

ON FINITE A-PERFECT ABELIAN GROUPS

M. M. NASRABADI* AND A. GHOLAMIAN

Communicated by Behrooz Khosravi

ABSTRACT. Let G be a group and $A = \text{Aut}(G)$ be the group of automorphisms of G . Then the element $[g, \alpha] = g^{-1}\alpha(g)$ is an autocommutator of $g \in G$ and $\alpha \in A$. Also, the autocommutator subgroup of G is defined to be $K(G) = \langle [g, \alpha] | g \in G, \alpha \in A \rangle$, which is a characteristic subgroup of G containing the derived subgroup G' of G . A group is defined as A-perfect, if it equals its own autocommutator subgroup. The present research is aimed at classifying finite abelian groups which are A-perfect.

1. Introduction

Let G be a group and $A = \text{Aut}(G)$ denote the group of automorphisms of G . As in [2], if $g \in G$ and $\alpha \in A$, then the element $[g, \alpha] = g^{-1}\alpha(g)$ is an *autocommutator* of g and α . Now, using this notation, the *autocommutator subgroup* of G is defined as:

$$K(G) = [G, A] = \langle [g, \alpha] | g \in G, \alpha \in A \rangle,$$

which is a characteristic subgroup of G . Particularly, if α is taken to be an inner automorphism, then the autocommutator subgroup is the derived subgroup G' of G . A group G is said to be *perfect* if $G = G'$. Here, the concept of A-perfect groups would be introduced. A group G would be known as *A-perfect*, if $G = K(G)$.

Example 1.1. *All non abelian simple groups are A-perfect.*

Example 1.2. *Let $G = D_8$, dihedral group of order 8. Then $K(G) \simeq \mathbb{Z}_4$ and hence, G is not A-perfect.*

Perfect groups are A-perfect, but the converse is not true in general.

MSC(2010): 20D45, 20D25.

Keywords: Automorphism, Autocommutator subgroup, A-perfect group, Finite abelian group.

Received: 17 November 2011, Accepted: 5 April 2012.

*Corresponding author.

Example 1.3. *One can easily check that*

$K(\mathbb{Z}_3) = \mathbb{Z}_3$, $K(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $K(Q_8) = Q_8$, where Q_8 is generalized quaternion group of order 8.

The present paper is aimed at classifying finite abelian groups which are A-perfect.

2. Preliminary Results

We begin with some useful results that will be used in the proof of our main theorem. Throughout this paper $\mathbb{Z}_n = \{[0], [1], [2], \dots, [n-1]\}$ for any natural number n . To be brief, $([k]_n, [k']_m)$ of group $\mathbb{Z}_n \times \mathbb{Z}_m$ would be indicated as (k, k') , where $k \in \{0, 1, 2, \dots, n-1\}$ and $k' \in \{0, 1, 2, \dots, m-1\}$.

Lemma 2.1. *i) Let H and T be two groups. Then,*

$$K(H) \times K(T) \subseteq K(H \times T).$$

ii) Let H and T be finite groups such that $(|H|, |T|) = 1$. Then,

$$K(H) \times K(T) = K(H \times T).$$

Proof. i) For $\alpha \in \text{Aut}(H)$ and $\beta \in \text{Aut}(T)$ we define the automorphism $\alpha \times \beta$ of group $H \times T$, given by $(\alpha \times \beta)((h, t)) = (\alpha(h), \beta(t))$ for all $h \in H$ and $t \in T$. It is easy to check that $([h, \alpha], [t, \beta]) = [(h, t), \alpha \times \beta]$. This implies the result.

ii) It is sufficient to prove $K(H \times T) \subseteq K(H) \times K(T)$. It is easy to check that $\gamma|_H \in \text{Aut}(H)$ and $\gamma|_T \in \text{Aut}(T)$, for all $\gamma \in \text{Aut}(H \times T)$. Now $[(h, t), \gamma] = ([h, \gamma|_H], [t, \gamma|_T])$, for all $h \in H$ and $t \in T$ and $\gamma \in \text{Aut}(H \times T)$. This implies the result. \square

In lemma 2.1(i), the equality is not true in general and the converse of lemma 2.1(ii) do not hold, as can be seen by the following examples.

Example 2.2. *Let $H = T = \mathbb{Z}_2$. Then $K(H) = K(T) = \langle 0 \rangle$, but $K(H \times T) = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Example 2.3. *Let $H = T = \mathbb{Z}_3$. Then $K(H) = K(T) = \mathbb{Z}_3$ and $K(H \times T) = \mathbb{Z}_3 \times \mathbb{Z}_3$.*

The following corollary is the immediate result of the above lemma.

Corollary 2.4. *If H and T are A-perfect groups, then so is $H \times T$.*

Lemma 2.5. *([1]) If G is a finite cyclic group, then $K(G) = G^2$.*

Corollary 2.6. *If G is a finite abelian group of odd order, then G is A-perfect.*

Proof. G is a direct product of finitely many \mathbb{Z}_{p^t} , where p is an odd prime number and $t \geq 1$. Hence, the result is true due to lemma 2.5 and corollary 2.4. \square

Lemma 2.7. *([1]) For all nonnegative integers $n > m_1 \geq \dots \geq m_k$,*

$$K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}}) = \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}}.$$

Lemma 2.8. *For all nonnegative integers $n \geq m_1 \geq m_2 \geq \dots \geq m_k$,*

$$K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}}) = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}}.$$

Proof. We define the automorphisms $\alpha, \alpha', \beta_1, \beta_2, \dots, \beta_k$ of the group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}}$, given by

$$\alpha((a, b, c_1, \dots, c_k)) = (a + b, b, c_1, \dots, c_k), \quad \alpha'((a, b, c_1, \dots, c_k)) = (a, a + b, c_1, \dots, c_k),$$

$$\beta_1((a, b, c_1, \dots, c_k)) = (a, b, a + c_1, c_2, \dots, c_k),$$

$$\beta_2((a, b, c_1, \dots, c_k)) = (a, b, c_1, a + c_2, c_3, \dots, c_k),$$

\vdots

$$\beta_k((a, b, c_1, \dots, c_k)) = (a, b, c_1, c_2, \dots, c_{k-1}, a + c_k),$$

for all $a, b \in \{0, 1, 2, \dots, 2^n - 1\}$ and $c_i \in \{0, 1, 2, \dots, 2^{m_i} - 1\}, 1 \leq i \leq k$. Clearly

$$(a, 0, \dots, 0) = [(0, a, 0, \dots, 0), \alpha], \quad (0, b, 0, \dots, 0) = [(b, 0, \dots, 0), \alpha'],$$

$$(0, 0, c_1, 0, \dots, 0) = [(c_1, 0, \dots, 0), \beta_1],$$

$$(0, 0, 0, c_2, 0, \dots, 0) = [(c_2, 0, \dots, 0), \beta_2],$$

\vdots

$$(0, 0, 0, \dots, 0, c_k) = [(c_k, 0, \dots, 0), \beta_k].$$

These imply that

$$\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}} \subseteq K(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}})$$

and lead to the result. \square

3. The Main Results

Theorem 3.1. *A finite abelian group G is A-perfect if and only if $G \simeq \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}} \times H$ for some nonnegative integers $n \geq m_1 \geq m_2 \geq \dots \geq m_k$, where H is a finite abelian group of odd order.*

Proof. The necessary condition follows from Lemma 2.8, Corollary 2.6 and Corollary 2.4. Now, for the reverse conclusion, we assume that G is not a product of $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \dots \times \mathbb{Z}_{2^{m_k}} \times H$. So, it is $\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^{s_1}} \times \mathbb{Z}_{2^{s_2}} \times \dots \times \mathbb{Z}_{2^{s_r}} \times N$, for all nonnegative integers $t > s_1 \geq s_2 \geq \dots \geq s_r$, where N is a finite abelian group of odd order. Lemma 2.1 implies that $K(\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^{s_1}} \times \mathbb{Z}_{2^{s_2}} \times \dots \times \mathbb{Z}_{2^{s_k}} \times N) = K(\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^{s_1}} \times \mathbb{Z}_{2^{s_2}} \times \dots \times \mathbb{Z}_{2^{s_k}}) \times K(N)$. Now the group G is not A-perfect due to Lemma 2.7. Hence, the proof is completed. \square

The following corollary is the immediate result of the above theorem.

Corollary 3.2. *For all natural numbers n and k , such that $k > 1$, if $G = \underbrace{\mathbb{Z}_n \times \dots \times \mathbb{Z}_n}_{k\text{-times}}$, then G is A-perfect.*

REFERENCES

- [1] C. Chiš, M. Chiš, and G. Silberberg, Abelian groups as autocommutator groups, *Arch. Math. (Basel)*., **90** no. 6 (2008) 490–492.
- [2] P. Hegarty, The absolute centre of a group, *J. Algebra*, **169** no. 3 (1994) 929–935.

Mohammad Mehdi Nasrabadi

Department of Mathematics, University of Birjand, Birjand, Iran

Email: mnasrabadi@birjand.ac.ir

Ali Gholamian

Department of Mathematics, University of Birjand, Birjand, Iran

Email: ali.ghfath@gmail.com

Archive of SID