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A NEW CHARACTERIZATION OF $PSL(2,25)$

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Abstract. Let G be a finite group and $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set nse(G):=$\{m_k | k \in \pi_e(G)\}$. In this paper, we prove that if G is a group such that nse(G)=nse($PSL(2,25)$), then $G \cong PSL(2,25)$.

1. **Introduction**

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Set m_i=$m_i(G)$=$|\{g \in G|$ the order of g is $i\}$. In fact, m_i is the number of elements of order i in G, and nse(G) := $\{m_i | i \in \pi_e(G)\}$, the set of sizes of elements with the same order. Throughout this paper, we denote by $\phi(n)$ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow $q-$ subgroup of G and $n_q(G)$ is the number of Sylow $q-$subgroups of G, that is, $n_q(G)$=$|Syl_q(G)|$. All further unexplained notations are standard and refer to [1], for example. In [2], it is proved that all simple K_4-groups can be uniquely determined by nse(G) and $|G|$. But, in [3], it is proved that the groups A_4, A_5 and A_6, and in [4], the groups $PSL(2,q)$, for $q \in \{7,8,11,13\}$ are uniquely determined by only nse(G). In [4], the authors gave the following problem:

Problem: Let G be a group such that nse(G) =nse($PSL(2,q)$), where q is a prime power. Is G isomorphic to $PSL(2,q)$?

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In this paper, we give a positive answer to this problem and show that the group $PSL(2, q)$ is characterizable by only nse(G) for $q = 25$. In fact the main theorem of our paper is as follow:

Main Theorem: Let G be a group such that nse(G)=nse($PSL(2, 25)$), then $G \cong PSL(2, 25)$.

In this paper, we use a new technique for the proof of our main result. Also we apply the technique of used in [4].

2. Preliminary Results

In this section we bring some preliminary lemmas to be used in the proof of main theorem theorem.

Lemma 2.1. [6] Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1} \dots q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, 2, \dots, s\}$:

(1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.

(2) The order of some chief factor of G is divisible by $q_i^{\beta_i}$.

Lemma 2.2. [7] If G is a simple K_3- group, then G is isomorphic to one of the following groups: A_5, A_6, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 17)$, $PSL(3, 3)$, $PSU(3, 3)$ or $PSU(4, 2)$.

Lemma 2.3. [8] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:
(1) A_7, A_8, A_9, A_{10}.
(2) M_{11}, M_{11}, J_2.
(3) (a) $L_2(r)$, where r is a prime and satisfies $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, $v > 3$, is a prime;
(b) $L_2(2^m)$, where satisfies $2^m - 1 = u$, $2^m + 1 = 3t^b$, with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$; (c) $L_2(3^m)$, where m satisfies $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$;
(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, $2F_4(2)'$.

Lemma 2.4. [2] Let G be a finite group, $P \in Syl_p(G)$, where $p \in \pi(G)$. Let G have a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \nmid |K|$, then the following hold:
(1) $N_{G/K}(PK/K) = N_G(P)K/K$;
(2) $|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;
(3) $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t, and $|N_K(P)|t = |K|$.

Lemma 2.5. [5] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.

Lemma 2.6. [3] Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k | k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Let G be a group such that nse(G)=nse($PSL(2,25)$). By Lemma 2.6, we can assume that G is finite. Let m_n be the number of elements of order n . We note that m_n= $k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \pi_e(G)$, then by Lemma 2.5 and the above notation we have:

$$\begin{cases} \phi(n) \mid m_n \\ \\ n \mid \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we almost apply $(*)$ and the above comments.

3. Proof of the Main Theorem

Let G be a group, such that nse(G)=nse($PSL(2,25)$)=$\{1, 325, 624, 650, 1300, 3600\}$. At first, we prove that $\pi(G) \subseteq \{2, 3, 5, 13\}$. Since $325 \in$ nse(G), it follows that by $(*)$, $2 \in \pi(G)$ and $m_2 = 325$. Let $2 \neq p \in \pi(G)$, by $(*)$, $p|(1 + m_p)$ and $(p-1)|m_p$, which implies that $p \in \{3, 5, 13, 1301\}$. If $1301 \in \pi(G)$, then by $(*)$, $m_{1301} = 3600$. On the other hand, by $(*)$, we conclude that if $2602 \in \pi_e(G)$, then $m_{2602} = 3600$ and $2602|(1 + m_2 + m_{1301} + m_{2602})$, and hence $2602 \mid 7526$, which is a contradiction. That is $2602 \notin \pi_e(G)$. Thus P_{1301} acts fixed point freely on the set of elements of order 2, and $|P_{1301}| \mid m_2$, which is a contradiction. Hence $1301 \notin \pi(G)$. If 3, 5 and $13 \in \pi_e(G)$, then $m_3 = 650$, $m_5 = 624$ and $m_{13} = 3600$, by $(*)$. Now let $3 \in \pi(G)$, we can see easily that G does not contain any elements of order 9. Hence $\exp(P_3) = 3$. By Lemma 2.5, with considering $m = |P_3|$, we have $|P_3| \mid (1 + m_3) = 651$. Hence $|P_3| = 3$, then $n_3 = m_3/\phi(3) = 325 \mid |G|$. Therefore if $3 \in \pi(G)$, then 5 and $13 \in \pi(G)$. If $13 \in \pi(G)$, then we can see easily that G does not contain any elements of order 169. Hence $\exp(P_3) = 13$. By Lemma 2.5, we have $|P_{13}| \mid (1 + m_{13}) = 3601$. Hence $|P_{13}| = 13$, then $n_{13} = m_{13}/\phi(13) = 300 \mid |G|$. Therefore if $13 \in \pi(G)$, then 3 and $5 \in \pi(G)$. By above discussion in follow, we show that $\pi(G)$ could not be the sets $\{2\}$ and $\{2, 5\}$, and $\pi(G)$ must be equal to $\{2, 3, 5, 13\}$.

Case a. Let $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 2^2, \ldots, 2^5\}$. Hence $|G| = 2^m = 6500 + 624k_1 + 650k_2 + 1300k_3 + 3600k_4$, where m, k_1, k_2, k_3 and k_4 are non-negative integers and $k_1 + k_2 + k_3 + k_4 = 0$. It is easy to check that the equation has no solution. Therefore this case is impossible.

<u>Case b.</u> Let $\pi(G) = \{2, 5\}$. Since $5^3 \notin \pi_e(G)$, we have $exp(P_5) = 5$ or 25. If $exp(P_5) = 5$, then $|P_5| \mid (1 + m_5) = 625$. Hence $|P_5| \mid 5^4$. If $|P_5| = 5$, then $n_5 = m_5/\phi(5) = 156 \mid |G|$, since $3 \notin \pi(G)$, we get a contradiction. If $|P_5| = 25$, then $|G| = 2^m \times 25 = 6500+624k_1+650k_2+1300k_3+3600k_4$, where m, k_1, k_2, k_3 and k_4 are non-negative integers. Since $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5, 2^5\} \bigcup \{5, 5 \times 2^2, 5 \times 2^3\}$, then $0 \leq k_1 + k_2 + k_3 + k_4 \leq 4$. Therefore $6500 \leq |G| \leq 3600 \times 4 + 6500$, then $m = 9$. Hence $6300 = 624k_1 + 650k_2 + 1300k_3 + 3600k_4$. It is easy to check that the equation has no solution. If $|P_5| = 125$, then $|G| = 2^m \times 125 = 6500 + 624k_1 + 650k_2 + 1300k_3 + 3600k_4$, then $m = 6$ or 7. Therefore $1500 = 624k_1 + 650k_2 + 1300k_3 + 3600k_4$ or $9500 = 624k_1 + 650k_2 + 1300k_3 + 3600k_4$, it is easy to check that these equation has no solution. Similarly if $|P_5| = 625$, we can get a contradiction. Now suppose that $exp(P_5) = 25$. By $(*)$, we have $m_{25} = 650$, 1300 or 3600. Since $|P_5| \mid (1 + m_5 + m_{25})$, we can conclude that $|P_5| = 25$, then $n_5 = m_{25}/\phi(25)$. As $m_{25} = 650$, 1300 or 3600, by Sylow theorem we get a contradiction.

Therefore $\pi(G) = \{2, 3, 5, 13\}$. Now we show that G does not contain any element of order 5×13. Suppose that $5 \times 13 \in \pi_e(G)$, we know that if P and Q are Sylow 13−subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore $m_{5\times13} = \phi(5 \times 13).n_{13}.k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_{13})$. Since $n_{13} = m_{13}/\phi(13) = 300$, we have $3600 \mid m_{5\times13}$. On the other hand we have, $5 \times 13 \mid (1 + m_5 + m_{13} + m_{5\times13}) = 7824$, which is a contradiction. Hence $5 \times 13 \notin \pi_e(G)$. Since $5 \times 13 \notin \pi_e(G)$, then the group P_5 acts fixed point freely on the set of elements of order 13, and so $|P_5| \mid m_{13} = 3600$, which implies that $|P_5| \mid 25$. Also we can prove that $26 \notin \pi_e(G)$, then the group P_2 acts fixed point freely on the set of elements of order 13, and so $|P_2| \mid m_{13} = 3600$, which implies that $|P_2| \mid 2^4$. We had $|P_3| = 3$ and $|P_{13}| = 13$, since $6500 \leq |G|$, hence $|G| = 2^4 \times 3 \times 25 \times 13$ or $|G| = 2^3 \times 3 \times 25 \times 13$. If $|G| = 2^4 \times 3 \times 25 \times 13$, we claim that G is unsolvable group. Suppose G is a solvable group, since $n_{13} = 300$, then by Lemma 2.1, $4 \equiv 1 \pmod{13}$, which is a contradiction. Hence G is a unsolvable group. Since G is a unsolvable group and $p \parallel |G|$ for $p \in \{3, 13\}$, then G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G and H/N is an unsolvable minimal normal subgroup of G/N. Then H/N is a non-abelian simple K_3−group or K_4−group. Let H/N be a non-abelian simple K_3−group, then by Lemma 2.2, $H/N \cong A_5$. If $P_3 \in Syl_3(G)$, then $P_3N/N \in Syl_3(H/N)$ and $n_3(H/N)t = n_3(G)$ for some positive integer t and $3 \nmid t$, by Lemma 2.4. Since $n_3(H/N) = n_3(A_5) = 10$, then $325 = 10t$, which is a contradiction. Hence H/N is a non-abelian simple K_4−group. By Lemma 2.3, $H/N \equiv PSL(2, 25)$. Now set $\overline{H} := H/N \cong PSL(2, 25)$ and $\overline{G} := G/N$. On the other hand, we have:

$$PSL(2, 25) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq Aut(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $PSL(2, 25) \leq G/K \leq Aut(PSL(2, 25))$, then $G/K \cong PSL(2, 25), PGL(2, 25), P\Gamma L(2, 25)$ or $P\Sigma L(2, 25)$. If $G/K \cong P\Sigma L(2, 25)$ or $P\Gamma L(2, 25)$, since $|G| = 2^4 \times 3 \times 25 \times 13$, which is a contradiction. If $G/K \cong PGL(2, 25)$, since $|G| = 2\,|PSL(2, 25)|$, then $|K| = 1$ and $G \cong PGL(2, 25)$, on the other hand nse$(G) \neq$nse$(PGL(2, 25))$, we get a contradiction. If $G/K \cong PSL(2, 25)$, then $|K| = 2$. We have $N \leq K$, and N is a maximal solvable normal subgroup of G, then $N = K$. Hence $G/N \cong PSL(2,25)$ and G has a normal subgroup G of order 2, generated by a central involution z. Let x be an element of order 13 in G. Since $xz = zx$ and $(o(x), o(z)) = 1$, then $o(xz) = 26$, which is a contradiction. Therefore $|G| = 2^3 \times 3 \times 13 \times 25$. Now we have $|G| = |PSL(2,25)|$ and $\mathrm{nse}(G) = \mathrm{nse}(PSL(2,25))$. By [2], since PSL(2, 25) is simple K_4-group, we can conclude that $G \cong PSL(2,25)$, and the proof is completed.

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