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# THE PROLONGATION OF CENTRAL EXTENSIONS

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**Abstract.** The aim of this paper is to study the $(\alpha, \gamma)$-prolongation of central extensions. We obtain the obstruction theory for $(\alpha, \gamma)$-prolongations and classify $(\alpha, \gamma)$-prolongations thanks to low-dimensional cohomology groups of groups.



## 1. **Introduction**

A description of group extensions by means of factor sets leads to a close relationship between the extension problem of a type of algebras and the corresponding cohomology theory. This allows to study extension problems using cohomology as an effective method [3, 5, 9].

For a group extension

$$\mathcal{E} : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$$

and for any homomorphism $\gamma : C' \to C$, from the existence of the pull-back of a pair $(\gamma, \beta)$, there is always an extension $\mathcal{E}' = \mathcal{E}\gamma$ making the following diagram

$$\begin{array}{lccccccccc}
\mathcal{E}' : & 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 1 \\
 & & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\gamma} & & \\
\mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 1
\end{array}$$

commute.

This shows the contravariance of a functor $\mathrm{Ext}(C, A)$ in terms of the first variable. The notion of pull-backs has been widely applied in works related to group extensions (see [6, 7]).

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Given an extension $\mathcal{E}'$ and a homomorphism $\gamma : C' \to C$, the problem here is that of finding whether there is any corresponding extension $\mathcal{E}$ of $A$ by $C$ such that $\mathcal{E}' = \mathcal{E}\gamma$. This problem is still unsolved for the general case. However, a description where the morphism $id : A \to A$ in the above diagram is replaced by a homomorphism $\alpha : A' \to A$, and $A', A$ are abelian groups, is presented in [8]. In this paper, our purpose is to show a better description when $\mathcal{E}'$ is a central extension. We study the obstruction theory for such extensions and classify those extensions due to low-dimensional cohomology groups.

Firstly, we introduce the notion of $(\alpha, \gamma)$-prolongations of central extensions $\mathcal{E}_0$ and show that each such $(\alpha, \gamma)$-prolongation induces a crossed module. The relationship between group extensions and crossed modules leads to many interesting results (see [1, 4, 9]). Here, the notion of pre-prolongation of $\mathcal{E}_0$ is derived from the induced crossed module. The obstruction of such a pre-prolongation is an element in the cohomology group $H^3(\operatorname{Coker}\gamma, A)$ whose vanishing is necessary and sufficient for there to exist an $(\alpha, \gamma)$-prolongation (Theorem 3.5). Moreover, each such $(\alpha, \gamma)$-prolongation is a central extension (Theorem 4.3). Finally, we state the Schreier theory for $(\alpha, \gamma)$-prolongations (Theorem 4.8).

## 2. $(\alpha, \gamma)$-prolongations of central extensions

Given a diagram of group homomorphisms

$$\begin{array}{lccccccccc}
\mathcal{E}_0 : & 0 & \longrightarrow & A_0 & \xrightarrow{j_0} & B_0 & \xrightarrow{p_0} & G_0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
\mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & G & \longrightarrow & 0,
\end{array}$$

where the rows are exact, $j_0 A_0 \subset ZB_0$, $\gamma$ is a normal monomorphism (in the sense that $\gamma G_0$ is a normal subgroup of $G$) and $\alpha$ is an epimorphism. Then $\mathcal{E}$ is said to be an $(\alpha, \gamma)$-*prolongation of* $\mathcal{E}_0$.

No loss of generality in assuming that $j_0$ is an inclusion map and $A_0$ can be identified with the subgroup $j_0 A_0$ of $B_0$. In addition, we denote $\Pi_0 = \operatorname{Coker}\gamma$, $E_0 = B_0/\operatorname{Ker}\alpha$ and let $\sigma : G \to \Pi_0$ be the natural projection. Obviously, $\operatorname{Ker}\beta = j_0 \operatorname{Ker}\alpha$.

For convenience, we write the operation in $\Pi_0$ as multiplication and in other groups as addition, even though the groups $B_0, G_0, B, G$ are non-necessarily abelian.

The factor group $\operatorname{Coker}\gamma$ plays a fundamental role in our study, as well as in the first literature [5] and in the recent ones [2].

**Lemma 2.1.** *Any $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$ induces an exact sequence of group homomorphisms*

$$0 \to E_0 \xrightarrow{\varepsilon} B \xrightarrow{\sigma p} \Pi_0 \to 1, \tag{2.1}$$

*where* $\varepsilon(b_0 + \operatorname{Ker}\alpha) = \beta(b_0)$.

*Proof.* First, we show that the sequence

$$B_0 \xrightarrow{\beta} B \xrightarrow{\sigma p} \Pi_0 \to 1$$

is exact. In fact, since $\sigma p\beta = (\sigma\gamma)p_0 = 1$, the above sequence is semi-exact. Further, for any $b \in \operatorname{Ker}(\sigma p)$, $(\sigma p)(b) = 1$. It follows that

$$p(b) \in \operatorname{Ker}\sigma = \operatorname{Im}\gamma \Rightarrow p(b) = \gamma(g_0),$$

for some $g_0 \in G_0$. Then, there is $b_0 \in B_0$ such that $p_0(b_0) = g_0$, and hence $p(b) = p\beta(b_0)$. This implies

$$b = \beta(b_0) + j\alpha(a_0) = \beta(b_0) + \beta(a_0) = \beta(b_0 + a_0)$$

for $a_0 \in A_0$. Thus $b \in \operatorname{Im}\beta$. This proves that the above sequence is exact.

The homomorphism $\beta$ induces the unique monomorphism

$$\varepsilon : E_0 \to B, \;\; \varepsilon(b_0 + \operatorname{Ker}\alpha) = \beta(b_0)$$

and one has $\operatorname{Im}\varepsilon = \operatorname{Im}\beta$. Therefore, the sequence (2.1) is exact. □

Since $A_0/\operatorname{Ker}\alpha \cong A$ via the canonical isomorphism $a_0 + \operatorname{Ker}\alpha \mapsto \alpha(a_0)$, Lemma 2.1 yields the following commutative diagram

(2.2)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & E_0 & \xrightarrow{\pi} & G_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & G & \longrightarrow & 0,
\end{array}$$

where

$$i(\alpha a_0) = a_0 + \operatorname{Ker}\alpha, \;\; \pi(b_0 + \operatorname{Ker}\alpha) = p_0(b_0).$$

**Definition 2.2** ([1]). A *crossed module* is a quadruple $(B, D, d, \theta)$, where $d : B \to D, \theta : D \to \operatorname{Aut} B$ are group homomorphisms satisfying the following relations:

$C_1$. $\theta d(b) = \mu_b, b \in B$,

$C_2$. $d(\theta_x(b)) = \mu_x(d(b)), x \in D, b \in B$,

where $\mu_x$ is the inner automorphism given by conjugation with $x$.

**Theorem 2.3.** *Any $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$ induces a homomorphism $\theta : G \to \operatorname{Aut} E_0$ such that the quadruple $(E_0, G, \gamma\pi, \theta)$ is a crossed module.*

*Proof.* The exact sequence (2.1) induces the group homomorphism

$$\phi : B \to \operatorname{Aut}E_0, \;\; b \mapsto \phi_b$$

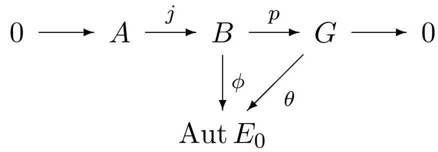
given by

(2.3)
$$\phi_b(e_0) = \varepsilon^{-1}\mu_b(\varepsilon e_0), \;\; e_0 \in E_0.$$

It is easy to see that $\phi j = id_{E_0}$. In fact, for all $a \in A$, $\phi j(a) = \phi_{ja}$. Since $ia \in ZE_0$,

$$\begin{aligned}\phi_{ja}(e_0) &\overset{(2.3)}{=} \varepsilon^{-1}\mu_{ja}(\varepsilon e_0) \overset{(2.2)}{=} \varepsilon^{-1}\mu_{\varepsilon ia}(\varepsilon e_0) \\ &= \mu_{ia}(e_0) = e_0.\end{aligned}$$

Then, by the universal property of Coker, there is a group homomorphism $\theta : G \to \operatorname{Aut} E_0$ such that the following diagram




commutes. The homomorphism $\theta$ is defined by

$$\theta_g = \phi_b, \; pb = g. \tag{2.4}$$

The homomorphisms $\theta : G \to \operatorname{Aut} E_0$ and $\gamma\pi : E_0 \to G$ satisfy the rules $C_1, C_2$ in the definition of a crossed module, that is,

$$\theta(\gamma\pi) = \mu, \tag{2.5}$$

$$(\gamma\pi)\theta_g(e_0) = \mu_g(\gamma\pi(e_0)), \quad g \in G, \quad e_0 \in E_0. \tag{2.6}$$

In fact, for $e_0, c \in E_0$, we get

$$\begin{aligned} \theta\gamma\pi(e_0)(c) &\overset{(2.2)}{=} \theta p\varepsilon(e_0)(c) \overset{(2.4)}{=} \phi_{\varepsilon(e_0)}(c) \\ &\overset{(2.3)}{=} \varepsilon^{-1}\mu_{\varepsilon e_0}(\varepsilon c) = \mu_{e_0}(c). \end{aligned}$$

Now, we show that the relation (2.6) holds. Let $g = pb$, then

$$\begin{aligned} \gamma\pi\theta_g(e_0) &\overset{(2.4)}{=} \gamma\pi\phi_b(e_0) \overset{(2.2)}{=} p\varepsilon\phi_b(e_0) \overset{(2.3)}{=} p[\mu_b(\varepsilon e_0)] \\ &= \mu_{pb}(p\varepsilon(e_0)) \overset{(2.4)}{=} \mu_g(\gamma\pi(e_0)), \end{aligned}$$

and the proof is completed. □

**Corollary 2.4.** *If $\mathcal{E}_0$ has an $(\alpha,\gamma)$-prolongation, then the homomorphism $\theta : G \to \operatorname{Aut} E_0$ induces the homomorphism $\theta^* : G \to \operatorname{Aut} A$ given by $\theta^*_g(a) = i^{-1}\theta_g(ia)$. Further, $A$ is $\Pi_0$-module with action*

$$xa = \theta^*_{u_x}(a),$$

*where $u_x \in G$, $\sigma(u_x) = x$.*

*Proof.* By Theorem 2.3, the quadruple $(E_0, G, \gamma\pi, \theta)$ is a crossed module. Then, it is easy to see that if $e_0 \in \operatorname{Ker}(\gamma\pi) = \operatorname{Ker}\pi$, then $\theta_g(e_0) \in \operatorname{Ker}\pi$, and hence for any $g \in G$, the restriction of $\theta_g$ to $\operatorname{Ker}\pi$ is an endomorphism of $\operatorname{Ker}\pi$. Since $iA = \operatorname{Ker}\pi$, each such endomorphism also induces an endomorphism of $A$.

We now can check that the correspondence $\Pi_0 \to \operatorname{Aut} A, x \mapsto \theta^*_{u_x}$, is a homomorphism. Therefore, $A$ is a $\Pi_0$-module with action

$$xa = i^{-1}\theta_{u_x}(ia) = \theta^*_{u_x}(a),$$

as required. □

## 3. Obstructions of $(\alpha, \gamma)$ -prolongations

Given a diagram of group homomorphisms

$$\begin{array}{lccccccccc} \mathcal{E}_0: & 0 & \longrightarrow & A_0 & \xrightarrow{j_0} & B_0 & \xrightarrow{p_0} & G_0 & \longrightarrow & 0, \\ & & & \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\gamma} & & \\ & & & A & & & & G & & \end{array}$$

where the row is exact, $j_0A_0 \subset ZB_0$, $\gamma$ is a normal monomorphism, $\alpha$ is an epimorphism, and a group homomorphism $\theta : G \to \operatorname{Aut}(B_0/\operatorname{Ker}\alpha)$ such that the quadruple $(B_0/\operatorname{Ker}\alpha, G, \gamma\pi, \theta)$ is a crossed module. These data denoted by the triple $(\alpha, \gamma, \theta)$ is said to be the *pre-prolongation* of $\mathcal{E}_0$. An $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$ inducing $\theta$ is called a *covering* of the pre-prolongation $(\alpha, \gamma, \theta)$.

The "prolongation problem" is that of finding whether there is any covering of the pre-prolongation $(\alpha, \gamma, \theta)$ of $\mathcal{E}_0$ and, if so, how many.

First, we show an obstruction of an $(\alpha, \gamma)$-prolongation. For any $x \in \Pi_0$, choose a representative $u_x$ in $G$ such that $\sigma(u_x) = x$, in particular, choose $u_1 = 0$. This set of representatives yields a factor set $f(x, y) \in \gamma G_0$, that is,

$$u_x + u_y = f(x, y) + u_{xy}, \ \forall x, y \in \Pi_0.$$

Because $u_1 = 0$, $f(x, y)$ satisfies the normalized condition $f(x, 1) = f(1, y) = 0$.

The associativity of the operation in $G$ implies

$$\mu_{u_x} f(y, z) + f(x, yz) = f(x, y) + f(xy, z), \tag{3.1}$$

where $\mu_{u_x}$ is the inner automorphism of $G$ given by conjugation with $u_x$.

The given homomorphism $\theta$ induces the homomorphism $\varphi : \Pi_0 \overset{u}{\to} G \overset{\theta}{\to} \operatorname{Aut} E_0$, that is,

$$\varphi(x) = \theta_{u_x}. \tag{3.2}$$

Hereafter, we refer to $u_x, f(x, y), \varphi(x)$ as before.

Now, we choose $h(x, y) \in E_0$ such that

$$\gamma\pi[h(x, y)] = f(x, y), \tag{3.3}$$

in particular, choose $h(x, 1) = h(1, y) = 0$. Thus,

$$\mu_{u_x} f(y, z) \overset{(3.3)}{=} \mu_{u_x}[\gamma\pi h(y, z)] \overset{(2.6)}{=} \gamma\pi\theta_{u_x}(h(y, z)) \overset{(3.2)}{=} \gamma\pi\varphi(x)h(y, z).$$

Take inverse image in $E_0$ for two sides of the equation (3.1) via the homomorphism $\gamma\pi : E_0 \to G$, we obtain

$$\varphi(x)h(y, z) + h(x, yz) = h(x, y) + h(xy, z) + k(x, y, z), \tag{3.4}$$

where $k(x, y, z) \in \operatorname{Ker}(\gamma\pi) = \operatorname{Ker}\pi = A \subset ZE_0$.

The relation (3.4) can be formally written in the form $k = \delta h$, even though $E_0$ is non-abelian.

**Lemma 3.1.** *The function $k$ given by (3.4) is a 3-cocycle in $Z^3(\Pi_0, A)$.*

*Proof.* Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_0 & \xrightarrow{\gamma} & G & \xrightarrow{\sigma} & \Pi_0 & \longrightarrow & 1, \\ & & & & \downarrow{\scriptstyle\theta} & & & & \\ & & & & \mathrm{Aut}\, E_0 & & & & \end{array}$$

by the condition (2.5), $(\theta\gamma)G_0 = \mu E_0$. Then, there is a homomorphism $\psi : \Pi_0 \to \mathrm{AutExt}E_0$ making the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_0 & \xrightarrow{\gamma} & G & \xrightarrow{\sigma} & \Pi_0 & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} & & \\ & & & & \mathrm{Aut}\, E_0 & \xrightarrow{\nu} & \mathrm{AutExt}E_0 & \longrightarrow & 1 \end{array}$$

commute, where $\nu$ is the natural projection. Thus, $k$ is just an obstruction of the abstract kernel $(\Pi_0, E_0, \psi)$. Due to Lemma 8.4 ([5] -Chapter IV), $k$ is a 3-cocycle of $B(Z\Pi_0)$. Moreover, because of its construction, $k$ takes values in $A$, as required. □

**Lemma 3.2.** *For given representatives $u_x$ in $G$, a change in the choice of $h : \Pi_0^2 \to E_0$ replaces $k$ by a cohomologous cocycle. Moreover, by suitably changing the choice of $h$, $k$ may be replaced by any cohomologous cocycle.*

*Proof.* In the proof of Lemma 8.5 ([5] -Chapter IV), replacing the functions with values in the central $C$ by those in $A$, we obtain the proof of Lemma 3.2. □

**Lemma 3.3.** *A change in the choice of $u_x$ in $G$ may be followed by a suitable new selection of $h$ satisfying* (3.3) *such as to leave the function $k$ unchanged.*

*Proof.* If $u_x$ is replaced by $u'_x$ such that $u'_1 = 0$, then $u'_x = g_x + u_x$, where $g : \Pi_0 \to \gamma G_0$ satisfies $g_1 = 0$. Thus, there is a function $t : \Pi_0 \to E_0$ such that $\gamma\pi(t_x) = g_x$.

Now, one determines a function $h' : \Pi_0^2 \to E_0$, given by

$$h'(x, y) = t_x + \theta_{u_x}(t_y) + h(x, y) - t_{xy}. \tag{3.5}$$

Thanks to the condition (2.6), it is easy to check that

$$\gamma\pi h'(x, y) = u'_x + u'_y - u'_{xy} = f'(x, y).$$

Hence, $h'(x, y)$ is just a factor set of $B$ induced by the representatives $u'_x$ in $G$. Thanks to (2.5) and (3.5), we can transform $\varphi(x)[h'(y, z)] + h'(x, yz)$ into $h'(x, y) + h'(xy, z) + k(x, y, z)$. This proves that $k$ is unchanged. □

From the above proved lemmas, we obtain the following proposition.

**Proposition 3.4.** *For any triple $(\alpha, \gamma, \theta)$, the cohomology class $[k] \in H^3(\Pi_0, A)$, where $k$ is given by* (3.4)*, does not depend on the choice of the representatives $u_x$ and the factor set $h(x, y)$.*

The cohomology class $[k] \in H^3(\Pi_0, A)$ is called an *obstruction* to an $(\alpha, \gamma)$-prolongation, and we denote $[k] = \mathrm{Obs}(\alpha, \gamma, \theta)$.

**Theorem 3.5.** *An extension* $\mathcal{E}_0$ *has an* $(\alpha,\gamma)$*-prolongation if and only if* $\mathrm{Obs}(\alpha,\gamma,\theta)$ *vanishes in* $H^3(\Pi_0, A)$.

*Proof. Necessary condition.* Let $\mathcal{E}$ be an $(\alpha,\gamma)$-prolongation of $\mathcal{E}_0$ inducing $\theta$. Recall that for the representatives $u_x$ in $G$, we have a factor set $f(x,y)$ satisfying the relation (3.1). By Lemma 2.1, $B$ is an extension of $E_0$ by $\Pi_0$, and hence we can choose the representatives $v_x$ in $B$, $x\in\Pi_0$, such that $p(v_x)=u_x$. This set of representatives gives a factor set $\varepsilon h$, where $h:\Pi_0^2\to E_0$, that is,

$$\varepsilon h(x,y)=v_x+v_y-v_{xy}.$$

Then,

$$\gamma\pi h(x,y)\overset{(2.2)}{=}p\varepsilon h(x,y)=p(v_x+v_y-v_{xy})=u_x+u_y-u_{xy}=f(x,y),$$

that is, $h$ satisfies (3.3).

Since $\varepsilon h$ is a factor set of the extension $B$ corresponding to the representatives $v_x$, we obtain

$$\mu_{v_x}[\varepsilon h(y,z)]+\varepsilon h(x,yz)=\varepsilon h(x,y)+\varepsilon h(xy,z).$$

We need to turn the above equality into the equality (3.4) to determine the function $k$. Thanks to the monomorphic property of $\varepsilon$ and the relation

$$\mu_{v_x}[\varepsilon h(y,z)]\overset{(2.3)}{=}\varepsilon\phi_{v_x}h(y,z)\overset{(2.4)}{=}\varepsilon\theta_{u_x}h(y,z)\overset{(3.2)}{=}\varepsilon\varphi(x)h(y,z),$$

the above equality becomes

$$\varphi(x)h(y,z)+h(x,yz)=h(x,y)+h(xy,z). \tag{3.6}$$

According to the determination of $k$ in (3.4), we deduce that $[k]=0$.

*Sufficient condition.* Conversely, let $\mathrm{Obs}(\alpha,\gamma,\theta)=0$ in $H^3(\Pi_0,A)$, that is,

$$k=\delta l,\ l:\Pi_0^2\to A.$$

Now, for $h'=h-l$, we obtain

$$k'=\delta h'=\delta h-\delta l=k-k=0.$$

This means that one can choose $h:\Pi_0^2\to E_0$ such that $[k]=0$ in $Z^3(\Pi_0,A)$. Then, the relation (3.4) becomes $\delta h=0$.

According to the relations (2.5) and (3.3), the function $\varphi$ given by (3.2) satisfies

$$\varphi(x)\varphi(y)=\mu_{h(x,y)}\varphi(xy). \tag{3.7}$$

Clearly, $\varphi(1)=id_{E_0}$. Thus, we can construct the crossed product $B_h=[E_0,\varphi,h,\Pi_0]$, that is, $B_h=E_0\times\Pi_0$ under the operation

$$(e_0,x)+(e_0',y)=(e_0+\varphi(x)e_0'+h(x,y),xy),$$

and there is an exact sequence

$$0\to A\xrightarrow{j'}B_h\xrightarrow{p'}G\to 0,$$

where

$$j'(a) = (ia, 1), \quad p'(e_0, x) = \gamma\pi e_0 + u_x.$$

Moreover, the correspondence $\beta' : B_0 \to B_h$ given by

$$\beta'(b_0) = (\bar{b}_0, 1) \tag{3.8}$$

is a group homomorphism.

Now, it is easy to check that the following diagram

$$\begin{array}{llllllllll}
\mathcal{E}_0 : & 0 & \longrightarrow & A_0 & \xrightarrow{j_0} & B_0 & \xrightarrow{p_0} & G_0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma} & & \\
\mathcal{E}_h : & 0 & \longrightarrow & A & \xrightarrow{j'} & B_h & \xrightarrow{p'} & G & \longrightarrow & 0
\end{array}$$

commutes. Therefore, $\mathcal{E}_h$ is an $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$. This completes the proof. □

## 4. Classification theorem

**Definition 4.1.** Two $(\alpha, \gamma)$-prolongations of $\mathcal{E}_0$

$$0 \to A \xrightarrow{j} B \xrightarrow{p} G \to 0,$$

$$0 \to A \xrightarrow{j'} B' \xrightarrow{p'} G \to 0$$

are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{lllllllllll}
0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & G & \longrightarrow & 0, & \qquad & B_0 \xrightarrow{\beta} B \\
 & & \| & & \downarrow{\scriptstyle \beta^*} & & \| & & & & \\
0 & \longrightarrow & A & \xrightarrow{j'} & B' & \xrightarrow{p'} & G & \longrightarrow & 0, & \qquad & B_0 \xrightarrow{\beta'} B'
\end{array}$$

such that $\beta^*\beta = \beta'$.

**Lemma 4.2.** *Any $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$ is equivalent to a crossed product extension.*

*Proof.* Let $\mathcal{E}$ be an $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$ inducing $\theta : G \to \operatorname{Aut} E_0$. By the exact sequence (2.1), we choose representatives $v_x$ in $B$ such that $p(v_x) = u_x$. This set of representatives yields a function $h : \Pi_0^2 \to E_0$ satisfing (3.3). Then, due to Theorem 3.5 the functions $\varphi$ and $h$ satisfy the relations (3.6) and (3.7). Thus, there is the crossed product $B_h = [E_0, \varphi, h, \Pi_0]$, and hence the crossed product extension $\mathcal{E}_h$ is an $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$. Now we show that $\mathcal{E}$ is equivalent to $\mathcal{E}_h$.

Thanks to the exact sequence (2.1), each element of $B$ is written in the form $b = \varepsilon e_0 + v_x$. Then, it is easy to check that the correspondence

$$\beta^* : B \to B_h, \quad \varepsilon e_0 + v_x \mapsto (e_0, x)$$

is an isomorphism satisfying the following commutative diagram

$$\begin{array}{ccccccccccccc} \mathcal{E}:0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & G & \longrightarrow & 0, & & B_0 & \xrightarrow{\beta} & B \\ & & \| & & \downarrow{\beta^*} & & \| & & & & & & \\ \mathcal{E}_h:0 & \longrightarrow & A & \xrightarrow{j'} & B_h & \xrightarrow{p'} & G & \longrightarrow & 0, & & B_0 & \xrightarrow{\beta'} & B_h. \end{array}$$

Finally, for all $b_0 \in B_0$ we have

$$\beta^*\beta(b_0) = \beta^*\varepsilon(\overline{b_0}) = (\overline{b_0}, 1) \overset{(3.8)}{=} \beta'(b_0).$$

Therefore, two extensions $\mathcal{E}$ and $\mathcal{E}_h$ are equivalent, as claimed. □

**Theorem 4.3.** *Any $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$ is a central extension.*

*Proof.* First, it is easy to check that the crossed product extension $\mathcal{E}_h$ mentioned in the proof of Theorem 3.5 is a central extension. This follows directly from the definition of operation in the crossed product $B_h$ and the hypothesis $A_0 \subset ZB_0$. Now, if $\mathcal{E}$ is an $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$, then by Lemma 4.2, $\mathcal{E}$ is equivalent to $\mathcal{E}_h$. Therefore, $\mathcal{E}$ is a central extension and so the proof is completed. □

**Lemma 4.4.** *The function $h : \Pi_0^2 \to E_0$ satisfying* (3.3) *determines a 2-cocycle $h_*$ with values in $A$.*

*Proof.* For representatives $\{b_r | r \in G\}$ of an extension

$$0 \to A \xrightarrow{j} B \xrightarrow{p} G \to 0,$$

a factor set $\kappa(r, s) = b_r + b_s - b_{r+s}$ takes values in $jA$. Then, for $r = u_x, s = u_y$ choose $b_r = v_x, b_s = v_y$, one has

$$\varepsilon h(x, y) = v_x + v_y - v_{xy} = b_r + b_s - b_{r+s} = \kappa(r, s) \in jA.$$

Hence, we obtain a 2-cocycle $h_* = (j^{-1}\varepsilon)_* h : \Pi_0^2 \to A$. □

**Lemma 4.5.** *Let $\mathcal{E}$ be an $(\alpha, \gamma)$-prolongation of $\mathcal{E}_0$. Then, the 2-cocycle $h_* : \Pi_0^2 \to A$ in Lemma 4.4 is uniquely determined up to a coboundary $\delta(t_*) \in B^2(\Pi_0, A)$.*

*Proof.* If $v'_x$ is another representative of $\Pi_0$ in $B$ such that $p(v'_x) = p(v_x) = u_x$, then there exists a function $t : \Pi_0 \to E_0$ such that $v'_x = \varepsilon t_x + v_x$. This set of representatives gives a factor set $\varepsilon h'$, where $h' : \Pi_0^2 \to E_0$ is a function satisfying (3.3). Then,

$$\begin{aligned} \varepsilon h'(x, y) &= v'_x + v'_y - v'_{xy} \\ &= (\varepsilon t_x + v_x) + (\varepsilon t_y + v_y) - (\varepsilon t_{xy} + v_{xy}) \\ &= \varepsilon t_x + \mu_{v_x}(\varepsilon t_y) + v_x + v_y - v_{xy} - \varepsilon t_{xy} \\ &= \varepsilon t_x + \varepsilon\varphi(x)(t_y) + \varepsilon h(x, y) - \varepsilon t_{xy}. \end{aligned}$$

Since $\varepsilon t_{xy} \in \operatorname{Ker} p = \operatorname{Im} j$ and since $jA \subset Z(\beta B_0) = Z(\varepsilon E_0)$, one has

$$\varepsilon h'(x, y) = \varepsilon[t_x + \varphi(x)(t_y) - t_{xy}] + \varepsilon h(x, y).$$

Again, since $\varepsilon$ is a monomorphism so

$$h'(x,y) - h(x,y) = t_x + \varphi(x)(t_y) - t_{xy} = \delta t(x,y).$$

Set $t_* = (\varepsilon^{-1}j)_* t$, we have $h'_* - h_* = \delta(t_*)$, as claimed. □

It follows from Lemmas 4.4 and 4.5 that

**Corollary 4.6.** *The cohomology class of $h_*$ is uniquely determined in $H^2(\Pi_0, A)$.*

The following corollary is deduced from Lemma 4.2.

**Corollary 4.7.** *Two crossed products $B_h = [E_0, \varphi, h, \Pi_0]$ and $B_{h'} = [E_0, \varphi, h', \Pi_0]$ are equivalent if and only if the cohomology classes of $h$ and $h'$ are equal in $H^2(\Pi_0, A)$.*

Denote by $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ the set of all equivalence classes of $(\alpha, \gamma)$-prolongations of $\mathcal{E}_0$, we obtain the following main result.

**Theorem 4.8** (Schreier theory for $(\alpha, \gamma)$-prolongations of central extensions)**.** *If $\mathcal{E}_0$ has an $(\alpha, \gamma)$-prolongation, then the set $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ is torseur under the group $H^2(\Pi_0, A)$.*

*Proof.* Firstly, we show that $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ is torseur under the group $H^2(\Pi_0, \varepsilon^{-1}j(A))$. In fact, by Corollary 4.6, we define a map $\omega$ from $H^2(\Pi_0, \varepsilon^{-1}j(A))$ onto the group of transformations of $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ by formula

$$\omega(\overline{\tau})(\mathrm{cls}[E_0, \varphi, h, \Pi_0]) = \mathrm{cls}[E_0, \varphi, h + \tau, \Pi_0].$$

From the above arguments, $\omega$ is really an element of the group of transformations of $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$. Furthermore, $\omega$ is a group homomorphism.

To prove that $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ is a torseur under $H^2(\Pi_0, \varepsilon^{-1}j(A))$, we also point out that for any two $(\alpha, \gamma)$-prolongations $\mathcal{E}_1$, $\mathcal{E}_2$, there is always the only element $\overline{\tau} \in H^2(\Pi_0, \varepsilon^{-1}j(A))$ such that

$$\mathrm{cls}\mathcal{E}_2 = \omega(\overline{\tau})(\mathrm{cls}\mathcal{E}_1).$$

In fact, we have $\mathrm{cls}\mathcal{E}_i = \mathrm{cls}[E_0, \varphi, h_i, \Pi_0]$, $i = 1, 2$, where $\varphi(x) = \theta_{u_x}$, and $h_i$'s are functions defined by sets of representatives $v_x^i \in B_{h_i}$, where $v_x^i$'s satisfy $p(v_x^i) = u_x$. Then, thanks to the proof of Theorem 3.5, one has

$$\gamma\pi[h_i(x,y)] = f(x,y).$$

It follows that $h_2 = h_1 + r,\ r(x,y) \in \mathrm{Ker}(\gamma\pi) = \mathrm{Ker}\,\pi = \varepsilon^{-1}j(A)$.

By transformations based on the established formula, we have $\delta r = 0$, that is, $r \in Z^2(\Pi_0, \varepsilon^{-1}j(A))$.

Finally, by the canonical isomorphism

$$H^2(\Pi_0, \varepsilon^{-1}j(A)) \leftrightarrow H^2(\Pi_0, A),$$

the theorem is proved. □

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