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ON FINITE A-PERFECT ABELIAN GROUPS

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Abstract. Let G be a group and $A = Aut(G)$ be the group of automorphisms of G. Then the element $[g, \alpha] = g^{-1}\alpha(g)$ is an autocommutator of $g \in G$ and $\alpha \in A$. Also, the autocommutator subgroup of G is defined to be $K(G) = \langle [g, \alpha] | g \in G, \alpha \in A\rangle$, which is a characteristic subgroup of G containing the derived subgroup G' of G. A group is defined as A-perfect, if it equals its own autocommutator subgroup. The present research is aimed at classifying finite abelian groups which are A-perfect.

1. Introduction

Let G be a group and $A = Aut(G)$ denote the group of automorphisms of G. As in [2], if $g \in G$ and $\alpha \in A$, then the element $[g, \alpha] = g^{-1}\alpha(g)$ is an *autocommutator* of g and α. Now, using this notation, the *autocommutator subgroup* of G is defined as:

$$K(G) = [G, A] = \langle [g, \alpha] | g \in G, \alpha \in A\rangle,$$

which is a characteristic subgroup of G. Particularly, if α is taken to be an inner automorphism, then the autocommutator subgroup is the derived subgroup G' of G. A group G is said to be *perfect* if $G = G'$. Here, the concept of A-perfect groups would be introduced. A group G would be known as *A-perfect*, if $G = K(G)$.

Example 1.1. *All non abelian simple groups are A-perfect.*

Example 1.2. *Let $G = D_8$, dihedral group of order 8. Then $K(G) \simeq \mathbb{Z}_4$ and hence, G is not A-perfect.*

Perfect groups are A-perfect, but the converse is not true in general.

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Example 1.3. *One can easily check that*
$K(\mathbb{Z}_3) = \mathbb{Z}_3$, $K(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $K(Q_8) = Q_8$, *where* Q_8 *is generalized quaternion group of order 8.*

The present paper is aimed at classifying finite abelian groups which are A-perfect.

2. Preliminary Results

We begin with some useful results that will be used in the proof of our main theorem. Throughout this paper $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$ for any natural number n. To be brief, $([k]_n, [k']_m)$ of group $\mathbb{Z}_n \times \mathbb{Z}_m$ would be indicated as (k, k'), where $k \in \{0, 1, 2, \ldots, n-1\}$ and $k' \in \{0, 1, 2, \ldots, m-1\}$.

Lemma 2.1. *i) Let* H *and* T *be two groups. Then,*

$$K(H) \times K(T) \subseteq K(H \times T).$$

ii) Let H *and* T *be finite groups such that* $(|H|, |T|) = 1$. *Then,*

$$K(H) \times K(T) = K(H \times T).$$

Proof. i) For $\alpha \in Aut(H)$ and $\beta \in Aut(T)$ we define the automorphism $\alpha \times \beta$ of group $H \times T$, given by $(\alpha \times \beta)((h, t)) = (\alpha(h), \beta(t))$ for all $h \in H$ and $t \in T$. It is easy to check that $([h, \alpha], [t, \beta]) = [(h, t), \alpha \times \beta]$. This implies the result.
ii) It is sufficient to prove $K(H \times T) \subseteq K(H) \times K(T)$. It is easy to check that $\gamma|_H \in Aut(H)$ and $\gamma|_T \in Aut(T)$, for all $\gamma \in Aut(H \times T)$. Now $[(h, t), \gamma] = ([h, \gamma|_H], [t, \gamma|_T])$, for all $h \in H$ and $t \in T$ and $\gamma \in Aut(H \times T)$. This implies the result. □

In lemma 2.1(i), the equality is not true in general and the converse of lemma 2.1(ii) do not hold, as can be seen by the following examples.

Example 2.2. *Let* $H = T = \mathbb{Z}_2$. *Then* $K(H) = K(T) = \langle 0 \rangle$, *but* $K(H \times T) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 2.3. *Let* $H = T = \mathbb{Z}_3$. *Then* $K(H) = K(T) = \mathbb{Z}_3$ *and* $K(H \times T) = \mathbb{Z}_3 \times \mathbb{Z}_3$.

The following corollary is the immediate result of the above lemma.

Corollary 2.4. *If* H *and* T *are A-perfect groups, then so is* $H \times T$.

Lemma 2.5. *([1]) If* G *is a finite cyclic group, then* $K(G) = G^2$.

Corollary 2.6. *If* G *is a finite abelian group of odd order, then* G *is A-perfect.*

Proof. G is a direct product of finitely many $\mathbb{Z}_{p^t}$, where p is an odd prime number and $t \geq 1$. Hence, the result is true due to lemma 2.5 and corollary 2.4. □

Lemma 2.7. *([1]) For all nonnegative integers* $n > m_1 \geq \cdots \geq m_k$,

$$K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}}) = \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}}.$$

Lemma 2.8. *For all nonnegative integers $n \geq m_1 \geq m_2 \geq ... \geq m_k$,*

$$K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}}) = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}}.$$

Proof. We define the automorphisms α,α',$\beta_1, \beta_2, ..., \beta_k$ of the group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}}$, given by
$\alpha((a, b, c_1, \ldots, c_k)) = (a + b, b, c_1, \ldots, c_k)$, $\alpha'((a, b, c_1, \ldots, c_k)) = (a, a + b, c_1, \ldots, c_k)$,
$\beta_1((a, b, c_1, \ldots, c_k)) = (a, b, a + c_1, c_2, \ldots, c_k)$,
$\beta_2((a, b, c_1, \ldots, c_k)) = (a, b, c_1, a + c_2, c_3, \ldots, c_k)$,
$\vdots$
$\beta_k((a, b, c_1, \ldots, c_k)) = (a, b, c_1, c_2, \ldots, c_{k-1}, a + c_k)$,
for all $a, b \in \{0, 1, 2, \ldots, 2^n - 1\}$ and $c_i \in \{0, 1, 2, \ldots, 2^{m_i} - 1\}, 1 \leq i \leq k$. Clearly
$(a, 0, \ldots, 0) = [(0, a, 0, \ldots, 0), \alpha]$, $(0, b, 0, \ldots, 0) = [(b, 0, \ldots, 0), \alpha']$,
$(0, 0, c_1, 0, \ldots, 0) = [(c_1, 0, \ldots, 0), \beta_1]$,
$(0, 0, 0, c_2, 0, \ldots, 0) = [(c_2, 0, \ldots, 0), \beta_2]$,
$\vdots$
$(0, 0, 0, \ldots, 0, c_k) = [(c_k, 0, \ldots, 0), \beta_k]$.
These imply that

$$\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}} \subseteq K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}})$$

and lead to the result. □

3. The Main Results

Theorem 3.1. *A finite abelian group G is A-perfect if and only if $G \simeq \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}} \times H$ for some nonnegative integers $n \geq m_1 \geq m_2 \geq \cdots \geq m_k$, where H is a finite abelian group of odd order.*

Proof. The necessary condition follows from Lemma 2.8, Corollary 2.6 and Corollary 2.4. Now, for the reverse conclusion, we assume that G is not a product of $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_k}} \times H$. So, it is $\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^{s_1}} \times \mathbb{Z}_{2^{s_2}} \times \cdots \times \mathbb{Z}_{2^{s_r}} \times N$, for all nonnegative integers $t > s_1 \geq s_2 \geq \cdots \geq s_r$, where N is a finite abelian group of odd order. Lemma 2.1 implies that $K(\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^{s_1}} \times \mathbb{Z}_{2^{s_2}} \times ... \times \mathbb{Z}_{2^{s_k}} \times N) = K(\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^{s_1}} \times \mathbb{Z}_{2^{s_2}} \times \cdots \times \mathbb{Z}_{2^{s_k}}) \times K(N)$. Now the group G is not A-perfect due to Lemma 2.7. Hence, the proof is completed. □

The following corollary is the immediate result of the above theorem.

Corollary 3.2. *For all natural numbers n and k, such that $k > 1$, if $G = \underbrace{\mathbb{Z}_n \times \ldots \times \mathbb{Z}_n}_{k-times}$, then G is A-perfect.*

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