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ON HUPPERT'S CONJECTURE FOR $F_4(2)$

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Abstract. Let G be a finite group and let $\mathrm{cd}(G)$ be the set of all complex irreducible character degrees of G. B. Huppert conjectured that if H is a finite nonabelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$, then $G \cong H \times A$, where A is an abelian group. In this paper, we verify the conjecture for $F_4(2)$.

1. Introduction

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. We denote by $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ the set of character degrees of G. In [6], B. Huppert proposed the following conjecture.

Huppert Conjecture. *Let G be a finite group and let H be a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.*

Huppert himself verified the conjecture for the Suzuki groups, the family of simple groups $\mathrm{PSL}_2(q)$ for $q \geq 4$, and many of the sporadic simple groups. For recent results on this conjecture, see [6, 7, 9, 10, 11]. In this paper, we extend Huppert's arguments to verify the conjecture for the simple exceptional group of Lie type $F_4(2)$. We prove the following theorem.

Theorem 1.1. *Let G be a finite group such that the character degree sets of G and $F_4(2)$ are the same. Then $G \cong F_4(2) \times A$, where A is an abelian group.*

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We note that this group is singled out from the infinite family $F_4(q)$, where q is a prime power, since it has two distinct nontrivial power degrees (see Lemma 3.1) and an exceptional Schur cover. For the proof of Theorem 1.1, we will follow the steps outlined by Huppert in [6]. Huppert's proofs rely upon the completion of the following five steps.

(1) Show $G' = G''$. Hence if G'/M is a chief factor of G, then $G'/M \cong S^k$, where S is a nonabelian simple group.
(2) Identify H as a chief factor G'/M of G.
(3) Show that if $\theta \in \mathrm{Irr}(M)$ and $\theta(1) = 1$, then θ is G'-invariant, which implies that $[M, G'] = M'$.
(4) Show that $M = 1$.
(5) Show that $G \cong G' \times C_G(G')$. As $G/G' \cong C_G(G')$ is abelian and $G' \cong H$, Huppert's Conjecture is verified.

We note that a weaker version of Huppert's Conjecture which asserts that nonabelian simple groups are uniquely determined by their character degrees (counting multiplicities) or equivalently by their complex group algebras has been answered positively (see [8]).

Notation. If n is an integer then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a group, we will write $\pi(G)$ instead of $\pi(|G|)$ to denote the set of all prime divisors of the order of G. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. Finally, the set of all irreducible constituents of θ^G is denoted by $\mathrm{Irr}(G|\theta)$. Other notation is standard.

2. **Background Results**

In this section, we present some results that we will need for the proof of Huppert's Conjecture.

Lemma 2.1. ([6, Lemma 2]). *Suppose $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$.*

(*a*) *If $\chi_N = \theta_1 + \theta_2 + \cdots + \theta_k$ with $\theta_i \in \mathrm{Irr}(N)$, then k divides $|G/N|$. In particular, if $\chi(1)$ is prime to $|G/N|$ then $\chi_N \in \mathrm{Irr}(N)$.*

(*b*) (Gallagher's Theorem) *If $\chi_N \in \mathrm{Irr}(N)$, then $\chi\psi \in \mathrm{Irr}(G)$ for every $\psi \in \mathrm{Irr}(G/N)$.*

Lemma 2.2. ([6, Lemma 3]). *Suppose $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. Let $I = I_G(\theta)$. Then*

(*a*) *If $\theta^I = \sum_{i=1}^k \varphi_i$ with $\varphi_i \in \mathrm{Irr}(I)$, then $\varphi_i^G \in \mathrm{Irr}(G)$. Hence, $\varphi_i(1)|G : I| \in \mathrm{cd}(G)$.*

(*b*) *If $\rho \in \mathrm{Irr}(I)$ such that $\rho_N = e\theta$, then $\rho = \theta_0\tau_0$, where θ_0 is a character of an irreducible projective representation of I of degree $\theta(1)$ while τ_0 is the character of an irreducible projective representation of I/N of degree e.*

The following result will be used to verify step 1 of Huppert's method.

Lemma 2.3. ([7, Lemma 2.3]). *Let G/N be a solvable factor group of G, minimal with respect to being nonabelian. Then two cases can occur.*

(*a*) *G/N is an r-group for some prime r. Hence there exists $\psi \in \mathrm{Irr}(G/N)$ such that $\psi(1) = r^b > 1$. If $\chi \in \mathrm{Irr}(G)$ and $r \nmid \chi(1)$, then $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$.*

(*b*) *G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N. Then $f = |G : F| \in \mathrm{cd}(G)$ and $|F/N| = r^a$ for some prime r, and F/N is an irreducible module for the cyclic group*

G/F, hence a is the smallest integer such that $r^a \equiv 1 (mod\ f)$. If $\psi \in \mathrm{Irr}(F)$ then either $f\psi(1) \in \mathrm{cd}(G)$ or r^a divides $\psi(1)^2$. In the latter case, r divides $\psi(1)$.

(1) If no proper multiple of f is in $\mathrm{cd}(G)$, then $\chi(1)$ divides f for all $\chi \in \mathrm{Irr}(G)$ such that $r \nmid \chi(1)$, and if $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \nmid f$, then $r^a \mid \chi(1)^2$.

(2) If $\chi \in \mathrm{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\mathrm{cd}(G)$, then either f divides $\chi(1)$ or r^a divides $\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G, then $f = \chi(1)$ or $r^a \mid \chi(1)^2$.

Let $\chi \in \mathrm{Irr}(G)$. We say that $\chi(1)$ is an *isolated degree* of G if $\chi(1)$ is divisible by no proper nontrivial character degree of G, and no proper multiple of $\chi(1)$ is a character degree of G.

Lemma 2.4. *If S is a nonabelian simple group, then there exists a nontrivial irreducible character θ of S that extends to $\mathrm{Aut}(S)$. Moreover the following hold:*

(i) if S is an alternating group of degree at least 7, then S has two characters of consecutive degrees $n(n-3)/2$ and $(n-1)(n-2)/2$ that both extend to $\mathrm{Aut}(S)$.

(ii) if S is a sporadic simple group or the Tits group, then S has two nontrivial irreducible characters of coprime degrees which both extend to $\mathrm{Aut}(S)$.

(iii) if S is a simple group of Lie type then the Steinberg character St_S of S of degree $|S|_p$ extends to $\mathrm{Aut}(S)$.

Proof. This is [1, Theorems 2, 3, 4]. □

Lemma 2.5. ([1, Lemma 5]). *Let N be a minimal normal subgroup of G such that $N \cong S^k$, where S is a nonabelian simple group. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta^k \in \mathrm{Irr}(N)$ extends to G.*

3. Verifying Huppert's Conjecture for $F_4(2)$

Assume from now on that $H \cong F_4(2)$. Using [3], we collect some properties of the character degree set of H in the following lemma. We first have that

$$\pi(F_4(2)) = \{2, 3, 5, 7, 13, 17\}.$$

Lemma 3.1. *Let a and b be nontrivial character degrees of H. The following hold.*

(i) If $(13 \cdot 17, a) = 1$, then $a = 2^{24}$ or $a = 2^2 \cdot 3^6 \cdot 7^2$.

(ii) The only nontrivial proper power degrees of G are 2^{24} and $2^2 \cdot 3^6 \cdot 7^2$.

(iii) H has no consecutive degrees.

(iv) H possesses two irreducible characters $\chi_i, i = 1, 2$, with degrees $3^6 \cdot 5^2 \cdot 7^2 \cdot 13$ and $3^6 \cdot 5^2 \cdot 7^2 \cdot 17$, respectively, such that for each i, no proper multiples of $\chi_i(1)$ is a degree of H.

(v) If $a \neq 2^{24}$, then the 2-part of a is at most 2^{12}.

Proof. These results are easily checked given the character table of $F_4(2)$ in [3]. □

We define a "mixed" degree of G to be a character degree of G which is divisible by 2 but is not a power of 2.

3.1. **Verifying Step 1.** Suppose that $G' \neq G''$. Then there exists a solvable factor group G/N of G minimal with respect to being nonabelian. By Lemma 2.3, G/N is an r-group or a Frobenius group.

Case 1: G/N is an r-group for some prime r. Then there exists $\psi \in \mathrm{Irr}(G/N)$ such that $\psi(1) = r^b > 1$. As 2^{24} is the only nontrivial prime power degree of H, we deduce that $\psi(1) = 2^{24}$ and so $r = 2$. By [3], G has a nonlinear character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 7^2 \cdot 17$ is odd. As $(\chi(1), |G : N|) = 1$, by Lemma 2.1, we have $\chi_N \in \mathrm{Irr}(N)$ and hence by Gallagher's Theorem, we obtain that $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$. Thus $2^{24} < 2^{24}\chi(1) \in \mathrm{cd}(G)$, which is impossible.

Case 2: G/N is a Frobenius group with elementary abelian Frobenius kernel F/N, where $|F : N| = r^a$ for some prime r. In addition, $f = |G : F| \in \mathrm{cd}(G)$ and f divides $r^a - 1$.

Subcase 2(a): $r \neq 2$. As 2^{24} is an isolated degree of G and $r \nmid 2^{24}$, we deduce from Lemma 2.3(b) that $f = 2^{24}$, and hence no multiples of f is in $\mathrm{cd}(G)$. Let $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = 7^2 \cdot 17$ and let $\varphi \in \mathrm{Irr}(F)$ be an irreducible constituent of ψ when restricted to F. As $\psi(1)/\varphi(1)$ divides $f = 2^{24}$, we deduce that $\varphi(1) > 1$ and hence $\varphi(1)f$ is not a degree of G. Thus $r^a \mid \varphi(1)^2_{2'} = \psi(1)^2_{2'} = 7^4 \cdot 17^2$, and so $r^a \leq 7^4 \cdot 17^2$. As $f \mid r^a - 1$, we obtain that $f = 2^{24} \leq r^a - 1 \leq 7^4 \cdot 17^2$, which is impossible.

Subcase 2(b): $r = 2$. Let $\chi_i, i = 1, 2$ be two irreducible characters of G in Lemma 3.1(iv). As both $\chi_i(1)$ are odd, by Lemma 2.3(b), we have that $f \mid \chi_i(1)$ for all i, and hence f divides the greatest common divisor of $\chi_i(1)$, which is $3^6 \cdot 5^2 \cdot 7^2$. It follows that $(f, 13 \cdot 17) = 1$ and so by Lemma 3.1(i), $f = 2^{24}$ or $f = 2^2 \cdot 3^6 \cdot 7^2$. However both cases are impossible as f is odd. Thus $G' = G''$.

3.2. **Verifying Step 2.** We continue by proving Step 2 of Huppert's method. Recall that Step 2 asserts that if G'/M is a chief factor of G, then $G'/M \cong H$. Suppose G'/M is a chief factor of G. As $G' = G''$ by Step 1, $G'/M \cong S^k$, where S is a nonabelian simple group. We need to show that $G'/M \cong F_4(2)$.

(a) Eliminating the alternating groups of degree at least 7. By Lemma 2.4, S possesses two irreducible characters θ_1 and θ_2 of degrees $n(n-3)/2$ and $n(n-3)/2 + 1 = (n-1)(n-2)/2$, respectively and both θ_i extend to S_n. By Lemma 2.5, both θ_i^k extend to G and so $\theta_i^k(1)$ are degrees of G. Assume first that $k \geq 2$. Then both $\theta_i^k(1)$ are proper nontrivial power degrees and so by Lemma 3.1(ii), each $\theta_i^k(1)$ is either 2^{24} or $2^2 \cdot 3^6 \cdot 7^2$. As $(n-1, n-2) = 1$, $(n, n-3) = (3, n)$ and $n \geq 7$, we deduce that both $\theta_i^k(1)$ cannot be 2-powers. Therefore both $\theta_i^k(1)$ must be $2^2 \cdot 3^6 \cdot 7^2$, which is impossible as $\theta_1^k(1)$ and $\theta_2^k(1)$ are distinct. Thus $k = 1$. But then by Lemma 3.1(iii), G has no consecutive degrees which is a contradiction as $\theta_2(1) = \theta_1(1) + 1$. Therefore this case cannot happen.

(b) Eliminating the sporadic simple groups or the Tits group. We have $\pi(F_4(2)) = \{2, 3, 5, 7, 13, 17\}$. Since every character degree of S divides some degree of $F_4(2)$, we deduce that $\pi(S) \subseteq \pi(F_4(2))$. Using [3], we only need to consider two cases $S \cong J_2$ or $S \cong {}^2F_4(2)'$. In each case, the group S possesses an irreducible character θ of degree $2^2 \cdot 3^2$ and 3^3, respectively and this character extends to its corresponding automorphism group. Applying Lemma 2.5, we have that each $\theta^k(1)$ is a degree of G, which is a nontrivial proper power. However 3^{3k} and 6^{2k} can never be 2^{24} nor $2^2 \cdot 3^6 \cdot 7^2$, contradicting Lemma 3.1(ii).

(c) Eliminating simple groups of Lie type.

If S is a simple group of Lie type and St_S is the Steinberg character of S, then $\mathrm{St}_S(1)$ is a power of the prime p, where p is the defining characteristic of the group. Since St_S extends to the automorphism group of S, by Lemma 2.5 we have that $\mathrm{St}_S(1)^k$ is a degree of G. As the only composite power of a prime among degrees of G is 2^{24}, we must have that $\mathrm{St}_S(1)^k = 2^{24}$. Thus, the defining characteristic of the simple group S must be 2. Let $S = S(q_1)$ be defined over a field of q_1 elements, where q_1 is a power of the prime 2. Assume that $\mathrm{St}_S(1) = q_1^j$.

We first claim that $k = 1$. By way of contradiction, assume that $k \geq 2$. Let $\tau \in \mathrm{Irr}(S)$ such that $1 \neq \tau(1) \neq |S|_2$, and let $\psi = \tau \times \mathrm{St}_S \times \cdots \times \mathrm{St}_S \in \mathrm{Irr}(G'/M)$. Then this degree which is not the Steinberg character of $F_4(2)$ must divide some degree of G and so the 2-part of this degree which is $q_1^{j(k-1)}$ is at most 2^{12}. As $q_1^{jk} = 2^{24}$, we deduce that $j(k-1) \leq jk/2$, which implies that $k \leq 2$. Thus $k = 2$. Let C be a normal subgroup of G such that $C/M = C_{G/M}(G'/M)$. Then $G'C/C \cong S^2$ is a unique minimal normal subgroup of G/C so G/C embeds into $\mathrm{Aut}(S) \wr \mathbb{Z}_2$, where $\mathbb{Z}_2$ is a cyclic group of order 2. Let $B = \mathrm{Aut}(S)^2 \cap G/C$. Then we obtain that $|G/C : B| = 2$. Let $\psi = 1 \times \mathrm{St}_S \in \mathrm{Irr}(G'C/C)$. Since both 1 and St_S extend to $\mathrm{Aut}(S)$, we deduce that ψ extends to $\mathrm{Aut}(S)^2$ and thus ψ extends to B as $G'C/C \cong S^2 \leq B \leq \mathrm{Aut}(S)^2$. In particular ψ is B-invariant and since it is not G/C-invariant, we deduce that B is the inertia group of ψ in G/C. By Lemma 2.2(a), $|G/C : B|\psi(1) = 2\psi(1) \in \mathrm{cd}(G)$. Hence $2\mathrm{St}_S(1) \in \mathrm{cd}(G)$. However $1 < 2\mathrm{St}_S(1) = 2^{13} < 2^{24}$ and 2^{13} is a degree of G, which contradicts Lemma 3.1(ii). Thus $k = 1$.

We will examine each of the families of simple groups of Lie type individually. We will show that if S is a simple group of Lie type in characteristic 2 and $S \neq {}^2F_4(2)'$, then $S \cong F_4(2)$. We will prove this by eliminating other possibilities for S. Assume that S is a simple group of Lie type in characteristic 2 and S is not the Tits group. We have shown that $G'/M \cong S$ and $|S|_2 = 2^{24}$. Observe that if $\theta \in \mathrm{Irr}(S)$ is extendible to $\mathrm{Aut}(S)$, then θ extends to $G/C_G(G')$ and thus $\theta(1) \in \mathrm{cd}(G)$. In fact, we will choose θ to be a unipotent character of S, and so by results of Lusztig, θ is extendible to $\mathrm{Aut}(S)$ apart from some explicit exceptions. We refer to [2, 13.8, 13.9] for the classification of unipotent characters and the notion of symbols. In Table 1, for each simple group of Lie type S in characteristic p, we list the p-part of some unipotent character of S that is extendible to $\mathrm{Aut}(S)$.

Case 1: $S \cong G_2(q_1)$. Then $2^{24} = {q_1}^6$, which implies $q_1 = 2^4$. By [2, 13.9], S possesses a unipotent character labeled by the symbol $\phi'_{1,3}$ of S of degree

$$\frac{1}{3}q_1({q_1}^4 + q_1^2 + 1) = \frac{1}{3}2^4(2^{16} + 2^8 + 1).$$

However this degree divides no degree of G.

Case 2: $S \cong F_4(q_1)$. We have that $2^{24} = {q_1}^{24}$ and so $q_1 = 2$. As the degrees of S are identical to the degrees of G, we have that S is possibly $F_4(2)$.

Case 3: $S \cong {}^2B_2({q_1}^2)$ or $S \cong {}^2F_4({q_1}^2)$, ${q_1}^2 = 2^{2n+1}$, $n \geq 1$. Then ${q_1}^4 = 2^{24}$ or $q_1^{24} = 2^{24}$, and hence $q_1 = 2^6$ or $q_1 = 2$, respectively. But q_1 is not an integer, a contradiction.

Case 4: $S \cong {}^2G_2({q_1}^2)$, ${q_1}^2 = 3^{2m+1}$, $m \geq 1$. This case cannot happen as the characteristic of S is 3.

Case 5: $S \cong {}^3D_4(q_1)$. We have that ${q_1}^{12} = 2^{24}$, so $2^2 = q_1$. Now by [4] S has a degree $(q_1^6 - 1)^2$ and hence this degree must divide a degree of G, so $(2^{12} - 1)^2$ must divide a degree of G, which is impossible.

Case 6: S is isomorphic to one of the remaining simple groups of exceptional Lie type. For the remaining simple groups of exceptional Lie type, we will use the same general argument. Let $\Phi_y := \Phi_y(q_1)$ denote the y^{th} cyclotomic polynomial. Recall $S = S(q_1)$ is a simple group of exceptional Lie type defined over a field of q_1 elements. Suppose the Steinberg character of S has degree ${q_1}^j$. By Lemma 2.5, $2^{24} = {q_1}^j$, so $2 = {q_1}^{j/24}$. For each of the remaining possibilities for S, there is a mixed degree of S whose power on q_1 is greater than $12j/24$ (see Table 1). As the mixed degrees of G have power at most $12j/24$ on q_1, we have a contradiction.

Case 7: $S \cong L_n^\epsilon(2^b)$, where $b \geq 1$ and $n \geq 2$. We have $bn(n-1) = 48$. If $n = 2$ then $b = 24$ and so $S = L_2(2^{24})$ and hence S has a character of degree $2^{24} + 1$. Obviously this degree divides no degree of G. Next if $n = 3$ then $b = 8$ and so $S = L_3^\epsilon(2^8)$. By [2, 13.8], S possesses a unipotent character parametrized by the partition $(1, 2)$ of degree $2^8(2^8 + \epsilon 1)$. However $F_4(2)$ has no such degree. If $n = 4$, then $b = 4$. So, $S = L_4^\epsilon(2^4)$. In this case, the unipotent character parametrized by the partition $(2, 2)$ has degree $2^8(2^8 + 1)$. As above, this degree does not belong to $\mathrm{cd}(G)$. Thus we can assume that $n \geq 5$. By Table 1, S possesses a unipotent character χ different from the Steinberg character with $\chi(1)_2 = 2^{b(n-1)(n-2)/2}$. Then $b(n-1)(n-2)/2 \leq 12$. Multiplying both sides by $2n$, we obtain

$$bn(n-1)(n-2) = 48(n-2) \leq 24n,$$

and so $n \leq 4$, which is a contradiction.

Case 8: $S \cong S_{2n}(q_1)$, or $O_{2n+1}(q_1)$, where $q_1 = 2^b$, $b \geq 1$, $n \geq 2$ and $S \neq S_4(2)$. As $S_{2n}(2^b) \cong O_{2n+1}(2^b)$, we can assume $S = S_{2n}(q_1)$ and $S \neq S_4(2)$. We have $bn^2 = 24$. Then n^2 divides 24, where $n \geq 2$, which implies that $n = 2$ and $b = 6$, thus $S = S_4(2^6)$. By [2, 13.8], S possesses a unipotent character labeled by the symbol $\binom{0\,1\,2}{-}$ of degree $2^6(2^6 - 1)^2/2$. However $F_4(2)$ has no such character degree.

Case 9: $S \cong O_{2n}^\epsilon(q_1)$, where $q_1 = 2^b$, $b \geq 1$, and $n \geq 4$. We have $bn(n-1) = 24$. As $n(n-1)$ divides 24 and $n \geq 4$, we deduce that $n = 4$ and $b = 2$ and so $S = O_8^\epsilon(4)$. If $S = O_8^+(4)$, then S has a unipotent character χ with $\chi(1)_2 = 2^{2(4^2-3\cdot4+3)} = 2^{14}$ by Table 1. However G has no such degree by [3]. Assume next that $S = O_8^-(4)$. By [2, 13.8], S has a unipotent character χ labeled by the symbol $\binom{0\,1\,2\,4}{1\,2}$ with degree $\chi(1) = 2^{12}(4^4 + 4^2 + 1)$. However G has no such degree.

This completes the proof of Step 2.

3.3. **Verifying Step 3.** Let $\theta \in \mathrm{Irr}(M)$ with $\theta(1) = 1$ and let $I = I_{G'}(\theta)$. Suppose $I < G'$. Write $\theta^I = e_1\phi_1 + e_2\phi_2 + \cdots + e_s\phi_s$, where $e_i \geq 1$ and $\phi_i \in \mathrm{Irr}(I)$. Since $G'/M \cong F_4(2)$ and $M \trianglelefteq I < G'$, there exists $I \leq U < G'$ such that U/M is maximal in G'/M. Let $t = |U : I|$. Then $\phi_i^{G'}(1) = |G' : U|t\phi_i(1)$ must divide some character degree of G and so the index $|G'/M : U/M|$ divides some degree of $F_4(2)$. It follows that one of the following holds:

Table 1. Some unipotent characters of simple groups of Lie type

$S = S(p^b)$	Symbol	p-part of degree
$L_n^\epsilon(p^b), n \geq 3$	$(1^{n-2}, 2)$	$p^{b(n-1)(n-2)/2}$
$S_{2n}(p^b), p = 2$	$\binom{0\ 1\ 2\ \cdots\ n-2\ n-1\ n}{1\ 2 \cdots\ n-2}$	$2^{b(n-1)^2-1}$
$O_{2n}^+(p^b)$	$\binom{0\ 1\ 2\ \cdots\ n-3\ n-1}{1\ 2\ 3 \cdots\ n-2\ n-1}$	$p^{b(n^2-3n+3)}$
$O_{2n}^-(p^b)$	$\binom{0\ 1\ 2 \cdots\ n-2\ n}{1\ 2 \cdots\ n-2}$	$p^{b(n^2-3n+2)}$
${}^3D_4(p^b)$	$\phi''_{1,3}$	p^{7b}
$F_4(p^b)$	$\phi_{9,10}$	p^{10b}
${}^2F_4(q^2)$	${}^2B_2[a], \epsilon$	$\frac{1}{\sqrt{2}}q^{13}$
$E_6(p^b)$	$\phi_{6,25}$	p^{25b}
${}^2E_6(p^b)$	$\phi''_{2,16}$	p^{25b}
$E_7(p^b)$	$\phi_{7,46}$	p^{46b}
$E_8(p^b)$	$\phi_{8,91}$	p^{91b}

Case 1: $U/M \cong (2_+^{1+8} \times 2^6) : S_6(2)$. Then for each i, $\phi_i^U(1) = t\phi_i(1)$ divides one of the following numbers in $\mathcal{A}$, where

$$\mathcal{A} = \{5 \cdot 2^3, 7 \cdot 2^3, 2^3 \cdot 3^2, 2 \cdot 3^2 \cdot 7, 3^3 \cdot 5, 2 \cdot 3 \cdot 5 \cdot 7\}.$$

Let $L \trianglelefteq U$ be such that $L/M \cong 2_+^{1+8} \times 2^6$. As L/M is nonabelian, we can always choose $\lambda \in \mathrm{Irr}(L|\theta)$ with $\lambda(1) > 1$. Since $\lambda(1) = \lambda(1)/\theta(1)$ divides $|L/M| = 2^{15}$, we deduce that $\lambda(1) \mid 2^{15}$ and so $2 \mid \lambda(1)$. By the transitivity of character induction, we have that $\theta^U = (\theta^L)^U$. As λ is an irreducible constituent of θ^L, we deduce that $\theta^L = \lambda + \gamma$ for some character γ of L. It follows that $\theta^U = (\lambda + \gamma)^U = \lambda^U + \gamma^U$. Thus every irreducible constituent of λ^U is also an irreducible constituent of θ^U. So, if $\chi \in \mathrm{Irr}(U|\lambda)$, then $\chi = \phi_j^U$ for some j with $1 \leq j \leq s$. Therefore $\chi(1) = t\phi_j(1)$ divides a number in $\mathcal{A}$. Moreover $2 \mid \chi(1)$ for any $\chi \in \mathrm{Irr}(U|\lambda)$ as $\lambda(1) \mid \chi(1)$. It follows that the degree of every irreducible constituent of λ^U divides some number in $\mathcal{A}_1 = \mathcal{A} - \{3^3 \cdot 5\}$. Let $J = I_U(\lambda)$ be the inertia group of λ in U. Write $\lambda^J = \sum_{i=1}^{l} f_i\mu_i$, where $\mu_i \in \mathrm{Irr}(J|\lambda)$. Then for each i, $|U : J|\mu_i(1)$ divides some number in $\mathcal{A}_1$ and $\mu_i(1)$ is even. Note that $U/L \cong S_6(2)$ and $L \trianglelefteq J \leq U$.

Subcase $J \lneq U$. As $L \leq J < U$ and $U/L \cong S_6(2)$, we deduce that $J \leq K \lneq U$, where K/L is a maximal subgroup of $U/L \cong S_6(2)$. Hence some index of a maximal subgroup of $S_6(2)$ must divide one of the numbers in $\mathcal{A}_1$. Consulting the list of maximal subgroups of $S_6(2)$ in [3], we have that K/L is isomorphic to one of the following groups

$$U_4(2) \cdot 2,\ A_8 \cdot 2,\ \text{or}\ \ 2^5 : S_6.$$

Assume that $K/L \cong U_4(2) : 2$. Then $|U : K| = |U/L : K/L| = 2^2 \cdot 7$. It follows that $|U : K||K : J|\mu_j(1) = 2^2 \cdot 7 \cdot |K : J|\mu_j(1)$ divides $2^3 \cdot 7$, therefore $|K : J|\mu_j(1)$ divides 2 for all j. Hence $\mu_j(1)$ divides 2 for all j and $|K : J|$ divides 2 and thus $J \trianglelefteq K$ with index at most 2 and hence J is nonsolvable. By [5, Theorem B], J/L is solvable since all $\mu_j \in \mathrm{Irr}(J|\lambda)$ have 2-power degrees and λ is J-invariant. However this contradicts our previous claim that J/M is nonsolvable.

Assume that $K/L \cong A_8 \cdot 2$. Then $|U:K| = 2^2 \cdot 3^2$. It follows that $|U:K||K:J|\mu_j(1) = 2^2 \cdot 3^2 \cdot |K:J|\mu_j(1)$ divides $2^3 \cdot 3^2$ and so $|K:J|\mu_j(1)$ divides 2 for all j. Now we can apply the same argument as in the previous case to get a contradiction.

Assume that $K/L \cong 2^5 : S_6$. Then $|U:K| = |U/L : K/L| = 3^2 \cdot 7$. It follows that $|U:K||K:J|\mu_j(1) = 3^2 \cdot 7 \cdot |K:J|\mu_j(1)$ divides $2 \cdot 3^2 \cdot 7$, thus $|K:J|\mu_j(1)$ divides 2 for all j. Hence $\mu_j(1)$ divide 2 for all j and $|K:J|$ divides 2 and thus by applying the same argument as above, we get a contradiction.

Subcase $J = U$. For each i, $\mu_i(1) = f_i\lambda(1)$ divides one of the numbers in $\mathcal{A}_1$. If $f_j = 1$ for some j, then λ extends to $\lambda_0 \in \mathrm{Irr}(U)$, so $\lambda_0\tau \in \mathrm{Irr}(U|\lambda)$ for any $\tau \in \mathrm{Irr}(U/L) = \mathrm{Irr}(S_6(2))$, and hence $\tau(1)\lambda_0(1)$ divides one of the numbers in $\mathcal{A}_1$. By choosing $\tau \in \mathrm{Irr}(U/L)$ with $\tau(1) = 3^3$, we obtain a contradiction. Thus by Lemma 2.2(b), each $f_i > 1$ is a degree of a proper projective irreducible representation of $S_6(2)$, and each f_i divides one of the numbers in $\mathcal{A}_1$. By [3], we have that $f_i = 8$ for all i. Then $\mu_i(1) = f_i\lambda(1) = 8\lambda(1)$ and hence $\mu_i(1)/\lambda(1) = 8$ for all i, where λ is U-invariant, by applying [5, Theorem B], we obtain a contradiction as U/L is nonsolvable.

Case 2: $U/M \cong S_8(2)$. Then $t\phi_i(1)$ divides $2^2 \cdot 3 \cdot 7$ or $2 \cdot 5 \cdot 17$. Inspecting the list of maximal subgroups of $S_8(2)$, we deduce that $t = 1$ and so $I = U$. As the Schur multiplier of $S_8(2)$ is trivial, we deduce that θ extends to $\theta_0 \in \mathrm{Irr}(I)$. Gallagher's Theorem yields that $\tau\theta_0$ are all the irreducible constituents of θ^I, where $\tau \in \mathrm{Irr}(I/M)$. If we choose $\tau \in \mathrm{Irr}(I/M)$ such that $\tau(1) = 7 \cdot 5$, then $\tau(1)\theta(1)$ divides none of the numbers above.

Case 3: $U/M \cong [2^{20}] : (S_3 \times L_3(2))$. Then $t\phi_i(1)$ divides 3. Let $M \leq L \leq U$ and $L \leq V \leq U$ such that $L/M \cong [2^{20}]$ and $V/L \cong L_3(2)$. As $t\phi_i(1) \mid 3$, we deduce that t and $\phi_i(1)$ are odd, so $L \leq I$ and that θ extends to $\lambda \in \mathrm{Irr}(L|\theta)$. Inspecting the list of maximal subgroups of $L_3(2) \cong L_2(7)$, we can deduce that λ is V-invariant, and for any $\varphi \in \mathrm{Irr}(V|\theta)$, as $V \trianglelefteq I$, we deduce that $\varphi(1) \mid \phi_i(1)$ for some i and so $\varphi(1) \mid 3$. Now [5, Theorem B] yields that V/L is solvable, a contradiction.

3.4. **Step 4: Establishing** $M = 1$**.** We now prove that the subgroup M of G is trivial. By Step 2, we know that $G'/M \cong F_4(2)$. Hence, when paired with this step, we have that $G' \cong F_4(2)$. By Step 3, $M' = [M, G']$ so by [6, Lemma 6], we have that $|M:M'|$ divides the order of the Schur multiplier of $G'/M \cong F_4(2)$ which is 2. Thus $|M:M'|$ divides 2. Assume first that $|M:M'| = 2$. It follows that $G'/M' \cong 2 \cdot F_4(2)$. By [3], $2 \cdot F_4(2)$ possesses an irreducible character of degree $2^{10} \cdot 5^2 \cdot 7^2 \cdot 13$ labelled by χ_{169}, which divides none of the degrees of G. Thus $M = M'$. If M is abelian, then we are done. So assume that M is nonabelian. It follows that if M/N is a chief factor of G', then $M/N \cong S^k$, where S is a nonabelian simple group and $k \geq 1$. By Lemmas 2.4 and 2.5, M/N has a nontrivial irreducible character φ which extends to G' and so if $\psi \in \mathrm{Irr}(G'/M)$ with $\psi(1) = 2^{24}$, then by Gallagher's Theorem, we deduce that G' has an irreducible character of degree $\psi(1)\varphi(1) = 2^{24}\varphi(1) > 2^{24}$. However this is impossible as $2^{24}\varphi(1)$ divides no degrees of G. Hence $M = 1$.

3.5. **Step 5: Establishing** $G = G' \times C_G(G')$**.** We can now conclude Huppert's argument and verify the conjecture for the simple group $F_4(2)$. If $G' \times C_G(G') < G$, then two characters of $G' \cong F_4(2)$ of degree 1105 are fused in G, producing the forbidden degree 2210 ([3], page 169).

Hence $G = G' \times C_G(G')$. This concludes the verification of the five steps of Huppert's argument and proves Theorem 1.1.

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