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ON VARIETAL CAPABILITY OF INFINITE DIRECT PRODUCTS OF GROUPS

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ABSTRACT. Recently, the authors gave some conditions under which a direct product of finitely many groups is \mathcal{V} -capable if and only if each of its factors is \mathcal{V} -capable for some varieties \mathcal{V} . In this paper, we extend this fact to any infinite direct product of groups. Moreover, we conclude some results for \mathcal{V} -capability of direct products of infinitely many groups in varieties of abelian, nilpotent and polynilpotent groups.

1. Introduction

R. Baer [1] initiated an investigation of the question "which conditions a group G must fulfill in order to be the group of inner automorphisms of a group E ?", that is $G \cong E/Z(E)$. Following M. Hall and J. K. Senior [5], such a group G is called *capable*. Baer [1] determined all capable groups which are direct sums of cyclic groups. As P. Hall [4] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F. R. Beyl, U. Felgner and P. Schmid [2] proved that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in G of the center of some central extension of G . This $Z^*(G)$ is characteristic in G and is the image of the center of every stem cover of G . Moreover, $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable [2]. Hence G is capable if and only if $Z^*(G) = 1$. They showed that the class of all capable groups is closed under the direct products. Also, they presented a condition in which the capability of a direct product of finitely many of groups implies the capability of each of the factors. Moreover, they proved that if N

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is a central subgroup of G , then $N \subseteq Z^*(G)$ if and only if the mapping $M(G) \rightarrow M(G/N)$ induced by the natural epimorphism, is monomorphism.

Then M. R. R. Moghadam and S. Kayvanfar [10] generalized the concept of capability to \mathcal{V} -capability for a group G . They introduced the subgroup $(V^*)^*(G)$ which is associated with the variety \mathcal{V} defined by a set of laws V and a group G in order to establish a necessary and sufficient condition under which G can be \mathcal{V} -capable. They also showed that the class of all \mathcal{V} -capable groups is closed under the direct products. Moreover, they exhibited a close relationship between the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, where N is a normal subgroup contained in the marginal subgroup of G with respect to the variety \mathcal{V} . Using this relationship, they gave a necessary and sufficient condition for a group G to be \mathcal{V} -capable.

The authors [7] presented some conditions in which the \mathcal{V} -capability of a direct product of finitely many groups implies the \mathcal{V} -capability of each of its factors. In this paper, we extend this fact to direct product of an infinite family of groups. Also, we deduce some new results about the \mathcal{V} -capability of direct product of infinitely many groups, where \mathcal{V} is the variety of abelian, nilpotent, or polynilpotent groups.

2. Main Results

Suppose that \mathcal{V} is a variety of groups defined by the set of laws V . A group G is said to be \mathcal{V} -capable if there exists a group E such that $G \cong E/V^*(E)$, where $V^*(E)$ is the marginal subgroup of E , which is defined as follows [6]:

$$\{g \in E \mid v(x_1, x_2, \dots, x_n) = v(x_1, x_2, \dots, gx_i, x_{i+1}, \dots, x_n) \\ \forall x_1, x_2, \dots, x_n \in E, \forall i \in \{1, 2, \dots, n\}\}.$$

If $\psi : E \rightarrow G$ is a surjective homomorphism with $\ker \psi \subseteq V^*(E)$, then the intersection of all subgroups of the form $\psi(V^*(E))$ is denoted by $(V^*)^*(G)$. It is obvious that $(V^*)^*(G)$ is a characteristic subgroup of G contained in $V^*(G)$. If \mathcal{V} is the variety of abelian groups, then the subgroup $(V^*)^*(G)$ is the same as $Z^*(G)$ and in this case \mathcal{V} -capability is equal to capability [10]. In the following, there are some results which we need them in sequel.

Theorem 2.1. [10] (i) A group G is \mathcal{V} -capable if and only if $(V^*)^*(G) = 1$.
(ii) If $\{G_i\}_{i \in I}$ is a family of groups, then $(V^*)^*(\prod_{i \in I} G_i) \subseteq \prod_{i \in I} (V^*)^*(G_i)$.

As a consequence, if the G_i 's are \mathcal{V} -capable groups, then $G = \prod_{i \in I} G_i$ is also \mathcal{V} -capable. In the above theorem, the equality does not hold in general (see Example 2.3 of [7]).

Theorem 2.2. [10] Let \mathcal{V} be a variety of groups with a set of laws V . Let G be a group and N be a normal subgroup with the property $N \subseteq V^*(G)$. Then $N \subseteq (V^*)^*(G)$ if and only if the homomorphism induced by the natural map $\mathcal{V}M(G) \rightarrow \mathcal{V}M(G/N)$ is a monomorphism.

We recall that the Baer-invariant of a group G , with the free presentation F/R , with respect to the variety \mathcal{V} , denoted by $\mathcal{V}M(G)$, is

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

where $V(F)$ is the verbal subgroup of F with respect to \mathcal{V} and

$$[RV^*F] = \langle v(f_1, \dots, f_{i-1}, f_i r, f_{i+1}, \dots, f_n) v(f_1, \dots, f_i, \dots, f_n)^{-1} \mid r \in R,$$

$$f_i \in F, v \in V, 1 \leq i \leq n, n \in \mathbf{N} \rangle .$$

It is known that the Baer-invariant of a group G is always abelian and independent of the choice of the presentation of G . Also if \mathcal{V} is the variety of abelian groups, then the Baer-invariant of G will be $R \cap F' / [R, F] \cong M(G)$, where $M(G)$ is the Schur multiplier of G (see [6]).

Theorem 2.3. [7] *Let \mathcal{V} be a variety, A and B be two groups with $\mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B)$, then $(V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B)$. Consequently, $A \times B$ is \mathcal{V} -capable if and only if A and B are both \mathcal{V} -capable.*

Theorem 2.4. [8] *Let $\{G_i; \phi_i^j, I\}$ be a directed system of groups. Then, for a given variety \mathcal{V} , the Baer-invariant preserves direct limit, that is $\mathcal{V}M(\lim_{\rightarrow} G_i) = \lim_{\rightarrow} \mathcal{V}M(G_i)$.*

Lemma 2.5. *For any family of groups $\{G_i\}_{i \in I}$, consider the directed system $\{\mathcal{G}_{I_\lambda}, \phi_\lambda^{\lambda'}, \Lambda\}$ consisting of all finite direct products $\mathcal{G}_{I_\lambda} = \prod_{i \in I_\lambda} G_i$ (I_λ is a finite subset of I), with the natural embedding homomorphisms $\phi_\lambda^{\lambda'} : \mathcal{G}_{I_\lambda} \rightarrow \mathcal{G}_{I_{\lambda'}}$ ($I_\lambda \subseteq I_{\lambda'}$). Also, the index set Λ is ordered in a directed way so that for any $\lambda, \lambda' \in \Lambda$, $\lambda \leq \lambda'$ if and only if $I_\lambda \subseteq I_{\lambda'}$. Then the direct product $\mathcal{G}_I = \prod_{i \in I} G_i$ is a direct limit of this directed system.*

Proof. Let $\mathcal{G} = \lim_{\rightarrow} \mathcal{G}_{I_\lambda}$ be a direct limit of this directed system, with homomorphisms $\phi_\lambda : \mathcal{G}_{I_\lambda} \rightarrow \mathcal{G}$. Also, for any $\lambda \in \Lambda$, consider the embedding homomorphism $\tau_\lambda : \mathcal{G}_{I_\lambda} \rightarrow \mathcal{G}_I$. Clearly, for any $\lambda, \lambda' \in \Lambda$ with $\lambda \leq \lambda'$, $\tau_{\lambda'} \tau_\lambda^{\lambda'} = \tau_\lambda$. Now, by universal property of \mathcal{G} , there exists a unique homomorphism $\phi : \mathcal{G} \rightarrow \mathcal{G}_I$ such that for any $\lambda \in \Lambda$, $\phi \phi_\lambda = \tau_\lambda$. To define the inverse homomorphism $\tau : \mathcal{G}_I \rightarrow \mathcal{G}$, recall that for any $x = \{x_i\}_{i \in I} \in \mathcal{G}_I$, there exists a finite subset I_λ of I that for any $i \in I \setminus I_\lambda$, x_i is trivial in G_i . Hence we can consider x as an element of \mathcal{G}_{I_λ} and define $\tau(x) = \phi_\lambda(x)$. It is easy to see that for any $\lambda \in \Lambda$, $\tau \tau_\lambda = \phi_\lambda$. Finally, we see that for any $x \in \mathcal{G}_I$, $\phi \tau(x) = \phi \phi_\lambda(x)$, for some $\lambda \in \Lambda$; and so $\phi \tau(x) = \tau_\lambda(x) = x$. Conversely, the equation $\tau \phi = id_{\mathcal{G}}$ holds because of the universal property of the direct limit \mathcal{G} . \square

By the above notations, we conclude that $\prod_{i \in I} G_i$, $\prod_{i \in I} V^{**}(G_i)$, and $\prod_{i \in I} G_i / V^{**}(G_i)$ are direct limits of directed systems $\{\prod_{i \in I_\lambda} G_i, \phi_\lambda^{\lambda'}, \Lambda\}$, $\{\prod_{i \in I_\lambda} V^{**}(G_i), \bar{\phi}_\lambda^{\lambda'}, \Lambda\}$, and $\{\prod_{i \in I_\lambda} G_i / V^{**}(G_i), \psi_\lambda^{\lambda'}, \Lambda\}$ respectively, where $\bar{\phi}_\lambda^{\lambda'}$'s are restrictions of $\phi_\lambda^{\lambda'}$'s and $\psi_\lambda^{\lambda'}$'s are quotient homomorphisms induced by $\phi_\lambda^{\lambda'}$'s.

Now, suppose that $\{G_i\}_{i \in I}$ is a family of groups in which for any G_i and G_j ($i, j \in I$), $\mathcal{VM}(G_i \times G_j) \cong \mathcal{VM}(G_i) \times \mathcal{VM}(G_j)$. By Theorem 2.3, $\prod_{i \in I_\lambda} (V^*)^*(G_i) \subseteq (V^*)^*(\prod_{i \in I_\lambda} G_i)$, for any finite subset I_λ of I . Thus, using Theorem 2.2, we have the following monomorphism

$$\mathcal{VM}\left(\prod_{i \in I_\lambda} G_i\right) \hookrightarrow \mathcal{VM}\left(\frac{\prod_{i \in I_\lambda} G_i}{\prod_{i \in I_\lambda} (V^*)^*(G_i)}\right).$$

By the fact that direct limit of a directed system preserves exactness of a sequence [8], we obtain the following monomorphism

$$\lim_{\rightarrow} \mathcal{VM}\left(\prod_{i \in I_\lambda} G_i\right) \hookrightarrow \lim_{\rightarrow} \mathcal{VM}\left(\frac{\prod_{i \in I_\lambda} G_i}{\prod_{i \in I_\lambda} (V^*)^*(G_i)}\right).$$

Using Theorem 2.4, we conclude the monomorphism

$$\mathcal{VM}\left(\lim_{\rightarrow} \prod_{i \in I_\lambda} G_i\right) \hookrightarrow \mathcal{VM}\left(\lim_{\rightarrow} \frac{\prod_{i \in I_\lambda} G_i}{\prod_{i \in I_\lambda} (V^*)^*(G_i)}\right),$$

and so we have the monomorphism

$$\mathcal{VM}\left(\prod_{i \in I} G_i\right) \hookrightarrow \mathcal{VM}\left(\frac{\prod_{i \in I} G_i}{\prod_{i \in I} (V^*)^*(G_i)}\right).$$

Finally, by Theorem 2.2, we conclude that

$$\prod_{i \in I} (V^*)^*(G_i) \subseteq (V^*)^*\left(\prod_{i \in I} G_i\right).$$

Using these notes, we deduce the following theorem.

Theorem 2.6. *Let \mathcal{V} be a variety, $\{G_i\}_{i \in I}$ be a family of groups such that for any $i, j \in I$, $\mathcal{VM}(G_i \times G_j) \cong \mathcal{VM}(G_i) \times \mathcal{VM}(G_j)$. Then $(V^*)^*\left(\prod_{i \in I} G_i\right) = \prod_{i \in I} (V^*)^*(G_i)$. Consequently, $\prod_{i \in I} G_i$ is \mathcal{V} -capable if and only if each G_i is \mathcal{V} -capable.*

Remark 2.7. (i) *In the above theorem, the sufficient condition*

$$\mathcal{VM}(A \times B) \cong \mathcal{VM}(A) \times \mathcal{VM}(B)$$

is not necessary (see Example 2.3(iii) of [7]). Also, this condition is essential and can not be omitted (see Example 2.3(i), (ii) of [7]).

(ii) *It is known that for varieties of abelian and nilpotent groups, and for any groups A and B , $\mathcal{VM}(A \times B) \cong \mathcal{VM}(A) \times \mathcal{VM}(B) \times T$, where T is an abelian group whose elements are tensor products of the elements of A^{ab} and B^{ab} [3], [9]. Hence in these known varieties, the isomorphism $\mathcal{VM}(A \times B) \cong \mathcal{VM}(A) \times \mathcal{VM}(B)$ holds, where both A^{ab} and B^{ab} have finite exponent with $(\exp(A^{ab}), \exp(B^{ab})) = 1$.*

In the following, using the main theorem and the above remark, we deduce some corollaries which are generalizations of some results of [7] (Remark 2.4(ii), Corollary 2.5 and Example 2.2).

Corollary 2.8. *Let $\{G_i\}_{i \in I}$ be a family of groups whose abelianizations have mutually coprime exponents. Then $\prod_{i \in I} G_i$ is capable (\mathcal{N}_c -capable) if and only if each G_i is capable (\mathcal{N}_c -capable).*

Corollary 2.9. *Suppose that $\{G_i\}_{i \in I}$ is a family of groups whose abelianizations have mutually coprime exponents. If $\prod_{i \in I} G_i$ is nilpotent of class at most c_1 , then it is $\mathcal{N}_{c_1, \dots, c_s}$ -capable if and only if every G_i is $\mathcal{N}_{c_1, \dots, c_s}$ -capable.*

Corollary 2.10. *If $\{G_i\}_{i \in I}$ is a family of perfect groups, then $\prod_{i \in I} G_i$ is \mathcal{V} -capable if and only if each G_i is \mathcal{V} -capable, where \mathcal{V} may be each of these three varieties:*

- (1) *variety of abelian groups,*
- (2) *variety of nilpotent groups,*
- (3) *variety of polynilpotent groups.*

REFERENCES

- [1] R. Baer, Groups with preassigned central and central quotient groups, *Trans. Amer. Math. Soc.*, **44** (1938) 378-412.
- [2] F. R. Beyl, U. Felgner, P. Schmid, On groups occurring as center factor groups, *J. Algebra*, **61** (1979) 161-177.
- [3] G. Ellis, On groups with a finite nilpotent upper central quotient, *Arch. Math.*, **70** (1998) 89-96.
- [4] P. Hall, The classification of prime-power groups, *J. reine angew. Math.*, **182** (1940) 130-141.
- [5] M. Hall, Jr., J. K. Senior, *The Groups of Order 2^n ($n \leq 6$)*, Macmillan, New York, 1964.
- [6] G. Karpilovsky, *The Schur Multiplier*, London Math. Soc. Monographs, New Series 2, Clarendon Press, Oxford University Press, Oxford, 1987.
- [7] H. Mirebrahimi, B. Mashayekhy, On varietal capability of direct product of groups, *J. Adv. Res. Pure Math.*, to appear.
- [8] M. R. R. Moghaddam, The Baer-invariant and the direct limit, *Monatsh. Math.*, **90** (1980) 37-43.
- [9] M. R. R. Moghaddam, The Baer-invariant of a direct product, *Arch. Math.*, **33** (1980) 504-511.
- [10] M. R. R. Moghaddam, S. Keyvanfar, A new notion derived from varieties of groups. *Algebra Colloq.*, **4** no. 1 (1997) 1-11.
- [11] D. J. S. Robinson, *A Course in the Theory of Groups*, Second Edition, Springer-Verlag, 1996.

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