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THE PROLONGATION OF CENTRAL EXTENSIONS

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Abstract. The aim of this paper is to study the (α, γ)-prolongation of central extensions. We obtain the obstruction theory for (α, γ)-prolongations and classify (α, γ)-prolongations thanks to low-dimensional cohomology groups of groups.

1. **Introduction**

A description of group extensions by means of factor sets leads to a close relationship between the extension problem of a type of algebras and the corresponding cohomology theory. This allows to study extension problems using cohomology as an effective method [3, 5, 9].

For a group extension

$$\mathcal{E} : 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 1$$

and for any homomorphism $\gamma : C' \to C$, from the existence of the pull-back of a pair (γ, β), there is always an extension $\mathcal{E}' = \mathcal{E}\gamma$ making the following diagram

$$\begin{array}{lccccccccc}
\mathcal{E}' : & 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 1 \\
& & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \gamma} & & \\
\mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 1
\end{array}$$

commute.

This shows the contravariance of a functor $\mathrm{Ext}(C, A)$ in terms of the first variable. The notion of pull-backs has been widely applied in works related to group extensions (see [6, 7]).

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Given an extension $\mathcal{E}'$ and a homomorphism $\gamma : C' \to C$, the problem here is that of finding whether there is any corresponding extension $\mathcal{E}$ of A by C such that $\mathcal{E}' = \mathcal{E}\gamma$. This problem is still unsolved for the general case. However, a description where the morphism $id : A \to A$ in the above diagram is replaced by a homomorphism $\alpha : A' \to A$, and A', A are abelian groups, is presented in [8]. In this paper, our purpose is to show a better description when $\mathcal{E}'$ is a central extension. We study the obstruction theory for such extensions and classify those extensions due to low-dimensional cohomology groups.

Firstly, we introduce the notion of (α, γ)-prolongations of central extensions $\mathcal{E}_0$ and show that each such (α, γ)-prolongation induces a crossed module. The relationship between group extensions and crossed modules leads to many interesting results (see [1, 4, 9]). Here, the notion of pre-prolongation of $\mathcal{E}_0$ is derived from the induced crossed module. The obstruction of such a pre-prolongation is an element in the cohomology group $H^3(\operatorname{Coker}\gamma, A)$ whose vanishing is necessary and sufficient for there to exist an (α, γ)-prolongation (Theorem 3.5). Moreover, each such (α, γ)-prolongation is a central extension (Theorem 4.3). Finally, we state the Schreier theory for (α, γ)-prolongations (Theorem 4.8).

2. (α, γ)-prolongations of central extensions

Given a diagram of group homomorphisms

$$\begin{array}{lccccccccc}
\mathcal{E}_0 : & 0 & \longrightarrow & A_0 & \xrightarrow{j_0} & B_0 & \xrightarrow{p_0} & G_0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
\mathcal{E} : & 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & G & \longrightarrow & 0,
\end{array}$$

where the rows are exact, $j_0 A_0 \subset ZB_0$, γ is a normal monomorphism (in the sense that γG_0 is a normal subgroup of G) and α is an epimorphism. Then $\mathcal{E}$ is said to be an (α, γ)-*prolongation of* $\mathcal{E}_0$.

No loss of generality in assuming that j_0 is an inclusion map and A_0 can be identified with the subgroup $j_0 A_0$ of B_0. In addition, we denote $\Pi_0 = \operatorname{Coker}\gamma$, $E_0 = B_0 / \operatorname{Ker}\alpha$ and let $\sigma : G \to \Pi_0$ be the natural projection. Obviously, $\operatorname{Ker}\beta = j_0 \operatorname{Ker}\alpha$.

For convenience, we write the operation in Π_0 as multiplication and in other groups as addition, even though the groups B_0, G_0, B, G are non-necessarily abelian.

The factor group $\operatorname{Coker}\gamma$ plays a fundamental role in our study, as well as in the first literature [5] and in the recent ones [2].

Lemma 2.1. *Any (α, γ)-prolongation of $\mathcal{E}_0$ induces an exact sequence of group homomorphisms*

$$0 \to E_0 \xrightarrow{\varepsilon} B \xrightarrow{\sigma p} \Pi_0 \to 1, \tag{2.1}$$

where $\varepsilon(b_0 + \operatorname{Ker}\alpha) = \beta(b_0)$.

Proof. First, we show that the sequence

$$B_0 \xrightarrow{\beta} B \xrightarrow{\sigma p} \Pi_0 \to 1$$

is exact. In fact, since $\sigma p\beta = (\sigma\gamma)p_0 = 1$, the above sequence is semi-exact. Further, for any $b \in \operatorname{Ker}(\sigma p)$, $(\sigma p)(b) = 1$. It follows that

$$p(b) \in \operatorname{Ker}\sigma = \operatorname{Im}\gamma \Rightarrow p(b) = \gamma(g_0),$$

for some $g_0 \in G_0$. Then, there is $b_0 \in B_0$ such that $p_0(b_0) = g_0$, and hence $p(b) = p\beta(b_0)$. This implies

$$b = \beta(b_0) + j\alpha(a_0) = \beta(b_0) + \beta(a_0) = \beta(b_0 + a_0)$$

for $a_0 \in A_0$. Thus $b \in \operatorname{Im}\beta$. This proves that the above sequence is exact.

The homomorphism β induces the unique monomorphism

$$\varepsilon : E_0 \to B, \;\; \varepsilon(b_0 + \operatorname{Ker}\alpha) = \beta(b_0)$$

and one has $\operatorname{Im}\varepsilon = \operatorname{Im}\beta$. Therefore, the sequence (2.1) is exact. □

Since $A_0/\operatorname{Ker}\alpha \cong A$ via the canonical isomorphism $a_0 + \operatorname{Ker}\alpha \mapsto \alpha(a_0)$, Lemma 2.1 yields the following commutative diagram

(2.2)
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E_0 \arrow[r, "\pi"] \arrow[d, "\varepsilon"] & G_0 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "j"] & B \arrow[r, "p"] & G \arrow[r] & 0,
\end{tikzcd}$$

where

$$i(\alpha a_0) = a_0 + \operatorname{Ker}\alpha, \;\; \pi(b_0 + \operatorname{Ker}\alpha) = p_0(b_0).$$

Definition 2.2 ([1]). A *crossed module* is a quadruple (B, D, d, θ), where $d : B \to D, \theta : D \to \operatorname{Aut} B$ are group homomorphisms satisfying the following relations:

C_1. $\theta d(b) = \mu_b, b \in B$,

C_2. $d(\theta_x(b)) = \mu_x(d(b)), x \in D, b \in B$,

where μ_x is the inner automorphism given by conjugation with x.

Theorem 2.3. *Any (α, γ)-prolongation of $\mathcal{E}_0$ induces a homomorphism $\theta : G \to \operatorname{Aut} E_0$ such that the quadruple $(E_0, G, \gamma\pi, \theta)$ is a crossed module.*

Proof. The exact sequence (2.1) induces the group homomorphism

$$\phi : B \to \operatorname{Aut}E_0, \;\; b \mapsto \phi_b$$

given by

$$\phi_b(e_0) = \varepsilon^{-1}\mu_b(\varepsilon e_0), \;\; e_0 \in E_0. \tag{2.3}$$

It is easy to see that $\phi j = id_{E_0}$. In fact, for all $a \in A$, $\phi j(a) = \phi_{ja}$. Since $ia \in ZE_0$,

$$\begin{aligned} \phi_{ja}(e_0) &\overset{(2.3)}{=} \varepsilon^{-1}\mu_{ja}(\varepsilon e_0) \overset{(2.2)}{=} \varepsilon^{-1}\mu_{\varepsilon ia}(\varepsilon e_0) \\ &= \mu_{ia}(e_0) = e_0. \end{aligned}$$

Then, by the universal property of Coker, there is a group homomorphism $\theta : G \to \operatorname{Aut} E_0$ such that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] & B \arrow[r, "p"] \arrow[d, "\phi"] & G \arrow[r] \arrow[dl, "\theta"] & 0 \\ & & \operatorname{Aut} E_0 & & \end{tikzcd}$$

commutes. The homomorphism θ is defined by

$$\theta_g = \phi_b, \ pb = g. \tag{2.4}$$

The homomorphisms $\theta : G \to \operatorname{Aut} E_0$ and $\gamma\pi : E_0 \to G$ satisfy the rules C_1, C_2 in the definition of a crossed module, that is,

$$\theta(\gamma\pi) = \mu, \tag{2.5}$$

$$(\gamma\pi)\theta_g(e_0) = \mu_g(\gamma\pi(e_0)), \quad g \in G, \quad e_0 \in E_0. \tag{2.6}$$

In fact, for $e_0, c \in E_0$, we get

$$\begin{aligned} \theta\gamma\pi(e_0)(c) &\overset{(2.2)}{=} \theta p\varepsilon(e_0)(c) \overset{(2.4)}{=} \phi_{\varepsilon(e_0)}(c) \\ &\overset{(2.3)}{=} \varepsilon^{-1}\mu_{\varepsilon e_0}(\varepsilon c) = \mu_{e_0}(c). \end{aligned}$$

Now, we show that the relation (2.6) holds. Let $g = pb$, then

$$\begin{aligned} \gamma\pi\theta_g(e_0) &\overset{(2.4)}{=} \gamma\pi\phi_b(e_0) \overset{(2.2)}{=} p\varepsilon\phi_b(e_0) \overset{(2.3)}{=} p[\mu_b(\varepsilon e_0)] \\ &= \mu_{pb}(p\varepsilon(e_0)) \overset{(2.4)}{=} \mu_g(\gamma\pi(e_0)), \end{aligned}$$

and the proof is completed. □

Corollary 2.4. *If $\mathcal{E}_0$ has an (α, γ)-prolongation, then the homomorphism $\theta : G \to \operatorname{Aut} E_0$ induces the homomorphism $\theta^* : G \to \operatorname{Aut} A$ given by $\theta^*_g(a) = i^{-1}\theta_g(ia)$. Further, A is Π_0-module with action*

$$xa = \theta^*_{u_x}(a),$$

where $u_x \in G$, $\sigma(u_x) = x$.

Proof. By Theorem 2.3, the quadruple $(E_0, G, \gamma\pi, \theta)$ is a crossed module. Then, it is easy to see that if $e_0 \in \operatorname{Ker}(\gamma\pi) = \operatorname{Ker}\pi$, then $\theta_g(e_0) \in \operatorname{Ker}\pi$, and hence for any $g \in G$, the restriction of θ_g to $\operatorname{Ker}\pi$ is an endomorphism of $\operatorname{Ker}\pi$. Since $iA = \operatorname{Ker}\pi$, each such endomorphism also induces an endomorphism of A.

We now can check that the correspondence $\Pi_0 \to \operatorname{Aut} A, x \mapsto \theta^*_{u_x}$, is a homomorphism. Therefore, A is a Π_0-module with action

$$xa = i^{-1}\theta_{u_x}(ia) = \theta^*_{u_x}(a),$$

as required. □

3. Obstructions of (α, γ) -prolongations

Given a diagram of group homomorphisms

$$\begin{tikzcd} \mathcal{E}_0 : & 0 \arrow[r] & A_0 \arrow[r, "j_0"] \arrow[d, "\alpha"] & B_0 \arrow[r, "p_0"] & G_0 \arrow[r] \arrow[d, "\gamma"] & 0, \\ & & A & & G & \end{tikzcd}$$

where the row is exact, $j_0A_0 \subset ZB_0$, γ is a normal monomorphism, α is an epimorphism, and a group homomorphism $\theta : G \to \operatorname{Aut}(B_0/\operatorname{Ker}\alpha)$ such that the quadruple $(B_0/\operatorname{Ker}\alpha, G, \gamma\pi, \theta)$ is a crossed module. These data denoted by the triple (α, γ, θ) is said to be the *pre-prolongation* of $\mathcal{E}_0$. An (α, γ)-prolongation of $\mathcal{E}_0$ inducing θ is called a *covering* of the pre-prolongation (α, γ, θ).

The "prolongation problem" is that of finding whether there is any covering of the pre-prolongation (α, γ, θ) of $\mathcal{E}_0$ and, if so, how many.

First, we show an obstruction of an (α, γ)-prolongation. For any $x \in \Pi_0$, choose a representative u_x in G such that $\sigma(u_x) = x$, in particular, choose $u_1 = 0$. This set of representatives yields a factor set $f(x, y) \in \gamma G_0$, that is,

$$u_x + u_y = f(x, y) + u_{xy}, \ \forall x, y \in \Pi_0.$$

Because $u_1 = 0$, $f(x, y)$ satisfies the normalized condition $f(x, 1) = f(1, y) = 0$.

The associativity of the operation in G implies

$$\mu_{u_x} f(y, z) + f(x, yz) = f(x, y) + f(xy, z), \tag{3.1}$$

where μ_{u_x} is the inner automorphism of G given by conjugation with u_x.

The given homomorphism θ induces the homomorphism $\varphi : \Pi_0 \xrightarrow{u} G \xrightarrow{\theta} \operatorname{Aut} E_0$, that is,

$$\varphi(x) = \theta_{u_x}. \tag{3.2}$$

Hereafter, we refer to $u_x, f(x, y), \varphi(x)$ as before.

Now, we choose $h(x, y) \in E_0$ such that

$$\gamma\pi[h(x, y)] = f(x, y), \tag{3.3}$$

in particular, choose $h(x, 1) = h(1, y) = 0$. Thus,

$$\mu_{u_x} f(y, z) \overset{(3.3)}{=} \mu_{u_x}[\gamma\pi h(y, z)] \overset{(2.6)}{=} \gamma\pi\theta_{u_x}(h(y, z)) \overset{(3.2)}{=} \gamma\pi\varphi(x)h(y, z).$$

Take inverse image in E_0 for two sides of the equation (3.1) via the homomorphism $\gamma\pi : E_0 \to G$, we obtain

$$\varphi(x)h(y, z) + h(x, yz) = h(x, y) + h(xy, z) + k(x, y, z), \tag{3.4}$$

where $k(x, y, z) \in \operatorname{Ker}(\gamma\pi) = \operatorname{Ker}\pi = A \subset ZE_0$.

The relation (3.4) can be formally written in the form $k = \delta h$, even though E_0 is non-abelian.

Lemma 3.1. *The function k given by (3.4) is a 3-cocycle in $Z^3(\Pi_0, A)$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G_0 \arrow[r, "\gamma"] & G \arrow[r, "\sigma"] \arrow[d, "\theta"] & \Pi_0 \arrow[r] & 1, \\ & & \operatorname{Aut} E_0 & & \end{tikzcd}$$

by the condition (2.5), $(\theta\gamma)G_0 = \mu E_0$. Then, there is a homomorphism $\psi : \Pi_0 \to \mathrm{AutExt}E_0$ making the following diagram

$$\begin{tikzcd} 0 \arrow[r] & G_0 \arrow[r, "\gamma"] & G \arrow[r, "\sigma"] \arrow[d, "\theta"] & \Pi_0 \arrow[r] \arrow[d, "\psi"] & 1 \\ & & \operatorname{Aut} E_0 \arrow[r, "\nu"] & \mathrm{AutExt}E_0 \arrow[r] & 1 \end{tikzcd}$$

commute, where ν is the natural projection. Thus, k is just an obstruction of the abstract kernel (Π_0, E_0, ψ). Due to Lemma 8.4 ([5] -Chapter IV), k is a 3-cocycle of $B(Z\Pi_0)$. Moreover, because of its construction, k takes values in A, as required. □

Lemma 3.2. *For given representatives u_x in G, a change in the choice of $h : \Pi_0^2 \to E_0$ replaces k by a cohomologous cocycle. Moreover, by suitably changing the choice of h, k may be replaced by any cohomologous cocycle.*

Proof. In the proof of Lemma 8.5 ([5] -Chapter IV), replacing the functions with values in the central C by those in A, we obtain the proof of Lemma 3.2. □

Lemma 3.3. *A change in the choice of u_x in G may be followed by a suitable new selection of h satisfying* (3.3) *such as to leave the function k unchanged.*

Proof. If u_x is replaced by u'_x such that $u'_1 = 0$, then $u'_x = g_x + u_x$, where $g : \Pi_0 \to \gamma G_0$ satisfies $g_1 = 0$. Thus, there is a function $t : \Pi_0 \to E_0$ such that $\gamma\pi(t_x) = g_x$.

Now, one determines a function $h' : \Pi_0^2 \to E_0$, given by

$$h'(x, y) = t_x + \theta_{u_x}(t_y) + h(x, y) - t_{xy}. \tag{3.5}$$

Thanks to the condition (2.6), it is easy to check that

$$\gamma\pi h'(x, y) = u'_x + u'_y - u'_{xy} = f'(x, y).$$

Hence, $h'(x, y)$ is just a factor set of B induced by the representatives u'_x in G. Thanks to (2.5) and (3.5), we can transform $\varphi(x)[h'(y, z)] + h'(x, yz)$ into $h'(x, y) + h'(xy, z) + k(x, y, z)$. This proves that k is unchanged. □

From the above proved lemmas, we obtain the following proposition.

Proposition 3.4. *For any triple (α, γ, θ), the cohomology class $[k] \in H^3(\Pi_0, A)$, where k is given by* (3.4)*, does not depend on the choice of the representatives u_x and the factor set $h(x, y)$.*

The cohomology class $[k] \in H^3(\Pi_0, A)$ is called an *obstruction* to an (α, γ)-prolongation, and we denote $[k] = \mathrm{Obs}(\alpha, \gamma, \theta)$.

Theorem 3.5. *An extension* $\mathcal{E}_0$ *has an* (α, γ)*-prolongation if and only if* $\mathrm{Obs}(\alpha, \gamma, \theta)$ *vanishes in* $H^3(\Pi_0, A)$.

Proof. Necessary condition. Let $\mathcal{E}$ be an (α, γ)-prolongation of $\mathcal{E}_0$ inducing θ. Recall that for the representatives u_x in G, we have a factor set $f(x, y)$ satisfying the relation (3.1). By Lemma 2.1, B is an extension of E_0 by Π_0, and hence we can choose the representatives v_x in B, $x \in \Pi_0$, such that $p(v_x) = u_x$. This set of representatives gives a factor set εh, where $h : \Pi_0^2 \to E_0$, that is,

$$\varepsilon h(x, y) = v_x + v_y - v_{xy}.$$

Then,

$$\gamma\pi h(x, y) \overset{(2.2)}{=} p\varepsilon h(x, y) = p(v_x + v_y - v_{xy}) = u_x + u_y - u_{xy} = f(x, y),$$

that is, h satisfies (3.3).

Since εh is a factor set of the extension B corresponding to the representatives v_x, we obtain

$$\mu_{v_x}[\varepsilon h(y, z)] + \varepsilon h(x, yz) = \varepsilon h(x, y) + \varepsilon h(xy, z).$$

We need to turn the above equality into the equality (3.4) to determine the function k. Thanks to the monomorphic property of ε and the relation

$$\mu_{v_x}[\varepsilon h(y, z)] \overset{(2.3)}{=} \varepsilon\phi_{v_x} h(y, z) \overset{(2.4)}{=} \varepsilon\theta_{u_x} h(y, z) \overset{(3.2)}{=} \varepsilon\varphi(x)h(y, z),$$

the above equality becomes

$$\varphi(x)h(y, z) + h(x, yz) = h(x, y) + h(xy, z). \tag{3.6}$$

According to the determination of k in (3.4), we deduce that $[k] = 0$.

Sufficient condition. Conversely, let $\mathrm{Obs}(\alpha, \gamma, \theta) = 0$ in $H^3(\Pi_0, A)$, that is,

$$k = \delta l,\ l : \Pi_0^2 \to A.$$

Now, for $h' = h - l$, we obtain

$$k' = \delta h' = \delta h - \delta l = k - k = 0.$$

This means that one can choose $h : \Pi_0^2 \to E_0$ such that $[k] = 0$ in $Z^3(\Pi_0, A)$. Then, the relation (3.4) becomes $\delta h = 0$.

According to the relations (2.5) and (3.3), the function φ given by (3.2) satisfies

$$\varphi(x)\varphi(y) = \mu_{h(x,y)}\varphi(xy). \tag{3.7}$$

Clearly, $\varphi(1) = id_{E_0}$. Thus, we can construct the crossed product $B_h = [E_0, \varphi, h, \Pi_0]$, that is, $B_h = E_0 \times \Pi_0$ under the operation

$$(e_0, x) + (e_0', y) = (e_0 + \varphi(x)e_0' + h(x, y), xy),$$

and there is an exact sequence

$$0 \to A \xrightarrow{j'} B_h \xrightarrow{p'} G \to 0,$$

where

$$j'(a) = (ia, 1), \quad p'(e_0, x) = \gamma\pi e_0 + u_x.$$

Moreover, the correspondence $\beta' : B_0 \to B_h$ given by

$$\beta'(b_0) = (\bar{b}_0, 1) \tag{3.8}$$

is a group homomorphism.

Now, it is easy to check that the following diagram

$$\begin{tikzcd}
\mathcal{E}_0 : & 0 \arrow[r] & A_0 \arrow[r, "j_0"] \arrow[d, "\alpha"] & B_0 \arrow[r, "p_0"] \arrow[d, "\beta'"] & G_0 \arrow[r] \arrow[d, "\gamma"] & 0 \\
\mathcal{E}_h : & 0 \arrow[r] & A \arrow[r, "j'"] & B_h \arrow[r, "p'"] & G \arrow[r] & 0
\end{tikzcd}$$

commutes. Therefore, $\mathcal{E}_h$ is an (α, γ)-prolongation of $\mathcal{E}_0$. This completes the proof. □

4. Classification theorem

Definition 4.1. Two (α, γ)-prolongations of $\mathcal{E}_0$

$$0 \to A \xrightarrow{j} B \xrightarrow{p} G \to 0,$$

$$0 \to A \xrightarrow{j'} B' \xrightarrow{p'} G \to 0$$

are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, equal] & B \arrow[r, "p"] \arrow[d, "\beta^*"] & G \arrow[r] \arrow[d, equal] & 0, & B_0 \arrow[r, "\beta"] & B \\
0 \arrow[r] & A \arrow[r, "j'"] & B' \arrow[r, "p'"] & G \arrow[r] & 0, & B_0 \arrow[r, "\beta'"] & B'
\end{tikzcd}$$

such that $\beta^*\beta = \beta'$.

Lemma 4.2. *Any (α, γ)-prolongation of $\mathcal{E}_0$ is equivalent to a crossed product extension.*

Proof. Let $\mathcal{E}$ be an (α, γ)-prolongation of $\mathcal{E}_0$ inducing $\theta : G \to \operatorname{Aut} E_0$. By the exact sequence (2.1), we choose representatives v_x in B such that $p(v_x) = u_x$. This set of representatives yields a function $h : \Pi_0^2 \to E_0$ satisfing (3.3). Then, due to Theorem 3.5 the functions φ and h satisfy the relations (3.6) and (3.7). Thus, there is the crossed product $B_h = [E_0, \varphi, h, \Pi_0]$, and hence the crossed product extension $\mathcal{E}_h$ is an (α, γ)-prolongation of $\mathcal{E}_0$. Now we show that $\mathcal{E}$ is equivalent to $\mathcal{E}_h$.

Thanks to the exact sequence (2.1), each element of B is written in the form $b = \varepsilon e_0 + v_x$. Then, it is easy to check that the correspondence

$$\beta^* : B \to B_h, \quad \varepsilon e_0 + v_x \mapsto (e_0, x)$$

is an isomorphism satisfying the following commutative diagram

$$\begin{array}{ccccccccccccc}
\mathcal{E}:0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & G & \longrightarrow & 0, & \qquad & B_0 & \xrightarrow{\beta} & B \\
 & & \| & & \downarrow{\scriptstyle \beta^*} & & \| & & & & & & \\
\mathcal{E}_h:0 & \longrightarrow & A & \xrightarrow{j'} & B_h & \xrightarrow{p'} & G & \longrightarrow & 0, & \qquad & B_0 & \xrightarrow{\beta'} & B_h.
\end{array}$$

Finally, for all $b_0 \in B_0$ we have

$$\beta^*\beta(b_0) = \beta^*\varepsilon(\overline{b_0}) = (\overline{b_0}, 1) \overset{(3.8)}{=} \beta'(b_0).$$

Therefore, two extensions $\mathcal{E}$ and $\mathcal{E}_h$ are equivalent, as claimed. □

Theorem 4.3. *Any (α, γ)-prolongation of $\mathcal{E}_0$ is a central extension.*

Proof. First, it is easy to check that the crossed product extension $\mathcal{E}_h$ mentioned in the proof of Theorem 3.5 is a central extension. This follows directly from the definition of operation in the crossed product B_h and the hypothesis $A_0 \subset ZB_0$. Now, if $\mathcal{E}$ is an (α, γ)-prolongation of $\mathcal{E}_0$, then by Lemma 4.2, $\mathcal{E}$ is equivalent to $\mathcal{E}_h$. Therefore, $\mathcal{E}$ is a central extension and so the proof is completed. □

Lemma 4.4. *The function $h : \Pi_0^2 \to E_0$ satisfying* (3.3) *determines a 2-cocycle h_* with values in A.*

Proof. For representatives $\{b_r | r \in G\}$ of an extension

$$0 \to A \xrightarrow{j} B \xrightarrow{p} G \to 0,$$

a factor set $\kappa(r, s) = b_r + b_s - b_{r+s}$ takes values in jA. Then, for $r = u_x, s = u_y$ choose $b_r = v_x, b_s = v_y$, one has

$$\varepsilon h(x, y) = v_x + v_y - v_{xy} = b_r + b_s - b_{r+s} = \kappa(r, s) \in jA.$$

Hence, we obtain a 2-cocycle $h_* = (j^{-1}\varepsilon)_* h : \Pi_0^2 \to A$. □

Lemma 4.5. *Let $\mathcal{E}$ be an (α, γ)-prolongation of $\mathcal{E}_0$. Then, the 2-cocycle $h_* : \Pi_0^2 \to A$ in Lemma 4.4 is uniquely determined up to a coboundary $\delta(t_*) \in B^2(\Pi_0, A)$.*

Proof. If v'_x is another representative of Π_0 in B such that $p(v'_x) = p(v_x) = u_x$, then there exists a function $t : \Pi_0 \to E_0$ such that $v'_x = \varepsilon t_x + v_x$. This set of representatives gives a factor set $\varepsilon h'$, where $h' : \Pi_0^2 \to E_0$ is a function satisfying (3.3). Then,

$$\begin{aligned}
\varepsilon h'(x, y) &= v'_x + v'_y - v'_{xy} \\
&= (\varepsilon t_x + v_x) + (\varepsilon t_y + v_y) - (\varepsilon t_{xy} + v_{xy}) \\
&= \varepsilon t_x + \mu_{v_x}(\varepsilon t_y) + v_x + v_y - v_{xy} - \varepsilon t_{xy} \\
&= \varepsilon t_x + \varepsilon\varphi(x)(t_y) + \varepsilon h(x, y) - \varepsilon t_{xy}.
\end{aligned}$$

Since $\varepsilon t_{xy} \in \operatorname{Ker} p = \operatorname{Im} j$ and since $jA \subset Z(\beta B_0) = Z(\varepsilon E_0)$, one has

$$\varepsilon h'(x, y) = \varepsilon[t_x + \varphi(x)(t_y) - t_{xy}] + \varepsilon h(x, y).$$

Again, since ε is a monomorphism so

$$h'(x,y) - h(x,y) = t_x + \varphi(x)(t_y) - t_{xy} = \delta t(x,y).$$

Set $t_* = (\varepsilon^{-1}j)_* t$, we have $h'_* - h_* = \delta(t_*)$, as claimed. □

It follows from Lemmas 4.4 and 4.5 that

Corollary 4.6. *The cohomology class of h_* is uniquely determined in $H^2(\Pi_0, A)$.*

The following corollary is deduced from Lemma 4.2.

Corollary 4.7. *Two crossed products $B_h = [E_0, \varphi, h, \Pi_0]$ and $B_{h'} = [E_0, \varphi, h', \Pi_0]$ are equivalent if and only if the cohomology classes of h and h' are equal in $H^2(\Pi_0, A)$.*

Denote by $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ the set of all equivalence classes of (α, γ)-prolongations of $\mathcal{E}_0$, we obtain the following main result.

Theorem 4.8 (Schreier theory for (α, γ)-prolongations of central extensions)**.** *If $\mathcal{E}_0$ has an (α, γ)-prolongation, then the set $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ is torseur under the group $H^2(\Pi_0, A)$.*

Proof. Firstly, we show that $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ is torseur under the group $H^2(\Pi_0, \varepsilon^{-1}j(A))$. In fact, by Corollary 4.6, we define a map ω from $H^2(\Pi_0, \varepsilon^{-1}j(A))$ onto the group of transformations of $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ by formula

$$\omega(\overline{\tau})(\mathrm{cls}[E_0, \varphi, h, \Pi_0]) = \mathrm{cls}[E_0, \varphi, h + \tau, \Pi_0].$$

From the above arguments, ω is really an element of the group of transformations of $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$. Furthermore, ω is a group homomorphism.

To prove that $\mathrm{Ext}_{(\alpha,\gamma)}(G, A)$ is a torseur under $H^2(\Pi_0, \varepsilon^{-1}j(A))$, we also point out that for any two (α, γ)-prolongations $\mathcal{E}_1$, $\mathcal{E}_2$, there is always the only element $\overline{\tau} \in H^2(\Pi_0, \varepsilon^{-1}j(A))$ such that

$$\mathrm{cls}\mathcal{E}_2 = \omega(\overline{\tau})(\mathrm{cls}\mathcal{E}_1).$$

In fact, we have $\mathrm{cls}\mathcal{E}_i = \mathrm{cls}[E_0, \varphi, h_i, \Pi_0]$, $i = 1, 2$, where $\varphi(x) = \theta_{u_x}$, and h_i's are functions defined by sets of representatives $v^i_x \in B_{h_i}$, where v^i_x's satisfy $p(v^i_x) = u_x$. Then, thanks to the proof of Theorem 3.5, one has

$$\gamma\pi[h_i(x,y)] = f(x,y).$$

It follows that $h_2 = h_1 + r,\ r(x,y) \in \mathrm{Ker}(\gamma\pi) = \mathrm{Ker}\,\pi = \varepsilon^{-1}j(A)$.

By transformations based on the established formula, we have $\delta r = 0$, that is, $r \in Z^2(\Pi_0, \varepsilon^{-1}j(A))$.

Finally, by the canonical isomorphism

$$H^2(\Pi_0, \varepsilon^{-1}j(A)) \leftrightarrow H^2(\Pi_0, A),$$

the theorem is proved. □

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