ if and only if G is n-abelian and for every $g \in G$ there exists an element $x \in G$ such that $g = x^n$. We also set $\mathfrak{A}_n = \mathfrak{B}_n \cap \mathfrak{C}_n$. This class has been studied in [6], [7] and [5]. Of course, groups of exponent dividing $|n-1|$ are in $\mathfrak{A}_n$.

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For all integers $n \neq 0$, every divisible abelian group is in $\mathfrak{C}_n$. In particular, the additive group $\mathbb{Q}$ of rational numbers is in $\mathfrak{A}_n$, as well as every Prüfer group $\mathbb{Z}(p^\infty)$, with $\gcd(p,n)=1$. The class $\mathfrak{B}_n$ is subgroup closed, but the class $\mathfrak{C}_n$ is not: in fact the group $\mathbb{Z}$ of all integer numbers is not in $\mathfrak{C}_n$. The class $\mathfrak{C}_n$ is quotient closed, but the class $\mathfrak{B}_n$ is not: for example $\mathbb{Q}/\mathbb{Z}$ is not in $\mathfrak{B}_n$. Each of these classes is closed under forming direct products of its members. However, they are both not closed under extensions. For, let G be the wreath product of a cyclic group of order p by $\mathbb{Z}(p^\infty)$. Then G is an extension of groups in $\mathfrak{A}_n$, for all integers n with $\gcd(p,n)=1$. But $Z(G)$ is trivial and so G is not n-abelian when $n \neq 0, 1$.

We describe the structure of groups in $\mathfrak{B}_n$ and $\mathfrak{C}_n$ in Section 2. In Section 3, we provide an arithmetic description of the set of all integers n such that a group G is in $\mathfrak{A}_n$.

2. Groups in $\mathfrak{B}_n \cup \mathfrak{C}_n$

Given an integer n, a group G is said to be *n-central* if $[x^n, y] = 1$ for all $x, y \in G$. If $n \neq 0$ then n-central is equivalent to $G/Z(G)$ having finite exponent dividing $|n|$.

Lemma 2.1. *Let G be an n-abelian group. Then G is $(n-1)$-central if and only if it is $(n-1)$-abelian.*

Proof. Since G is n-abelian, for all $x, y \in G$, we get

$$x^n y^{n-1} x^{-1} = x^n y^n y^{-1} x^{-1} = (xy)^n (xy)^{-1} = (xy)^{n-1}.$$

Thus, the group G is $(n-1)$-abelian if and only if we have $x^n y^{n-1} x^{-1} = x^{n-1} y^{n-1}$, which is equivalent to $[y^{n-1}, x] = 1$. □

In the sequel, we always assume $|n| > 1$. Moreover, we denote by $\mathbb{P}$ the set of all (positive) primes and, with n integer, by π_n the set of all primes dividing n. Finally, if π is a set of primes, we set $\pi' = \mathbb{P} \setminus \pi$.

Proposition 2.2. *If $G \in \mathfrak{B}_n \cup \mathfrak{C}_n$, then G is $(n-1)$-central and $(n-1)$-abelian. In particular, G' has finite exponent dividing $|n-1|$.*

Proof. The group G is n-abelian, so $1 = [x^{n(n-1)}, y] = [x^{n-1}, y]^n$ for all $x, y \in G$. If $G \in \mathfrak{B}_n$, then G has no elements of order dividing $|n|$ and therefore $[x^{n-1}, y] = 1$. If $G \in \mathfrak{C}_n$, there exists $z \in G$ such that $x = z^n$. Hence $[x^{n-1}, y] = [z^{n(n-1)}, y] = 1$. In both cases G is $(n-1)$-abelian by Lemma 2.1. □

It is well-known that, for every n-abelian group G, the elements of finite order form a subgroup T such that G/T is abelian and T is the direct product of a π_{n-1}-group, a π_n-group and an abelian $\pi'_{n(n-1)}$-group (see [2]). This allows us to characterize torsion groups in $\mathfrak{B}_n \cup \mathfrak{C}_n$.

Theorem 2.3. *Let G be a torsion group. Then:*

(i) $G \in \mathfrak{B}_n$ if and only if $G = A \times B$ where A is an n-abelian π_{n-1}-group and B is an abelian $\pi'_{n(n-1)}$-group;

(ii) $G \in \mathfrak{C}_n$ if and only if $G = A \times B$ where A is an n-abelian π_{n-1}-group and $B = B^n$ is an abelian π'_{n-1}-group.

Proof. (i) Suppose $G \in \mathfrak{B}_n$. Since G has no elements of order dividing $|n|$, by Theorem A of [2], we get $G = A \times B$ where A is an n-abelian π_{n-1}-group and B is an abelian $\pi'_{n(n-1)}$-group. The converse is clear.

(ii) Let $G \in \mathfrak{C}_n$. Then $G = A \times C \times D$ where A is a π_{n-1}-group, C is π_n-group and D is an abelian $\pi'_{n(n-1)}$-group (see [2], Theorem A). In particular C is abelian by Proposition 2.2. Thus, $B = C \times D$ is an abelian π'_{n-1}-group. Since $G/A \in \mathfrak{C}_n$, we also have $B = B^n$.

Conversely, for all $g \in A$, there exist integers α and β, depending on g, such that $1 = \alpha|g| + \beta n$. Then $g = g^{\beta n}$ and $A = A^n$. This implies $G \in \mathfrak{C}_n$. □

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Let G be a torsion group in $\mathfrak{B}_n$. Then $G \in \mathfrak{A}_n$.*

Hence, if $\mathfrak{T}$ denotes the class of all torsion groups, we have

$$\mathfrak{A}_n \cap \mathfrak{T} = \mathfrak{B}_n \cap \mathfrak{T} \subseteq \mathfrak{C}_n \cap \mathfrak{T}$$

for all integers n, where the inclusion can be proper: for each prime p, the group $\mathbb{Z}(p^\infty)$ is in $\mathfrak{C}_p \setminus \mathfrak{B}_p$.

However, for groups of finite exponent we have:

Proposition 2.5. *Let $G \in \mathfrak{C}_n$ be a group of finite exponent. Then $G \in \mathfrak{A}_n$.*

Proof. Let $\exp(G) = k$ and suppose $\gcd(k, n) = d \neq 1$. Then $k = \alpha d$ for some $\alpha < k$ and $n = \beta d$. Now, for all $g \in G$, there exists $x \in G$ such that $g = x^n = (x^\beta)^d$, so $g^\alpha = 1$. Therefore $\exp(G)$ divides α, which is a contradiction. □

Now we use Theorem 2.3 to obtain characterizations of groups in $\mathfrak{B}_n$ and in $\mathfrak{C}_n$.

Theorem 2.6. *Let G be a group. Then $G \in \mathfrak{B}_n$ if and only if G is isomorphic to a subgroup of the direct product of an n-abelian π_{n-1}-group by an abelian group without elements of order dividing $|n|$.*

Proof. Assume $G \in \mathfrak{B}_n$. Let V be a maximal torsion-free subgroup of $Z(G)$ and let W/V be the subgroup consisting of all π_n-elements of $Z(G)/V$. Notice that G/W is torsion because so are $G/Z(G)$ and $Z(G)/V$. Furthermore, if $x \in W \cap G'$, then there exists a π_n-number m such that $x^m \in V$. As $\gcd(m, n-1) = 1$ and $x^{n-1} = 1$, we get $x = 1$. Hence, W has trivial intersection with G'.

Let $x \in G$. First, suppose $(xW)^n = W$. Then $x^n V \in W/V$ and $x^{ns} V = V$ for some π_n-number s. By Proposition 2.2 we have $x^{n-1} \in Z(G)$, so that $x \in Z(G)$. Thus $xV \in Z(G)/V$ is a π_n-element. Therefore $x \in W$ and G/W has no elements of order dividing $|n|$, that is $G/W \in \mathfrak{B}_n$. Suppose now $(xG')^n = G'$. Then $x^n \in G'$, so $x^{n(n-1)} = 1$ by Proposition 2.2. This implies $x^{n-1} = 1$. Hence $x \in G'$ and $G/G' \in \mathfrak{B}_n$.

Finally, since G is isomorphic to a subgroup of $G/W \times G/G'$, the claim follows by (i) of Theorem 2.3.

The converse is clear. □

Theorem 2.7. *Let G be an n-abelian group and denote by T its torsion group. Then $G \in \mathfrak{C}_n$ if and only if $T = A \times B$, where A is a π_{n-1}-group, $B = B^n$ is an abelian π'_{n-1}-group, and G/T is a p-divisible abelian group for every prime p dividing n.*

Proof. Assume $G \in \mathfrak{C}_n$ and let p be a prime dividing n. Then $n = \alpha p$ for some integer α and, for all $g \in G$, there exists $x \in G$ such that $g = x^n = (x^\alpha)^p \in G^p$. This means that G/T is in $\mathfrak{C}_p$. The rest follows by (*ii*) of Theorem 2.3.

Conversely, given $g \in G$, we have $gT = x^nT$ for some $x \in G$: in fact, G/T is n-divisible. Thus $g^{-1}x^n \in T$ and, by Corollary 2.4, there exists $y \in T$ such that $g^{-1}x^n = y^n$.
Therefore$g = x^n y^{-n} = (xy^{-1})^n$, that is $G = G^n$. □

Notice that in Theorem 2.7 one cannot replace the hypothesis that G is n-abelian by the weaker hypothesis that A is n-abelian. For example, consider the wreath product G of the cyclic group of order 2 by $\mathbb{Q}$. The torsion part T of G is the base group, that is an infinite group of exponent 2. Moreover, $G/T = \mathbb{Q}$. Finally $Z(G) = 1$ and so G is not n-abelian for all $n \neq 0, 1$. In particular, $G \notin \mathfrak{C}_n$.

3. The semigroup $\mathbb{A}(G)$

Let G be a group. In [4] F. W. Levi introduced the set

$$\mathbb{E}(G) = \{n \in \mathbb{Z} \,:\, (xy)^n = x^n y^n \text{ for all } x, y \in G\},$$

the so-called *exponent semigroup* of G. It is a multiplicative subsemigroup of $\mathbb{Z}$ containing 0 and 1; moreover, $n \in \mathbb{E}(G)$ if and only if $1 - n \in \mathbb{E}(G)$. It has been shown (see [4] and [3]) that $\mathbb{E}(G)$ is either $\{0, 1\}$, or $\mathbb{Z}$, or the union of certain residue classes modulo some integers depending on G.

Similarly, starting from the map $\phi_n : g \in G \mapsto g^n \in G$, we introduce the set

$$\mathbb{A}(G) = \{n \in \mathbb{Z} \,:\, \phi_n \in Aut\,(G)\}.$$

This is a subsemigroup of $\mathbb{E}(G)$ containing 1. Obviously $0 \in \mathbb{A}(G)$ if and only if $G = \{1\}$; in that case $\mathbb{A}(G) = \mathbb{Z}$. From now on, we assume $G \neq \{1\}$.

Lemma 3.1. *With H and K groups, we have $\mathbb{A}(H \times K) = \mathbb{A}(H) \cap \mathbb{A}(K)$.*

Proof. A direct product is in $\mathfrak{A}_n$ if and only if each of its direct factors is. □

Lemma 3.2. *Let G be a group and suppose that $\mathbb{A}(G)$ satisfies one of the following conditions:*

(*i*) $2 \in \mathbb{A}(G)$;
(*ii*) $3 \in \mathbb{A}(G)$;
(*iii*) $n \in \mathbb{A}(G)$ *and* $-n \in \mathbb{A}(G)$ *for some* $n \neq 0$;
(*iv*) $n \in \mathbb{A}(G)$ *and* $m \in \mathbb{A}(G)$ *with* $\gcd(n-1, m-1) \leq 2$.

Then G is abelian.

Proof. (*i*) is trivial, (*ii*) follows from Proposition 2.2.

(*iii*) The group G is n and $(n+1)$-abelian and so, by Lemma 2.1, it is n-central. It is also $(n-1)$-central by Proposition 2.2. This means that G is abelian.

(*iv*) Let $\gcd(n-1, m-1) = 1$. By Proposition 2.2 the quotient group $G/Z(G)$ has exponent dividing $n-1$ and $m-1$. Hence G is abelian. Assume $\gcd(n-1, m-1) = 2$. Then $G/Z(G)$ has exponent 2 by Proposition 2.2 and so G is nilpotent of class ≤ 2. In particular, we have $[x^2, y] = [x, y]^2 = 1$ for all $x, y \in G$. This implies that, for every integer k with $k \equiv 0 \pmod 4$ and for all $x, y \in G$,

$$(xy)^k = x^k y^k [y, x]^{k(k-1)/2} = x^k y^k.$$

Without loss of generality we can suppose $n \equiv -1 \pmod 4$. Thus G is $(n+1)$-abelian and, as in (iii), we conclude that G is abelian. □

Notice that $\mathbb{E}(G) = \mathbb{Z}$ if and only if G is abelian. On the other hand, it is easy to show that $\mathbb{A}(G) = \mathbb{Z} \setminus \{0\}$ if and only if G is isomorphic to a direct sum of copies of $\mathbb{Q}$. However, G is abelian if and only if $-1 \in \mathbb{A}(G)$. So, in that case, $-n \in \mathbb{A}(G)$ for all $n \in \mathbb{A}(G)$. Furthermore, if G is abelian, $n \in \mathbb{A}(G)$ if and only if $p \in \mathbb{A}(G)$ for all primes p dividing n. The semigroup $\mathbb{A}(G)$ is thus generated by -1 and all primes in $\mathbb{A}(G)$. Therefore, if $\pi(G)$ denotes the set of all primes involved in the decomposition of orders of elements of G, and $\delta(G)$ denotes the set of all primes p such that G is p-divisible, one can easily realize that:

Theorem 3.3. *Let G be an abelian group. Then $\mathbb{A}(G)$ is the multiplicative subsemigroup of $\mathbb{Z}$ generated by $(\delta(G) \setminus \pi(G)) \cup \{-1\}$. Moreover:*

- (i) *if G is torsion then $\mathbb{A}(G)$ is generated by $(\mathbb{P} \setminus \pi(G)) \cup \{-1\}$;*
- (ii) *if G is torsion-free then $\mathbb{A}(G)$ is generated by $\delta(G) \cup \{-1\}$.*

By Theorem 3.3, $\mathbb{A}(\mathbb{Z}) = \{-1, 1\} = \mathbb{A}(\mathbb{Q}/\mathbb{Z})$ and, if G is a torsion abelian group, then

$$\mathbb{A}(G) = \mathbb{Z} \setminus \cup_{p \in \pi(G)} p\mathbb{Z}.$$

Concerning (ii) of Theorem 3.3 we also point out that, given a set π of primes, there always exists a torsion-free abelian group G such that $\mathbb{A}(G)$ is the multiplicative subsemigroup of $\mathbb{Z}$ generated by $\pi \cup \{-1\}$. For example, the additive group of all rational numbers with a π-number as denominator has the above properties.

Now let G be a non-abelian group and suppose that G is n-abelian for some integer $n \neq 0, 1$, so $\{0, 1\} \subset \mathbb{E}(G) \subset \mathbb{Z}$. It has been shown in [4] (see also [3]) that the set $\mathbb{E}_0(G)$ of all integers n such that G is both n-abelian and n-central is an ideal of $\mathbb{Z}$. Let $\mathbb{E}_0(G) = w\mathbb{Z}$. Then $w > 2$, G is both w-abelian and w-central, and w is the least positive integer with such properties. Let $w = q_1 q_2 \ldots q_t$ be a factorization of w, with $t \geq 1$, $q_i > 2$ for all $i = 1, 2, \ldots, t$ and $\gcd(q_i, q_j) = 1$ for $i \neq j$. Then $\mathbb{E}(G)$ is the union of the 2^t residue classes modulo w which are the solutions of the 2^t systems of congruential equations $x \equiv \delta_i \pmod{q_i}$, where $i = 1, 2, \ldots, t$ and $\delta_i \in \{0, 1\}$. In [4] and [3] it has also been shown that, if $n, n+1 \in \mathbb{E}(G)$, then $n \equiv 0 \pmod w$. This, together with Proposition 2.2, gives $n - 1 \equiv 0 \pmod w$ for all $n \in \mathbb{A}(G)$. Therefore we have:

Theorem 3.4. *For each group G, $\mathbb{A}(G) \subseteq [1]_w$.*

In general, the equality does not hold in the theorem above. For example, consider a non-abelian group H of exponent 3, and the direct product G of H by the cyclic group of order 4. Clearly, $\mathbb{E}_0(G) = 3\mathbb{Z}$. However, $4 \notin \mathbb{A}(G)$ since G has an element of order 4. Thus $\mathbb{A}(G) \neq [1]_3$.

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