



www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 2 No. 1 (2013), pp. 11-35.
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REPLACEMENT AND ZIG-ZAG PRODUCTS, CAYLEY GRAPHS AND LAMPLIGHTER RANDOM WALK

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Communicated by Patrizia Longobardi

ABSTRACT. We investigate two constructions - the replacement and the zig-zag product of graphs - describing several fascinating connections with Combinatorics, via the notion of expander graph, Group Theory, via the notion of semidirect product and Cayley graph, and with Markov chains, via the Lamplighter random walk. Many examples are provided.

1. Introduction

In this paper, several constructions involving graphs, Markov chains, and groups are studied, with the aim of focusing on numerous interesting relations occurring between Combinatorics, Probability, and Harmonic Analysis. It is inspired by a series of new recent constructions in Group and Graph Theory, which have already helped to solve some interesting problems, but whose potential is far from being fully developed and which deserve to be investigated further. These connections between the above mentioned different approaches, combinatorial, analytic, probabilistic and algebraic, appear significantly when considering, for instance, random walks on graphs, Laplacians on graphs and their spectral properties, Cayley graphs of finitely generated groups.

The idea of constructing new graphs starting from smaller component graphs is very natural. Products of graphs were largely studied in the literature, for both their theoretical interest and practical applicability. Standard products include the Cartesian product, the direct product, the strong product, the lexicographic product [22, 30, 31]. See also the beautiful handbook [19]. Since several modern

MSC(2010): Primary: 05C76; Secondary: 20E22, 60J10, 05C81, 37A30, 43A85.

Keywords: Replacement and zig-zag product, expander graph, lamplighter random walk, Cayley graph, semidirect and wreath product.

Received: 24 September 2012, Accepted: 31 October 2012.

applications require sparse graphs, mainly in computer sciences, some new constructions have been introduced in order to satisfy this requirement: in this paper, we focus our attention on the replacement and the zig-zag products of graphs, describing connections with Cayley graphs of finitely generated groups and with the well known Lamplighter Markov chain.

In Section 2 we recall some basic definitions about the adjacency matrix of a regular graph and its spectral properties (Subsection 2.1). Then we present the definition of an *expander graph*: informally, a graph is expander if it is simultaneously sparse, i.e., it has relatively few edges, and highly connected (Subsection 2.2). Expander graphs have many interesting applications in different areas of computer science, such as design and analysis of communication networks and error correcting codes [21, 24]. They were firstly defined by Pinsker [28], who also coined the name. What is mostly fascinating about expander graphs, is the fact that the expansion property can be described from several points of view and with different approaches and interpretations. From a combinatorial point of view, the expansion property has an isoperimetric nature, described in terms of the *edge expansion ratio* or *Cheeger constant*: every subset of the vertex set of the graph which is not too large has a relatively large boundary. Algebraically, the expansion depends on the spectral properties of the graph: the spectral gap or, equivalently, the first positive eigenvalue of the Laplace operator, must be bounded away from 0. Finally, from a probabilistic point of view, an expander graph has the remarkable property that the simple random walk on it quickly converges to the limit stationary distribution. This fact has important applications in many computational problems, often solved by the Monte-Carlo algorithm, so that expander graphs play a crucial role also in Statistical Physics, Computational Group Theory, and Optimization.

In [29], a new construction is introduced in order to produce (infinite) families of constant-degree expanders of arbitrary size, starting from one constant-size expander (Subsection 3.3): this is the *zig-zag* product of two graphs (Subsection 3.2), strictly related to the simpler *replacement* product of graphs (Subsection 3.1). The replacement and the zig-zag product play also an important role in Geometric Group Theory, since it turns out that, when applied to Cayley graphs of two finite groups, they provide the Cayley graph of the semidirect product of these groups [3], with a suitable choice of the generating sets (Subsection 3.4). Hence, the semidirect product of groups becomes a tool for constructing large expander Cayley graphs starting from small graphs. Moreover, in [3], using the zig-zag construction, a negative answer is provided to a question formulated by Lubotzky and Weiss in [25], namely whether expansion is a property of the group, rather than of a particular choice of the generating set. In a work in progress with E. Sava-Huss, we are attacking the problem of studying sequences of zig-zag products of graphs, aiming at describing their spectral properties and investigating the existence, uniqueness, topological properties of the limit graph.

Section 4 is devoted to the study of the connections between replacement and zig-zag product of graphs, on the one hand, and the Lamplighter random walk, on the other hand. We recall the basic definitions for this model in Subsection 4.1, then in Subsection 4.2 we introduce a new model called "*Walk-switch-walk*", highlighting its connection with the combinatorial products of the associated graphs. Several examples are described. In Subsection 4.3, we describe the connection between the

Lamplighter random walk and the wreath product of graphs, then we pick out some relations between wreath, Cartesian, lexicographic and replacement products of graphs (Remark 4.6). In a future work with E. Candellero, we intend to pursue the investigation of the Lamplighter random walk on zig-zag product of graphs, also in connection with its spectral analysis, the study of the mixing time and the cut-off phenomenon [27].

2. Preliminaries

2.1. Regular graphs. Let $G = (V, E)$ be a finite undirected graph, so that E is a set of unordered pairs of type $\{u, v\}$, with $u, v \in V$. If $\{u, v\} \in E$, we say that the vertices u and v are adjacent in G , and we use the notation $u \sim v$. Observe that loops and multi-edges are allowed. A path in G is a sequence u_0, u_1, \dots, u_t of vertices such that $u_i \sim u_{i+1}$. The graph is connected if, for every $u, v \in V$, there exists a path u_0, u_1, \dots, u_t in G such that $u_0 = u$ and $u_t = v$.

Suppose $|V| = n$. We denote by $A_G = (a_{u,v})_{u,v \in V}$ the *adjacency matrix* of G , i.e., the square matrix of size n whose entry $a_{u,v}$ is equal to the number of edges joining u and v . As the graph G is undirected, A_G is a symmetric matrix and so it admits n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The *degree* of a vertex $u \in V$ is defined as $\deg(u) = \sum_{v \in V} a_{u,v}$. In particular, we say that G is *regular* of degree d , or d -regular, if $\sum_{v \in V} a_{u,v} = d$, for each $u \in V$. For such a graph G , the *normalized adjacency matrix* is defined as $A'_G = \frac{1}{d}A_G$.

Proposition 2.1 ([7, 12]). *Let $G = (V, E)$ be a finite d -regular graph, with $|V| = n$, and let A_G be its adjacency matrix. Then:*

- (1) d is an eigenvalue of A_G and $|\lambda_i| \leq d$, for each $i = 1, \dots, n$;
- (2) the multiplicity of d as an eigenvalue is 1 if and only if G is connected.

Recall that a graph G is bipartite if its vertex set V can be partitioned into two disjoint subsets V_+ and V_- such that, for every $u, v \in V$ such that $u \sim v$, if $u \in V_+$ (resp. V_-) then $v \in V_-$ (resp. V_+).

Proposition 2.2 ([7, 12]). *Let $G = (V, E)$ be a finite connected d -regular graph, with $|V| = n$. The following are equivalent:*

- (1) G is bipartite;
- (2) the spectrum of A_G is symmetric about 0;
- (3) $\lambda_n = -d$.

The eigenvalues d and $-d$, if the second occurs, are called the *trivial eigenvalues* of G . We also need to recall the definition of *second largest eigenvalue* of a d -regular graph G , given by

$$\lambda_G = \max\{|\lambda_i| : |\lambda_i| \neq d\}.$$

In other words, the second largest eigenvalue is the maximal nontrivial eigenvalue of A_G in absolute value.

2.2. Expander graphs. Let $G = (V, E)$ be a finite d -regular graph, with say $|V| = n$. Given $S \subseteq V$, the *boundary* of S is defined as $\partial S = \{\{u, v\} \in E \mid u \in S, v \notin S\}$. The *edge expansion ratio* (or *Cheeger constant*) of G is then defined by

$$(2.1) \quad h(G) = \min_{S \subseteq V, |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}.$$

Definition 2.3. For $\varepsilon > 0$, we say that G is an ε -expander if $h(G) \geq \varepsilon$.

In other words $h(G)$ is large if, for any partition of the vertex set V into two subsets V_1 and V_2 , there exist many links between V_1 and V_2 . Observe that a disconnected graph is clearly not expander, since the boundary of a connected component is empty.

In many applications, in particular to theoretical computer science, one is interested in constructing economical networks (i.e., with a fixed constant degree d , which must be as small as possible) of increasing size, for which the Cheeger constant is large. This leads to the notion of family of expander graphs.

Definition 2.4. A sequence of d -regular graphs $(G_n = (V_n, E_n))_{n \geq 1}$, with $|V_n| \rightarrow +\infty$ for $n \rightarrow +\infty$, is a family of expander graphs if there exists $\varepsilon > 0$ such that $h(G_n) \geq \varepsilon$, for every $n \geq 1$.

The first explicit construction of a family of expander graphs was given by Margulis in [26] as a family of regular graphs $(G_n)_{n \geq 1}$ of degree 8 for each $n \geq 1$.

Remark 2.5. There exists another notion of expansion, called *vertex expansion*, using a different notion of boundary where, for a given subset $S \subseteq V$, one counts the number of vertices of $V \setminus S$ which are adjacent to some vertex of S .

Now, let $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A_G of G . The difference $d - \lambda_2$ is called the *spectral gap* of G , and the following theorem shows how it provides an estimate of the expansion property of G . The theorem was first proven by Dodziuk [14] and then independently by Alon-Milman [4] and Alon [2].

Theorem 2.6. Let G be a connected d -regular graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)},$$

where $h(G)$ is the edge expansion ratio defined in (2.1).

In particular, the Cheeger constant is bounded away from 0 if and only if the spectral gap is bounded away from 0 and so a family $(G_n = (V_n, E_n))_{n \geq 1}$ of d -regular graphs, with $|V_n| \rightarrow +\infty$ for $n \rightarrow +\infty$, is a family of expander graphs if and only if there exists $\varepsilon > 0$ such that $d - \lambda_2(G_n) \geq \varepsilon$, for every $n \geq 1$.

3. Replacement and zig-zag product of graphs

This section is devoted to the description of the replacement and zig-zag products of graphs. These are noncommutative graph products producing a graph whose degree depends only on the degree of the second component graph, and providing large and sparse graphs. Moreover, we will see that the replacement and the zig-zag products both inherit the expansion properties of both the component graphs.

3.1. Replacement product of graphs. The replacement product of two graphs is a simple and intuitive construction, which is well known in the literature, where it was often used in order to reduce the vertex degree without losing the connectivity property. It has been widely used in many areas including Combinatorics, Probability, Group theory, in the study of expander graphs and graph-based coding schemes [8, 21, 23, 29]. It is worth mentioning that Gromov studied the second eigenvalue of an iterated replacement product of a d -dimensional cube with a lower dimensional cube [18].

We introduce some notation. Let $G = (V, E)$ be a d -regular graph over n vertices (loops and multi-edges are allowed). Suppose that we have a set of d colors (labels), that we identify with the set $\{1, 2, \dots, d\}$, that we are going to denote by $[d]$. We can assume that, for each vertex $v \in V$, the edges incident to v are labelled by a color $h \in [d]$ near v , and that any two distinct edges issuing from v have a different color near v . This allows to define the *rotation map* $\text{Rot}_G : V \times [d] \rightarrow V \times [d]$ such that

$$\text{Rot}_G(v, h) = (w, k), \quad \forall v \in V, h \in [d],$$

if there exists an edge joining v and w in G , which is colored by the color h near v and by the color k near w . Note that it may be $h \neq k$. Moreover, it follows from the definition that the composition $\text{Rot}_G \circ \text{Rot}_G$ is the identity map.

Definition 3.1. Let $G_1 = (V_1, E_1)$ be a d_1 -regular graph, with $|V_1| = n_1$, and let $G_2 = (V_2, E_2)$ be a d_2 -regular graph, satisfying the fundamental condition that $|V_2| = d_1$. The replacement product $G_1 \textcircled{R} G_2$ is the regular graph of degree $d_2 + 1$ with vertex set $V_1 \times V_2$, that we can identify with the set $V_1 \times [d_1]$, and whose edges are described by the following rotation map:

$$\text{Rot}_{G_1 \textcircled{R} G_2}((v, k), i) = \begin{cases} ((v, m), j) & \text{if } i \in [d_2] \text{ and } \text{Rot}_{G_2}(k, i) = (m, j) \\ (\text{Rot}_{G_1}(v, k), i) & \text{if } i = d_2 + 1, \end{cases}$$

for all $v \in V_1, k \in [d_1], i \in [d_2 + 1]$.

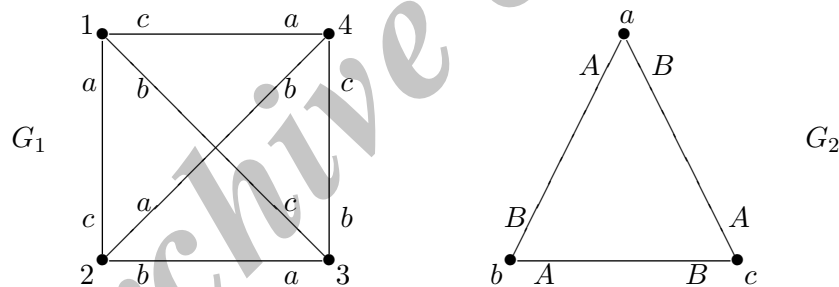
In other words, we can think that the vertex set of $G_1 \textcircled{R} G_2$ is partitioned into n_1 clouds, indexed by the vertices of G_1 , where by definition the v -cloud, with $v \in V_1$, is constituted by the vertices $(v, 1), (v, 2), \dots, (v, d_1)$. The idea is that we place a copy of G_2 around every vertex v of G_1 , but we keep edges of both G_1 and G_2 , so that every vertex of $G_1 \textcircled{R} G_2$ will be connected to its original neighbors within its cloud (by edges coming from G_2), but also to one vertex in a different cloud, which

depends on the rotation map of G_1 . Note that the degree of $G_1 \textcircled{F} G_2$ only depends on the degree of the second factor graph G_2 .

Remark 3.2. Observe that the definition of $G_1 \textcircled{F} G_2$ depends on the labelling of the edges around the vertices of G_1 . Hence, given a d_1 -regular graph G_1 on n_1 vertices and a d_2 -regular graph G_2 on d_1 vertices, there are $(d_1!)^{n_1}$ replacement products which are not necessarily isomorphic. See [1, Example 2.3], where two non isomorphic replacement products are constructed starting from the same factor graphs, endowed with different labellings. In particular, that example shows the following fact: if the graphs G_1 and H_1 have the same spectrum, and the graphs G_2 and H_2 have the same spectrum, then the graphs $G_1 \textcircled{F} G_2$ and $H_1 \textcircled{F} H_2$ do not have necessarily the same spectrum.

Example 3.3. Consider the graphs G_1 and G_2 in Fig. 1. G_1 is a 3-regular graph on $n_1 = 4$ vertices, with $V_1 = \{1, 2, 3, 4\}$, where the edges incident to every vertex $v \in V_1$ are labelled by a weight in $\{a, b, c\}$. G_2 is a 2-regular graph on $n_2 = d_1 = 3$ vertices, whose vertex set $V_2 \equiv \{a, b, c\}$ is identified with the set of possible labels of the edges of G_1 . For each vertex $v \in V_2$, the edges emanating from it are labelled by a weight in $\{A, B\}$. In Fig. 1, we use the convention that an edge connecting two vertices u and v of G_i is labelled by l near u and by l' near v if $\text{Rot}_{G_i}(u, l) = (v, l')$. In other words, that edge can be considered as the l -th edge incident to u and the l' -th edge incident to v .

FIGURE 1. The graphs G_1 and G_2 .

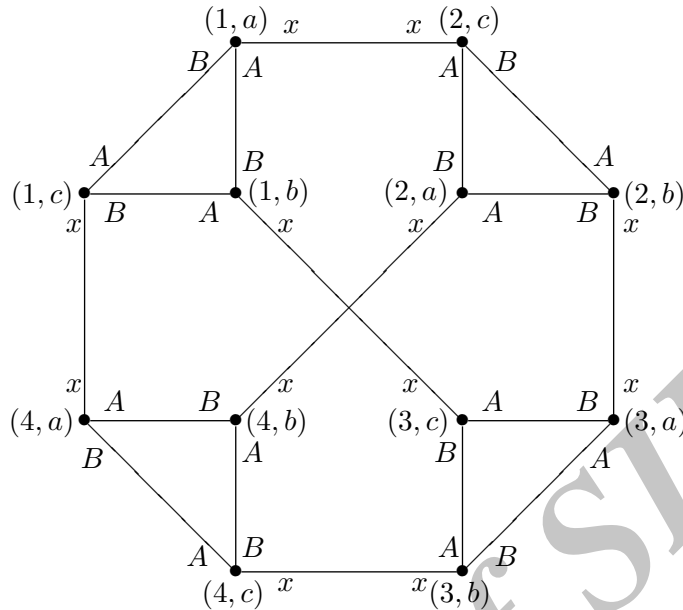


The adjacency relations in G_1 and G_2 are represented by the following rotation maps:

- $\text{Rot}_{G_1}(1, a) = (2, c)$ $\text{Rot}_{G_1}(1, b) = (3, c)$ $\text{Rot}_{G_1}(1, c) = (4, a)$
 $\text{Rot}_{G_1}(2, a) = (4, b)$ $\text{Rot}_{G_1}(2, b) = (3, a)$ $\text{Rot}_{G_1}(2, c) = (1, a)$
 $\text{Rot}_{G_1}(3, a) = (2, b)$ $\text{Rot}_{G_1}(3, b) = (4, c)$ $\text{Rot}_{G_1}(3, c) = (1, b)$
 $\text{Rot}_{G_1}(4, a) = (1, c)$ $\text{Rot}_{G_1}(4, b) = (2, a)$ $\text{Rot}_{G_1}(4, c) = (3, b)$
- $\text{Rot}_{G_2}(a, A) = (b, B)$ $\text{Rot}_{G_2}(a, B) = (c, A)$
 $\text{Rot}_{G_2}(b, A) = (c, B)$ $\text{Rot}_{G_2}(b, B) = (a, A)$
 $\text{Rot}_{G_2}(c, A) = (a, B)$ $\text{Rot}_{G_2}(c, B) = (b, A)$.

It follows from Definition 3.1 that the graph $G_1 \textcircled{F} G_2$, see Fig. 2, is the graph with vertex set $\{1, 2, 3, 4\} \times \{a, b, c\}$, whose edges are labelled by labels in $\{A, B, x\}$ (here x is the new color), and whose adjacencies are expressed by the following rotation map $\text{Rot}_{G_1 \textcircled{F} G_2}$ (since the rotation map is an involution, we only list half of the values taken by $\text{Rot}_{G_1 \textcircled{F} G_2}$):

FIGURE 2. The graph $G_1 \textcircled{F} G_2$.



$\text{Rot}_{G_1 \textcircled{F} G_2}((1, a), A) = ((1, b), B)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((1, a), B) = ((1, c), A)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((1, a), x) = ((2, c), x)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((1, b), A) = ((1, c), B)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((1, b), x) = ((3, c), x)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((1, c), x) = ((4, a), x)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((2, a), A) = ((2, b), B)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((2, a), B) = ((2, c), A)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((2, a), x) = ((4, b), x)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((2, b), A) = ((2, c), B)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((2, b), x) = ((3, a), x)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((3, a), A) = ((3, b), B)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((3, a), B) = ((3, c), A)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((3, b), A) = ((3, c), B)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((3, b), x) = ((4, c), x)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((4, a), A) = ((4, b), B)$
$\text{Rot}_{G_1 \textcircled{F} G_2}((4, a), B) = ((4, c), A)$	$\text{Rot}_{G_1 \textcircled{F} G_2}((4, b), A) = ((4, c), B)$

3.2. The zig-zag construction. The zig-zag product of two graphs was introduced in [29] as a construction allowing to produce, from a large graph G_1 and a small graph G_2 , a new graph $G_1 \textcircled{Z} G_2$ which inherits the size from the large one, the degree from the small one, and the expansion property from both the graphs. In [29], it is explicitly described how iteration of this new construction, together with the standard squaring, provides an infinite family of constant-degree expander graphs, starting from a particular graph representing the building block of this construction (Subsection 3.3).

Definition 3.4. Let $G_1 = (V_1, E_1)$ be a d_1 -regular graph, with $|V_1| = n_1$, and let $G_2 = (V_2, E_2)$ be a d_2 -regular graph such that $|V_2| = d_1$ (graphs are allowed to have loops or multi-edges). Let Rot_{G_1} (resp. Rot_{G_2}) denote the rotation map of G_1 (resp. G_2). The zig-zag product $G_1 \textcircled{Z} G_2$ is the regular graph of degree d_2^2 with vertex set $V_1 \times V_2$, that we identify with the set $V_1 \times [d_1]$, and whose edges are described by the rotation map

$$\text{Rot}_{G_1 \textcircled{Z} G_2}((v, k), (i, j)) = ((w, l), (j', i')),$$

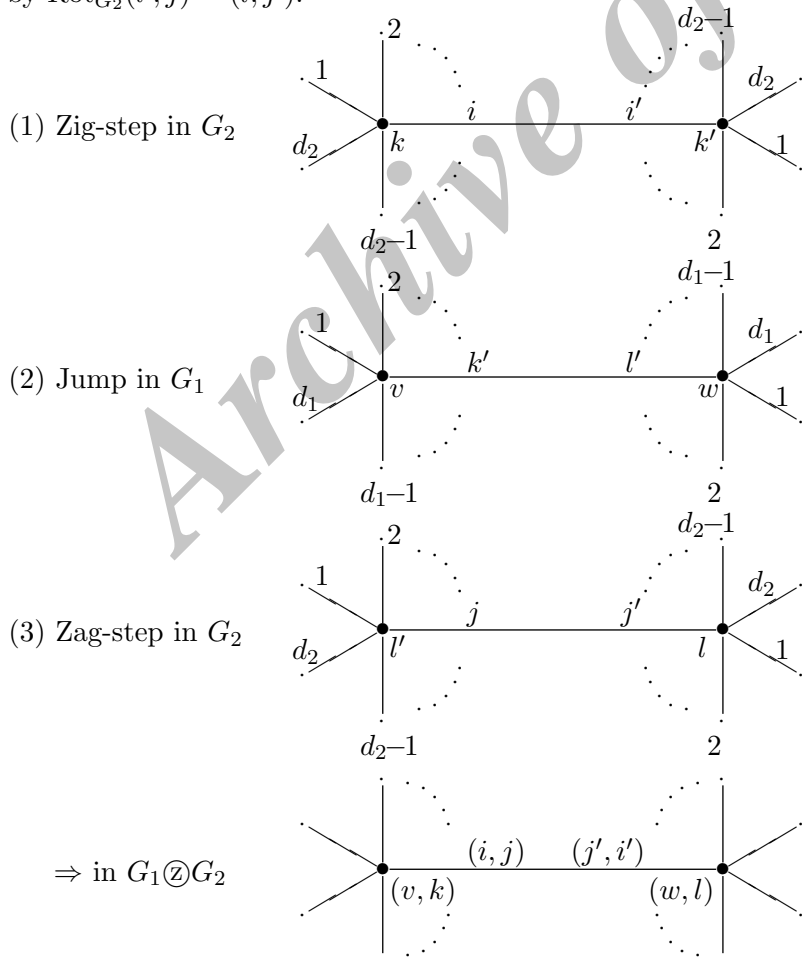
for all $v \in V_1, k \in [d_1], i, j \in [d_2]$, if:

- (1) $Rot_{G_2}(k, i) = (k', i')$,
- (2) $Rot_{G_1}(v, k') = (w, l')$,
- (3) $Rot_{G_2}(l', j) = (l, j')$.

where $w \in V_1, l, k', l' \in [d_1]$ and $i', j' \in [d_2]$.

Observe that labels in $G_1 \otimes G_2$ are elements from $[d_2]^2$. As in the case of the replacement product, the vertex set of $G_1 \otimes G_2$ is partitioned into n_1 clouds, indexed by the vertices of G_1 , where by definition the v -cloud, with $v \in V_1$, is constituted by the vertices $(v, 1), (v, 2), \dots, (v, d_1)$. Two vertices (v, k) and (w, l) of $G_1 \otimes G_2$ are adjacent in $G_1 \otimes G_2$ if it is possible to go from (v, k) to (w, l) by a sequence of three steps of the following form:

- (1) a first step “zig” within the initial cloud, from the vertex (v, k) to the vertex (v, k') , described by $Rot_{G_2}(k, i) = (k', i')$;
- (2) a second step jumping from the v -cloud to the w -cloud, from the vertex (v, k') to the vertex (w, l') , described by $Rot_{G_1}(v, k') = (w, l')$;
- (3) a third step “zag” within the new cloud, from the vertex (w, l') to the vertex (w, l) , described by $Rot_{G_2}(l', j) = (l, j')$.



Example 3.5. Consider again the graphs G_1 and G_2 in Fig. 1. It follows from Definition 3.4 that the graph $G_1 \circledast G_2$, see Fig. 3, is the graph with vertex set $\{1, 2, 3, 4\} \times \{a, b, c\}$, whose edges are labelled by ordered pairs (i, j) from $\{A, B\}^2$ (in order to simplify the notation in Fig. 3, we will write ij), and whose adjacency relations are expressed by the following rotation map $\text{Rot}_{G_1 \circledast G_2}$ (since $\text{Rot}_{G_1 \circledast G_2}$ is an involution, we only list half of the values taken by $\text{Rot}_{G_1 \circledast G_2}$):

$$\begin{aligned} \text{Rot}_{G_1 \circledast G_2}((1, a), (A, A)) &= ((3, a), (B, B)) & \text{Rot}_{G_1 \circledast G_2}((1, a), (A, B)) &= ((3, b), (A, B)) \\ \text{Rot}_{G_1 \circledast G_2}((1, a), (B, A)) &= ((4, b), (B, A)) & \text{Rot}_{G_1 \circledast G_2}((1, a), (B, B)) &= ((4, c), (A, A)) \\ \text{Rot}_{G_1 \circledast G_2}((1, b), (A, A)) &= ((4, b), (B, B)) & \text{Rot}_{G_1 \circledast G_2}((1, b), (A, B)) &= ((4, c), (A, B)) \\ \text{Rot}_{G_1 \circledast G_2}((1, b), (B, A)) &= ((2, a), (B, A)) & \text{Rot}_{G_1 \circledast G_2}((1, b), (B, B)) &= ((2, b), (A, A)) \\ \text{Rot}_{G_1 \circledast G_2}((1, c), (A, A)) &= ((2, a), (B, B)) & \text{Rot}_{G_1 \circledast G_2}((1, c), (A, B)) &= ((2, b), (A, B)) \\ \text{Rot}_{G_1 \circledast G_2}((1, c), (B, A)) &= ((3, a), (B, A)) & \text{Rot}_{G_1 \circledast G_2}((1, c), (B, B)) &= ((3, b), (A, A)) \\ \text{Rot}_{G_1 \circledast G_2}((2, a), (A, A)) &= ((3, b), (B, B)) & \text{Rot}_{G_1 \circledast G_2}((2, a), (A, B)) &= ((3, c), (A, B)) \\ \text{Rot}_{G_1 \circledast G_2}((2, b), (B, A)) &= ((4, c), (B, A)) & \text{Rot}_{G_1 \circledast G_2}((2, b), (B, B)) &= ((4, a), (A, A)) \\ \text{Rot}_{G_1 \circledast G_2}((2, c), (A, A)) &= ((4, c), (B, B)) & \text{Rot}_{G_1 \circledast G_2}((2, c), (A, B)) &= ((4, a), (A, B)) \\ \text{Rot}_{G_1 \circledast G_2}((2, c), (B, A)) &= ((3, b), (B, A)) & \text{Rot}_{G_1 \circledast G_2}((2, c), (B, B)) &= ((3, c), (A, A)) \\ \text{Rot}_{G_1 \circledast G_2}((3, a), (A, A)) &= ((4, a), (B, B)) & \text{Rot}_{G_1 \circledast G_2}((3, a), (A, B)) &= ((4, b), (A, B)) \\ \text{Rot}_{G_1 \circledast G_2}((3, c), (B, A)) &= ((4, a), (B, A)) & \text{Rot}_{G_1 \circledast G_2}((3, c), (B, B)) &= ((4, b), (A, A)). \end{aligned}$$

It follows from the definition of replacement product and zig-zag product that the edges of $G_1 \circledast G_2$ arise from paths of length 3 in $G_1 \circledcirc G_2$ of the following type:

- a first step within one cloud (the zig-step);
- a second step which is a jump to a new cloud;
- a third step within the new cloud (the zag-step).

For instance, the three consecutive steps $(1, a) \rightarrow (1, b) \rightarrow (3, c) \rightarrow (3, b)$ in the replacement product of Example 3.3 produce the edge connecting $(1, a)$ and $(3, b)$ in the zig-zag product of Example 3.5. In other words, $G_1 \circledast G_2$ is a regular subgraph of the graph obtained by taking the third power of $G_1 \circledcirc G_2$. This fact can also be expressed in terms of normalized adjacency matrices.

More precisely, let A'_1 (resp. A'_2) be the normalized adjacency matrix of the graph G_1 (resp. G_2). It follows from the definition of zig-zag product that the normalized adjacency matrix of $G_1 \circledast G_2$ is $M_{\circledast} = \tilde{A}_2 \tilde{A}_1 \tilde{A}_2$ (see [29]), where $\tilde{A}_2 = I_{n_1} \otimes A'_2$ and \tilde{A}_1 is the permutation matrix on $V_1 \times [d_1]$ associated with the map Rot_{G_1} , i.e.,

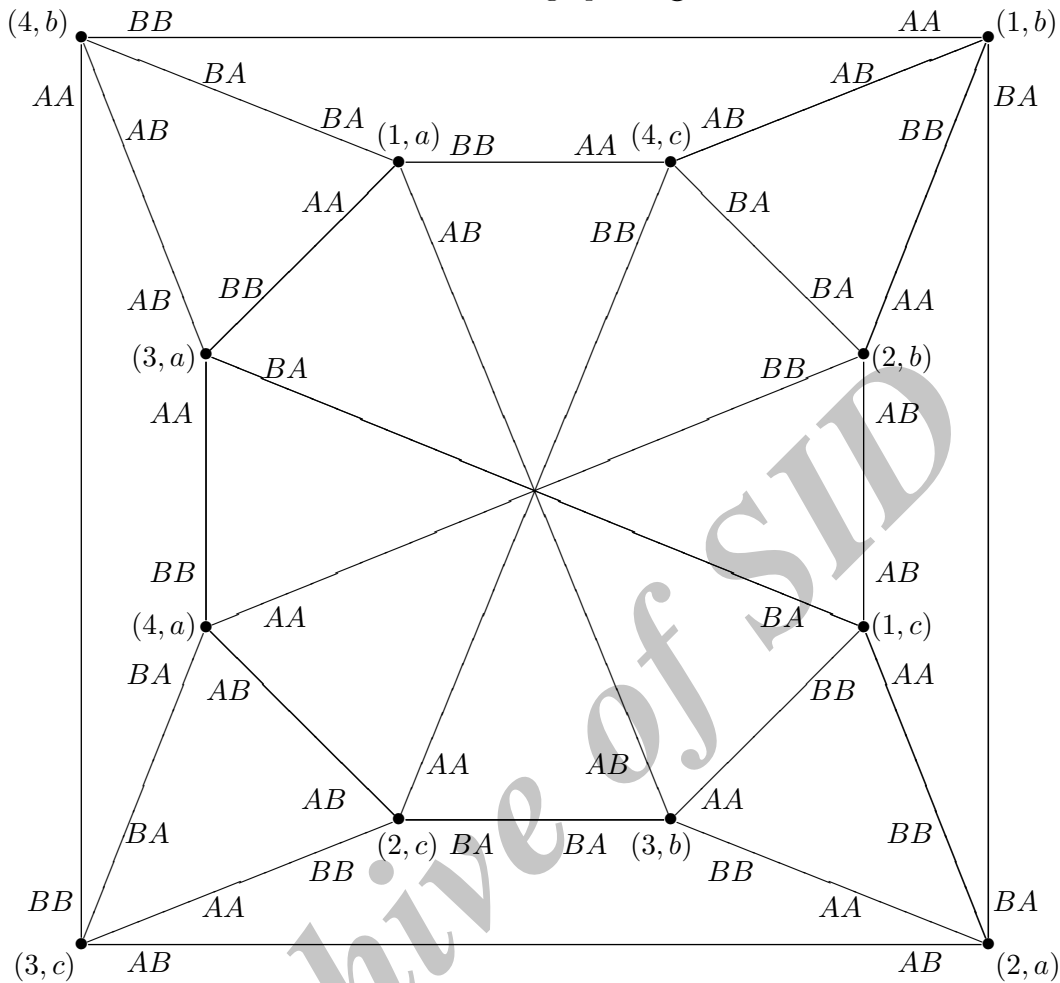
$$\tilde{A}_1(v,k),(w,l) = \begin{cases} 1 & \text{if } v \sim w \text{ in } G_1 \text{ by an edge labelled } k \text{ near } v \text{ and } l \text{ near } w \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, it follows from the definition of replacement product that the normalized adjacency matrix of $G_1 \circledcirc G_2$ is $M_{\circledcirc} = \frac{\tilde{A}_1 + d_2 \tilde{A}_2}{d_2 + 1}$, and the following decomposition holds:

$$M_{\circledcirc}^3 = \frac{d_2^2}{(d_2 + 1)^3} \tilde{A}_2 \tilde{A}_1 \tilde{A}_2 + \left(1 - \frac{d_2^2}{(d_2 + 1)^3} \right) C,$$

where C is the normalized adjacency matrix of a regular graph.

FIGURE 3. The graph $G_1 \otimes G_2$.



Remark 3.6. In [19], the definition of zig-zag product is reformulated in terms of Cartesian products and complete matchings of lexicographic products of the component graphs with suitable totally disconnected graphs (see Subsection 4.3 for the definition of these products).

3.3. Constructing expander graphs using zig-zag products. In this subsection we describe the construction presented in [29], where an infinite family of constant-degree expander graphs is obtained by using zig-zag products. More precisely, starting with a constant size expander graph, one can iterate zig-zag products and squaring of graphs in order to get larger and larger expanders of the same fixed degree. So we need to recall the basic notion of squaring graphs.

We will say that G is an (n, d, α) -graph if it is a d -regular graph on n vertices, and its normalized second largest eigenvalue λ'_G satisfies $\lambda'_G \leq \alpha$.

Now let $G = (V, E)$ be an (n, d, α) -graph. The graph obtained squaring G , denoted by G^2 , is defined as the graph whose vertex set coincides with V , and where edges are paths of length 2 in G . It follows that, if A_G is the adjacency matrix of G , then the adjacency matrix of G^2 is given by A_G^2 . As a consequence, if G is an (n, d, α) -graph, then G^2 is an (n, d^2, α^2) -graph. In terms of rotation map,

the graph G^2 can be described by

$$\text{Rot}_{G^2}(v_0, (k_1, k_2)) = (v_2, (l_2, l_1)),$$

if $\text{Rot}_G(v_0, k_1) = (v_1, l_1)$ and $\text{Rot}_G(v_1, k_2) = (v_2, l_2)$, with $v_i \in V$ and $k_i, l_i \in [d]$. The following crucial theorem has been proven in [29], and it shows that the zig-zag product $G_1 \otimes G_2$ is a good expander, if both G_1 and G_2 are.

Theorem 3.7. *Let G_1 be an (n_1, d_1, α_1) -graph and let G_2 be a (d_1, d_2, α_2) -graph. The zig-zag product $G_1 \otimes G_2$ is an $(n_1 \cdot d_1, d_2^2, f(\alpha_1, \alpha_2))$ -graph, where $f(\alpha_1, \alpha_2) \leq \alpha_1 + \alpha_2 + \alpha_2^2$ and $f(\alpha_1, \alpha_2) < 1$ when $\alpha_1, \alpha_2 < 1$.*

Observe that the degree of $G_1 \otimes G_2$ can also be much smaller than the degree of the large graph G_1 ; however, if both G_1 and G_2 are expanders, then also the zig-zag product $G_1 \otimes G_2$ will be expander.

We are now able to describe the construction performed in [29]. Let H be any $(d^4, d, \frac{1}{5})$ -graph, so that H is an expander (there exist a probabilistic argument showing that such an expander exists) and define

$$G_1 = H^2, \quad G_{n+1} = G_n^2 \otimes H, \quad \forall n \geq 1.$$

Proposition 3.8. *For each $n \geq 1$, the graph G_n is a $(d^{4n}, d^2, \frac{2}{5})$ -graph, and so the family $(G_n)_{n \geq 1}$ is expander.*

Proof. The proof works by induction on n . For $n = 1$, the claim is true by definition of G_1 . Now let us assume the statement to be true for n , so that G_n is a $(d^{4n}, d^2, \frac{2}{5})$ -graph. This implies that the graph G_n^2 is a $(d^{4n}, d^4, \frac{4}{25})$ -graph, then Theorem 3.7 ensures that G_{n+1} is an $(d^{4n+4}, d^2, \frac{4}{25} + \frac{1}{5} + \frac{1}{25}) = (d^{4(n+1)}, d^2, \frac{2}{5})$ -graph and the proof is completed. \square

The following analogous theorem holds for the replacement product [29].

Theorem 3.9. *Let G_1 be an (n_1, d_1, α_1) -graph and let G_2 be a (d_1, d_2, α_2) -graph. The replacement product $G_1 \oplus G_2$ is an $(n_1 \cdot d_1, d_2 + 1, g(\alpha_1, \alpha_2, d_2))$ -graph, where*

$$g(\alpha_1, \alpha_2, d_2) \leq \left(\frac{d_2^2}{(d_2 + 1)^3} + \left(1 - \frac{d_2^2}{(d_2 + 1)^3} \right) f(\alpha_1, \alpha_2) \right)^{1/3},$$

where f is the function of Theorem 3.7.

Remark 3.10. In [29], it is proven that the upper bound given by Theorem 3.7 and 3.9 for the normalized second largest eigenvalue can be improved by using the function

$$f(\alpha_1, \alpha_2) = \frac{1}{2}(1 - \alpha_2^2)\alpha_1 + \frac{1}{2}((1 - \alpha_2^2)^2\alpha_1^2 + 4\alpha_2^2)^{1/2},$$

satisfying the following nice properties:

- (1) $f(\alpha, 0) = f(0, \alpha) = \alpha$ and $f(\alpha, 1) = f(1, \alpha) = 1$, for all $\alpha \in [0, 1]$;
- (2) $f(\alpha_1, \alpha_2)$ is a strictly increasing function of both α_1 and α_2 (except when one of them is equal to 1);
- (3) $f(\alpha_1, \alpha_2) < 1$ if $\alpha_1, \alpha_2 < 1$;

$$(4) f(\alpha_1, \alpha_2) \leq \alpha_1 + \alpha_2, \text{ for all } \alpha_1, \alpha_2 \in [0, 1].$$

Remark 3.11. Using the function f of Remark 3.10, it is shown in [23] that starting from a graph G_1 of type $(n, d+1, \alpha_1)$ and a graph H of type $((d+1)^4, d, \alpha_2)$, with $\alpha_1, \alpha_2 \leq \frac{1}{5}$, $d \geq 6$, and defining by recursion

$$G_{i+1} = G_i^4 \circledast H, \quad \forall i \geq 1,$$

one gets that G_i is a $(n(d+1)^{4(i-1)}, d+1, \alpha)$ -graph, with $\alpha \leq \frac{43}{50}$, so that the family $(G_i)_{i \geq 1}$ is expander.

3.4. Zig-zag product of graphs and semidirect product of groups. In [3], it is shown that the zig-zag product of two graphs can be regarded as a generalization of the semidirect product of groups, via the notion of Cayley graph associated with a finitely generated (finite) group. A similar argument can be developed for the replacement product (see, for instance, [21, 23]).

Let us briefly recall the classical definition of semidirect product of groups. Let A and B be two finite groups, and suppose that an action by automorphisms of B on A is defined, i.e., there exists a group homomorphism $\phi : B \rightarrow \text{Aut}(A)$. For every $a \in A$ and $b \in B$, we denote by a^b the image of a under the action of $\phi(b)$ and, similarly, we denote by $a^B = \{a^b \mid b \in B\}$ the orbit of a under the action of the group B . The *semidirect product* $A \rtimes B$ is the group whose underlying set is $A \times B = \{(a, b) \mid a \in A, b \in B\}$, and whose group operation is defined by

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2^{b_1}, b_1 b_2), \quad \forall a_1, a_2 \in A, b_1, b_2 \in B.$$

It is easy to check that the identity of $A \rtimes B$ is given by $(1_A, 1_B)$, where 1_A and 1_B are the identity in A and B , respectively, and that

$$(a, b)^{-1} = ((a^{-1})^{b^{-1}}, b^{-1}), \quad \forall a \in A, b \in B.$$

We recall now the definition of *Cayley graph* $\text{Cay}(G, S)$ of a finite group G with respect to a generating set S (the same definition can be given for a finitely generated infinite group). Note that the generating set S is supposed to be symmetric (i.e., if $s \in S$, then also $s^{-1} \in S$) and it can be a multiset, i.e., repetitions are allowed. The graph $\text{Cay}(G, S)$ is the graph whose vertex set is G , and where two vertices g and g' are joined by an edge labelled $s \in S$ if the generator s satisfies $gs = g'$. It is clearly a regular graph of degree $|S|$. Note also that a rotation map can be naturally defined on $\text{Cay}(G, S)$ as $\text{Rot}_{\text{Cay}(G, S)}(g, s) = (g', s^{-1})$ if and only if $gs = g'$.

Now let A and B be two finite groups with symmetric generating sets S_A and S_B , respectively, such that $|B| = |S_A|$, and suppose that B acts on A in such a way that $S_A = x^B$, for some $x \in S_A$. In order to show the correspondence between replacement product of graphs and semidirect product of groups, we slightly modify the replacement construction by defining the product $\text{Cay}(A, S_A) \circledast \text{Cay}(S, S_B)$ as follows. Consider the graph $\text{Cay}(A, S_A)$. If an edge e connects two vertices a and a' , then by definition there exists $s_a \in S_A$ such that $as_a = a'$. If $s_a = x^b$ for some $b \in B$, we label e by the label b near a and near a' in $\text{Cay}(A, S_A)$. Hence, e can be regarded as an edge of type x labelled by b with respect to the vertex a , because $ax^b = a'$, but also as an edge of type x^{-1} labelled by b with respect to the vertex a' , because it is also true that $a'(x^{-1})^b = a$. In this way, we have $|B|$ edges of type x

and $|B|$ edges of type x^{-1} issuing from every vertex of $Cay(A, S_A)$, so that the degree becomes $2|B|$ and one cannot perform the classical replacement product between $Cay(A, S_A)$ and $Cay(B, S_B)$. The product $Cay(A, S_A) \circledast Cay(B, S_B)$ has vertex set $A \times B$, as in the classical construction; moreover, it has three types of edges, so the novelty is that we can use labels coming from the graph $Cay(B, S_B)$ (i.e., elements $s_b \in S_B$) and two new labels, which we identify with x and x^{-1} . More precisely, the vertex (a, b) will be joined:

- to the vertex (a, bs_b) by the edge labelled s_b (this is a step within the cloud indexed by a , corresponding to a step in $Cay(B, S_B)$);
- to the vertex (ax^b, b) by the edge labelled x (this is a jump from the cloud indexed by a to the cloud indexed by ax^b , corresponding to a step in $Cay(A, S_A)$ according with an edge of type x);
- to the vertex $(a(x^{-1})^b, b)$ by the edge labelled x^{-1} (this is a jump from the cloud indexed by a to the cloud indexed by $a(x^{-1})^b$, corresponding to a step in $Cay(A, S_A)$ according with an edge of type x^{-1}).

Therefore, the graph $Cay(A, S_A) \circledast Cay(B, S_B)$ has degree $|S_B| + 2$. The following theorem holds.

Theorem 3.12. *Let A and B be two finite groups with symmetric generating sets S_A and S_B , respectively, such that $|B| = |S_A|$, and suppose that B acts on A in such a way that $S_A = x^B$, for some $x \in S_A$. Then $S = \{(1_A, s) \mid s \in S_B\} \cup \{(x^\varepsilon, 1_B), \varepsilon = \pm 1\}$ is a symmetric generating set for $A \rtimes B$ and*

$$Cay(A \rtimes B, S) = Cay(A, S_A) \circledast Cay(B, S_B).$$

More generally, if $S_A = x_1^B \cup \dots \cup x_k^B$, with $x_1, \dots, x_k \in S_A$, then a generating set for $A \rtimes B$ is given by $S = \{(1_A, s) \mid s \in S_B\} \cup \{(x_i^\varepsilon, 1_B), \dots, (x_k^\varepsilon, 1_B), \varepsilon = \pm 1\}$ and $Cay(A \rtimes B, S)$ is the union of the products $Cay(A, x_i^B) \circledast Cay(B, S_B)$, for $i = 1, \dots, k$.

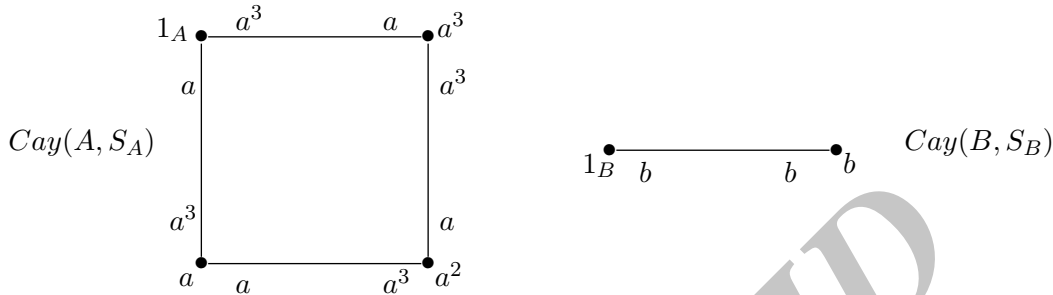
Proof. Consider the case $S_A = x^B$, for some $x \in A$. It is obvious that S is a symmetric set. We want to show that it generates the group $A \rtimes B$. Observe that each element $(a, b) \in A \rtimes B$ can be written as $(a, b) = (a, 1_B)(1_A, b)$ and so it can be written as a product of elements from the set $\{(1_A, s_b), (s_a, 1_B), s_a \in S_A, s_b \in S_B\}$. By assumption, $s_a = x^b$, for some $b \in B$, where b is a product of elements of S_B , by definition. Therefore the element $(s_a, 1_B)$ can be written as a product of elements from S (we are using the identity $(x^{bb'}, 1_B) = (1_A, bb')(x, 1_B)(1_A, (bb')^{-1})$, holding for every $b, b' \in B$).

Consider now the graph $Cay(A \rtimes B, S)$. This graph consists of clouds which are copies of the Cayley graph of B , indexed by the elements of A , and connections within each cloud are given by edges of $Cay(B, S_B)$, since $(a, b)(1_A, s_b) = (a, bs_b)$. Between two different clouds we have edges of the form $(a, b)(x^{\pm 1}, 1_B) = (a(x^{\pm 1})^b, b)$. So the cloud indexed by a is connected by one edge to each of the clouds indexed by a neighbor of a in the graph $Cay(A, S_A)$, according with the definition of the modified replacement product \circledast . In particular, the resulting graph will be regular of degree $|S_B| + 2$.

The proof in the case of k orbits can be given by repeating a similar argument for each orbit: in this case, one obtains a graph which is regular of degree $|S_B| + 2k$. □

Example 3.13. Let $A = \langle a \mid a^4 = 1_A \rangle$ be the cyclic group of order 4, with symmetric generating set $S_A = \{a, a^3\}$ and let $B = \langle b \mid b^2 = 1_B \rangle$ be the cyclic group of order 2, with symmetric generating set $S_B = \{b\}$. The corresponding Cayley graphs are shown in Fig. 4.

FIGURE 4. The Cayley graphs $Cay(A, S_A)$ and $Cay(B, S_B)$.



We define an action of B on A by

$$1_A^b = 1_A \quad a^b = a^3 \quad (a^2)^b = a^2 \quad (a^3)^b = a.$$

In particular, S_A coincides with the orbit generated by a under the action of B , so that we can chose $x = a$ and $x^{-1} = a^3$. When we apply the strategy described to prove Theorem 3.12, we get the situation in Fig. 5. For example, we have drawn an edge joining a and a^2 labelled by 1_{B+} near a

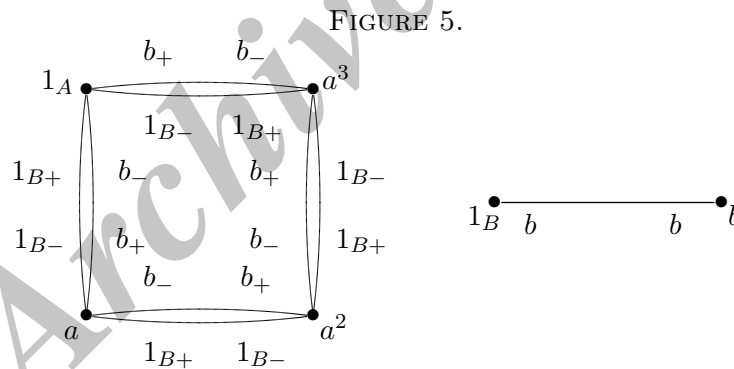


FIGURE 5.

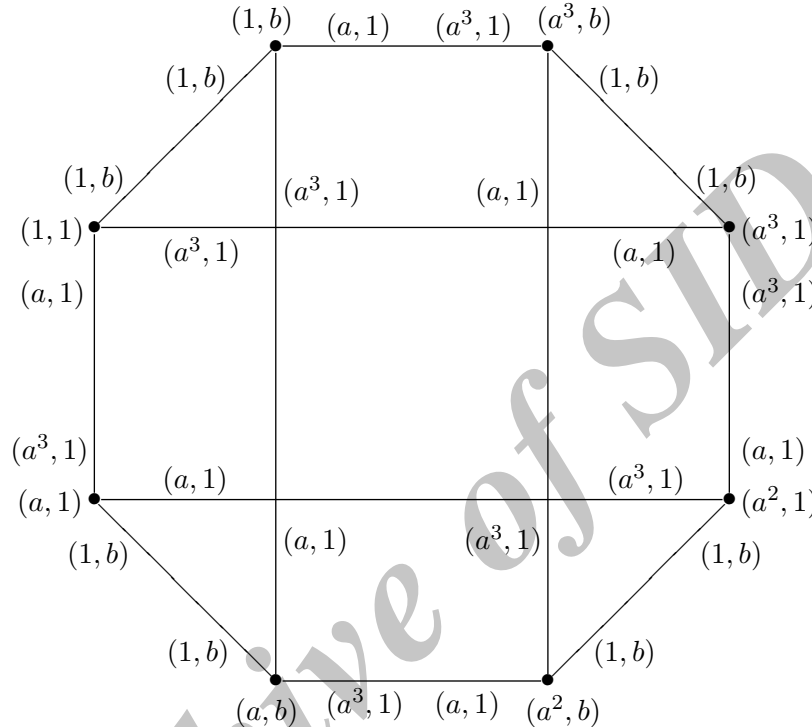
and by 1_{B-} near a^2 , meaning that this edge is of type x labelled by 1_B with respect to a , because $ax^{1_B} = a^2$, but it is of type x^{-1} labelled by 1_B with respect to a^2 , because $a^2(x^{-1})^{1_B} = a$. If we want to establish, for instance, which are the neighbors of the vertex $(a, 1_B)$ in $Cay(A, S_A) \widehat{\otimes} Cay(B, S_B)$, we have to proceed in the following way. When we leave $(a, 1_B)$ by the edge labelled $(1_A, b)$, we move to $(a, 1_B)(1_A, b) = (a, b)$. When we leave $(a, 1_B)$ by the edge labelled $(x, 1_B)$, we move to $(a, 1_B)(x, 1_B) = (ax^{1_B}, 1_B) = (a^2, 1_B)$. Finally, when we leave $(a, 1_B)$ by the edge labelled $(x^{-1}, 1_B)$, we move to $(a, 1_B)(x^{-1}, 1_B) = (a(x^{-1})^{1_B}, 1_B) = (1_A, 1_B)$.

On the other hand, if we move from the vertex (a^3, b) , the following three possibilities occur. If we follow the edge labelled $(1_A, b)$, we move to $(a^3, b)(1_A, b) = (a^3 1_A^b, 1_B) = (a^3, 1_B)$. If we follow the edge labelled $(x, 1_B)$, we go to $(a^3, b)(x, 1_B) = (a^3 x^b, b) = (a^2, b)$. Finally, following the edge labelled

$(x^{-1}, 1_B)$, we move to $(a^3, b)(x^{-1}, 1_B) = (a^3(x^{-1})^b, b) = (1_A, b)$.

The product $\widehat{\mathcal{R}}$ produces the graph in Fig. 6, which is the Cayley graph of the dihedral group $D_4 = \langle a, b \mid a^4 = b^2 = 1_{D_4}, ab = ba^{-1} \rangle$ of order 8, with respect to the symmetric generating set $\{(a^{\pm 1}, 1_B), (1_A, b)\}$.

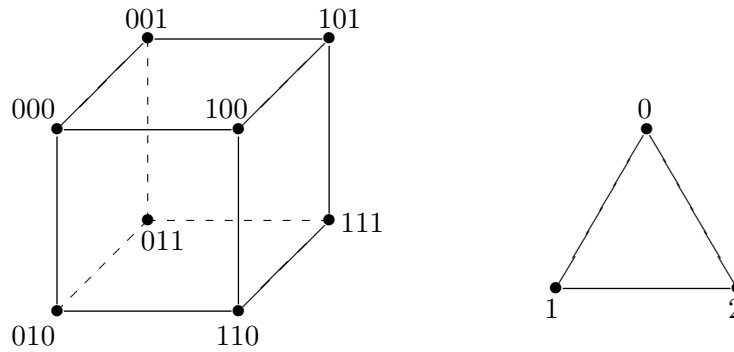
FIGURE 6. The graph $Cay(A, S_A) \widehat{\mathcal{R}} Cay(B, S_B)$.



Example 3.14. Another example is provided by the action of the cyclic group $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, with generating set $\{\pm 1\}$, on the d -dimensional cube \mathbb{Z}_2^d , with generating set $\mathbf{E}_d = \{\mathbf{e}_i, i = 0, \dots, d-1\}$, where \mathbf{e}_i denotes the d -tuple with 1 at the i -th coordinate and 0 elsewhere. The action of \mathbb{Z}_d on \mathbb{Z}_2^d is described by $(v_0, \dots, v_{d-1})^m = (v_{-m}, \dots, v_{d-1-m})$, for every $(v_0, \dots, v_{d-1}) \in \mathbb{Z}_2^d$ and $m \in \mathbb{Z}_d$ (observe that the coordinates are indexed by integers modulo d). Therefore, one can construct the semidirect product $\mathbb{Z}_2^d \rtimes \mathbb{Z}_d$.

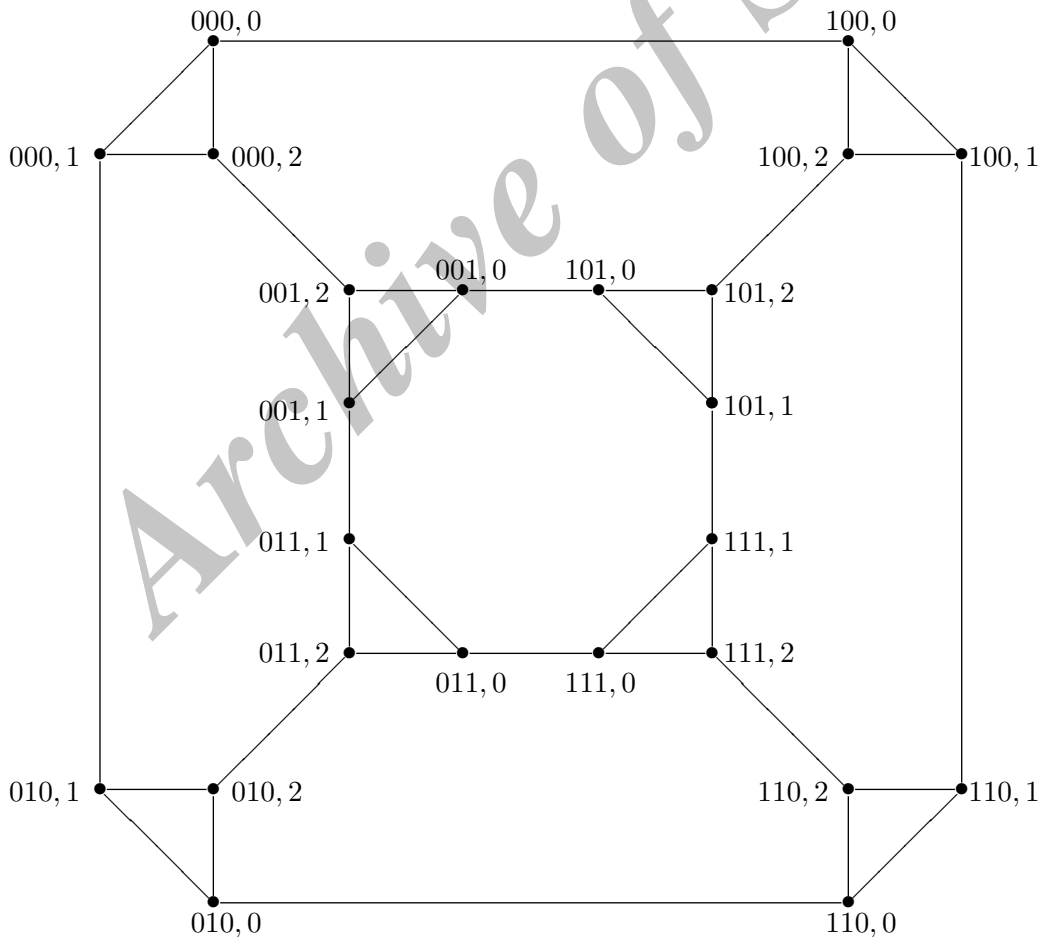
The graphs $Cay(\mathbb{Z}_2^d, \mathbf{E}_d)$ and $Cay(\mathbb{Z}_d, \{\pm 1\})$ are the d -dimensional cube and the d -cycle, respectively. In Fig. 7, the case of $d = 3$ is represented. On the other hand, since the degree of the d -dimensional cube is d , which is the number of vertices in the d -cycle, one can construct the product $Cay(\mathbb{Z}_2^d, \mathbf{E}_d) \widehat{\mathcal{R}} Cay(\mathbb{Z}_d, \{\pm 1\})$. Note that the generating set \mathbf{E}_d is a single orbit under the action of \mathbb{Z}_d , and a representative element is given by \mathbf{e}_0 . Note also that $\mathbf{e}_i = \mathbf{e}_i^{-1}$ for each $i = 0, \dots, d-1$, so that the products $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{R}}$ coincide in this case. In the replacement product, every vertex of \mathbb{Z}_2^d is replaced by a cloud of d vertices representing a copy of \mathbb{Z}_d . We also connect each vertex of a cloud to exactly one vertex of a neighboring cloud. More precisely, if an edge connects two vertices u, v of \mathbb{Z}_2^d representing d -tuples which differ in the i -th coordinate, then we label by \mathbf{e}_i in $Cay(\mathbb{Z}_2^d, \mathbf{E}_d)$ that edge near both u and v . With this choice, we connect in $Cay(\mathbb{Z}_2^d, \mathbf{E}_d) \widehat{\mathcal{R}} Cay(\mathbb{Z}_d, \{\pm 1\})$ the vertex (v, i) to

FIGURE 7. The graphs $Cay(\mathbb{Z}_2^3, \mathbf{E}_3)$ and $Cay(\mathbb{Z}_3, \{\pm 1\})$.



the vertex $(v + \mathbf{e}_i, i)$. The resulting graph for $d = 3$ is depicted in Fig. 8. From the Cayley graph point of view, this graph represents the semidirect product $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3$, with respect to the generating set $\{(000, \pm 1), (\mathbf{e}_0, 0)\}$.

FIGURE 8. The graph $Cay(\mathbb{Z}_2^3, \mathbf{E}_3) \oplus Cay(\mathbb{Z}_3, \{\pm 1\})$.



The construction that we have introduced in Theorem 3.12, where one has 2 ($2k$ in the case of k orbits) possibilities for the step in the Cayley graph of A , according with the choice of x or x^{-1} in

the generating set, is very close to the construction introduced in [3], where the definition of zig-zag product $G \circledast H$ introduced in [29] is modified in order to adapt it to the construction of Cayley graphs. It is shown there that, with a suitable choice of the generating sets for the groups $A, B, A \times B$, the Cayley graph of $A \times B$ is the zig-zag product of the Cayley graphs of A and B .

In this modified version, the “small graph” $H = (V_H, E_H)$ is a graph of type (m, d, α) and the “large graph” $G = (V_G, E_G)$ is of type (n, cm, β) , where c is a positive integer (which is equal to 1 in the original definition), so that G appears as the (edge)-disjoint union of c regular graphs of degree m , say G_i , for $i = 1, \dots, c$, with the same vertex set of cardinality n . We also suppose that in each graph G_i the edges around every vertex are labelled in a fixed way by labels from the set $[m]$. The zig-zag product $G \circledast H$ is the regular graph of degree cd^2 with vertex set $V_G \times [m]$, where two vertices (v, k) and (w, l) are connected if there exist $k', l' \in [m]$ and $i \in [c]$ such that:

- (1) there is an edge in H connecting the vertices k and k' ;
- (2) there is an edge connecting the vertices v and w in G_i , which is labelled k' near v and l' near w ;
- (3) there is an edge in H connecting the vertices l' and l .

The novelty is that in this model the middle step, occurring in G_i , is stochastic and not deterministic as in the classical definition, since one now has c possibilities for leaving the vertex v using an edge labelled k' near v . An analogue of Theorem 3.7 is true also in this case.

Theorem 3.15 ([3]). *The zig-zag product $G \circledast H$ is an $(nm, cd^2, f(\alpha, \beta))$ -graph, where $f(\alpha, \beta)$ can be chosen as in Theorem 3.7 or in Remark 3.10.*

Theorem 3.12 can be reformulated for the zig-zag product as follows.

Theorem 3.16 ([3]). *Let A and B be two finite groups with symmetric generating sets S_A and S_B , respectively, such that $|B| = |S_A|$, and suppose that B acts on A in such a way that $S_A = x_1^B \cup \dots \cup x_k^B$, with $x_1, \dots, x_k \in S_A$. Then a generating set for $A \times B$ is given by*

$$S = \{(1_A, s)(x_i^\varepsilon, 1_B)(1_A, s') \mid s, s' \in S_B, \varepsilon = \pm 1, i = 1, \dots, k\}$$

and $\text{Cay}(A \times B, S)$ is the union of the zig-zag products $\text{Cay}(A, x_i^B) \circledast \text{Cay}(B, S_B)$, for $i = 1, \dots, k$.

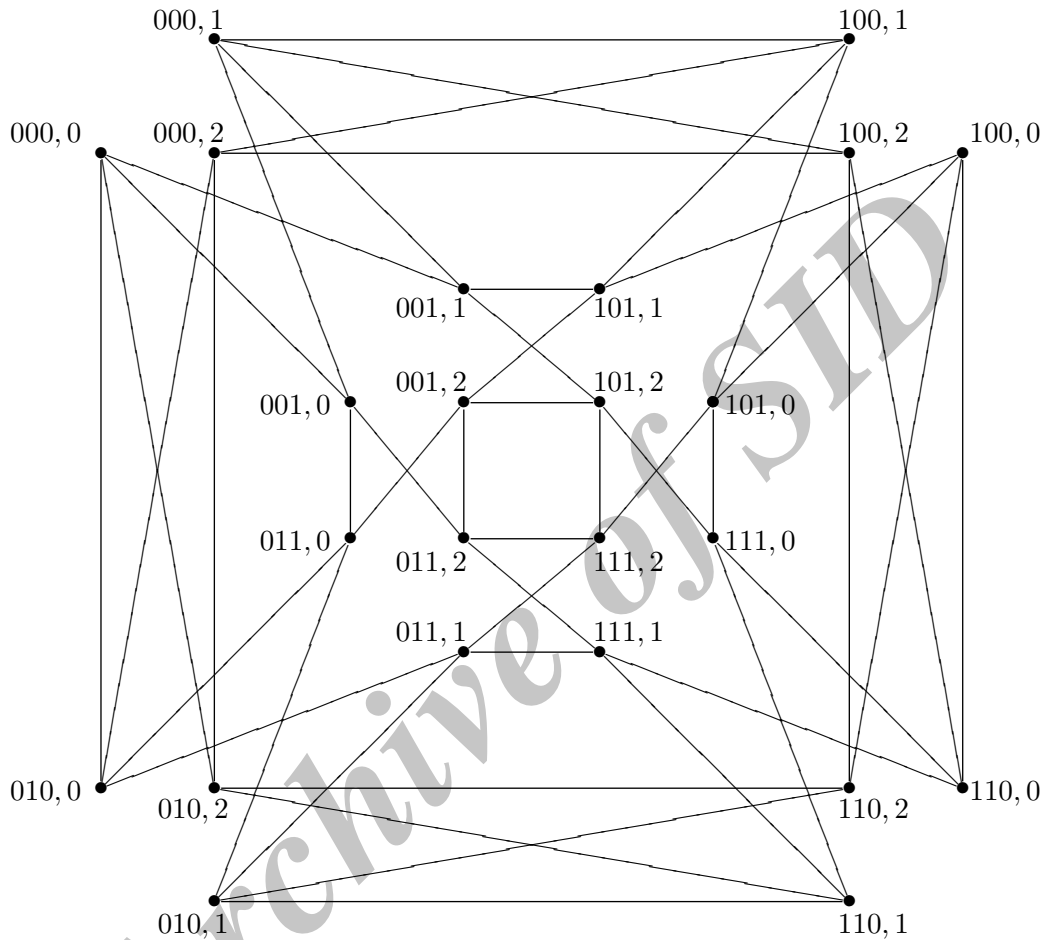
Proof. The proof is similar to the proof of Theorem 3.12. The fact that $\text{Cay}(A \times B, S)$ is obtained as the union of k zig-zag products of $\text{Cay}(A, x_i^B)$ and $\text{Cay}(B, S_B)$, with $i = 1, \dots, k$ easily follows from the definition. More precisely, an edge in $\text{Cay}(A \times B, S)$ labelled by $(1_A, s)(x_i^\varepsilon, 1_B)(1_A, s')$ leads from (\bar{a}, \bar{b}) to $(\bar{a}, \bar{b})(1_A, s)(x_i^\varepsilon, 1_B)(1_A, s') = (\bar{a}(x_i^\varepsilon)^{\bar{b}s}, \bar{b}s s')$. On the other hand, if we consider the zig-zag product, we have that the zig-step in B leads from (\bar{a}, \bar{b}) to $(\bar{a}, \bar{b}s)$, then the middle step goes from $(\bar{a}, \bar{b}s)$ to $(\bar{a}(x_i^\varepsilon)^{\bar{b}s}, \bar{b}s)$ and finally the zag-step moves from $(\bar{a}(x_i^\varepsilon)^{\bar{b}s}, \bar{b}s)$ to $(\bar{a}(x_i^\varepsilon)^{\bar{b}s}, \bar{b}s s')$. \square

Example 3.17. In Fig. 9, the zig-zag product of the 3-dimensional cube and the triangle of Fig. 7 is represented. It follows from Theorem 3.16 that this graph is isomorphic to the Cayley graph of the semidirect product $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3$, with respect to the generating set $\{(\mathbf{e}_1, 2), (\mathbf{e}_1, 0), (\mathbf{e}_2, 0), (\mathbf{e}_2, 1)\}$.

Observe that one edge in this graph is obtained as a sequence of three steps in the graph $\mathbb{Z}_2^3 \circledast \mathbb{Z}_3$ in

Fig. 8: the first step within some initial cloud, the second step jumping to a new cloud and finally a third step within the new cloud. For instance, the three steps $(001, 2) \rightarrow (001, 1) \rightarrow (011, 1) \rightarrow (011, 0)$ produce the edge connecting the vertices $(001, 2)$ and $(011, 0)$ in $\mathbb{Z}_2^3 \otimes \mathbb{Z}_3$.

FIGURE 9. The graph $Cay(\mathbb{Z}_2^3, \mathbf{E}_3) \otimes Cay(\mathbb{Z}_3, \{\pm 1\})$.



Example 3.18. Consider again the graphs in Example 3.3. The graph G_1 can be regarded as the Cayley graph of the Klein group $V_4 = \langle a, a' \mid a^2 = (a')^2 = 1_{V_4}, aa' = a'a \rangle$ with respect to the generating set $S_1 = \{a, a', a''\}$, with $a'' := aa'$, and the graph G_2 can be regarded as the Cayley graph of the cyclic group $C_3 = \langle b \mid b^3 = 1_{C_3} \rangle$ of order 3 with respect to the generating set $S_2 = \{b, b^2\}$ (see Fig. 10). Define an action of C_3 on V_4 by

$$1_{V_4}^b = 1_{V_4}, \quad a^b = a', \quad (a')^b = a'', \quad (a'')^b = a,$$

so that S_1 is a unique orbit under the action on B . Theorem 3.16 ensures that the graph $G_1 \otimes G_2$ is the Cayley graph of the semidirect product $V_4 \rtimes C_3$ with respect to the symmetric generating set $S = \{(a', 1_B), (a'', 1_B), (a', b^2), (a'', b)\}$ (see Fig. 11).

Remark 3.19. It is worth mentioning that in [3] the authors give a negative answer to the following conjecture formulated by Lubotzky and Weiss [25], asking whether expansion is a group property:

FIGURE 10. The graphs $Cay(V_4, S_1)$ and $Cay(C_3, S_2)$.

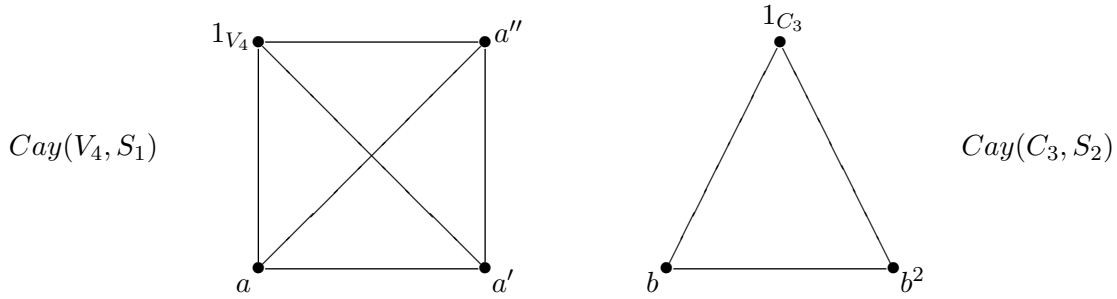
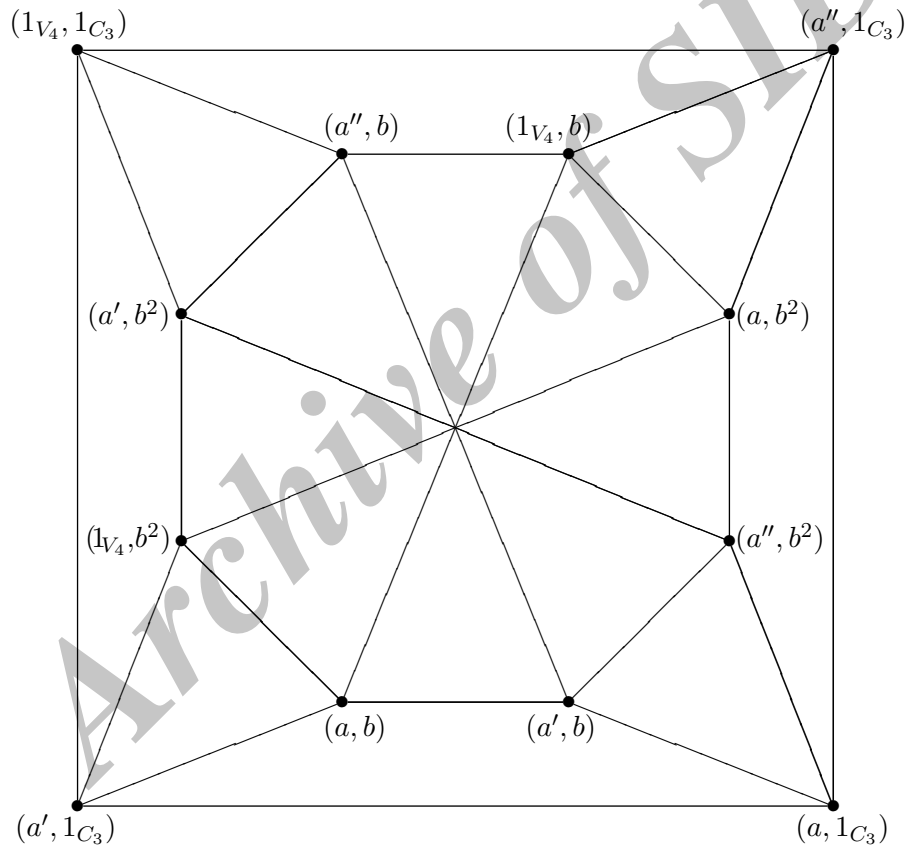


FIGURE 11. The graph $Cay(V_4, S_1) \otimes Cay(C_3, S_2)$.



“Fix an integer d . Let A_i be any family of finite groups, and for each i take any two symmetric sets S_i, \widehat{S}_i of generators of size at most d . Let $\lambda_i, \widehat{\lambda}_i$ be the second largest eigenvalues of the random walks on the Cayley graphs of A_i with generators S_i, \widehat{S}_i , respectively. Is it true that the sequence λ_i is uniformly bounded above by a fixed constant less than 1 if and only if the sequence $\widehat{\lambda}_i$ is?”

Remark 3.20. In [5, Subsection 3.2], the correspondence between zig-zag product of graphs and semidirect product of groups is used in the study of the Euclidean distortion of the cyclic lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}_n$.

4. The Lamplighter random walk

The Lamplighter random walk is a model which is well known in the literature, and several papers have been devoted to its analysis, mainly in the case where the underlying graph is the discrete line. Spectral computations for random walks on lamplighter groups and related graphs have been recently developed in the infinite setting in [6, 17, 35], and in the finite setting in [32, 33, 34].

4.1. Generalities. We restrict here our attention to the finite case. Let X and L be two finite connected graphs. An L -valued configuration on X is a function $\varphi : X \rightarrow L$. Observe that, by a small abuse of notation, we write $\varphi : X \rightarrow L$ meaning a function from the vertex set of X into the vertex set of L . We will denote by L^X the set of all L -valued configurations.

The *Lamplighter graph* $X \wr L$ is defined as the graph whose vertex set is $L^X \times X$, and where the adjacency relations are described by:

$$(\varphi_1, x_1) \sim (\varphi_2, x_2) \quad \text{if:}$$

- (1) either $x_1 \sim x_2$ in X and $\varphi_1(x) = \varphi_2(x)$ for all $x \in X$;
- (2) or $x_1 = x_2$, $\varphi_1(x) = \varphi_2(x)$ for all $x \neq x_1$ and $\varphi_1(x_1) \sim \varphi_2(x_1)$ in L .

It follows from the definition that the graph $X \wr L$ is connected, since both X and L are connected. Moreover, if X is a regular graph on n_X vertices with degree d_X and L is regular graph on n_L vertices with degree d_L , then the Lamplighter graph $X \wr L$ is a $(d_X + d_L)$ -regular graph on $n_X \cdot n_L^{n_X}$ vertices. It is called the Lamplighter graph since the following interpretation in terms of random walk on $X \wr L$ can be given: suppose that at each vertex of X there is a lamp, whose possible states (or colors) are represented by the vertices of L , so that an element $(\varphi, x) \in X \wr L$ represents the configuration of the n_X lamps (the lamp at the vertex $\bar{x} \in X$ is in the state $\varphi(\bar{x}) \in L$) together with the position x of a lamplighter on the graph X . In a single step, the lamplighter may either go to a neighbor of the current vertex $x \in X$ and leave all lamps unchanged (this situation corresponds to edges of type (1) in $X \wr L$), or he may stay at the vertex x of X , but he changes the state of the lamp which is in x to a neighbor state in L (this situation corresponds to edges of type (2) in $X \wr L$): this model is called the “*Walk or switch*” model.

Notice that there exists also a second model of Lamplighter random walk, called “*Switch-walk-switch*”: in this case, if the lamplighter stands at the vertex x and the actual configuration is φ , he first may change the state of the lamp at x , then he makes a step to some neighbor vertex $x' \in X$, and finally he may change the state of the lamp at x' .

4.2. Replacement product, zig-zag product and Lamplighter random walk. In this section we investigate connections between the replacement and zig-zag products of graphs, on the one hand, and the Lamplighter random walk, on the other hand.

Suppose that the graph X whose vertices are all the possible positions of the lamplighter is a d -regular graph on n vertices. Moreover, assume that at each vertex x of X there is a lamp which can

be in exactly 2 states, let us call them 0 and 1 (we can think that the lamp is “off” if it is in the state 0 and is “on” otherwise), so that the graph L of the states of the lamps reduces in this case to the graph $L_2 = Cay(\mathbb{Z}_2, \{1\})$ in Fig. 12. Now construct the n -dimensional cube H_n , which can be identified

FIGURE 12. The graph $L_2 = Cay(\mathbb{Z}_2, \{1\})$.



with the graph $Cay(\mathbb{Z}_2^n, \mathbf{E}_n)$, where two vertices u and v are adjacent if and only if they differ exactly in one coordinate. It follows that H_n is an n -regular graph over 2^n vertices. Observe that the vertices of H_n are functions from X to L_2 , so that each vertex of H_n represents a configuration of lamps over the vertices of X , i.e., the i -th coordinate of the vertex v is the state of the lamp which is at the i -th vertex of X . Moreover, two configurations are adjacent in H_n if and only if they differ exactly at one vertex.

Since the degree of H_n is equal to the number of vertices of X , we can construct the replacement product $H_n \textcircled{R} X$, which is a $(d + 1)$ -regular graph over $n \cdot 2^n$ vertices. Therefore, a vertex of $H_n \textcircled{R} X$ is a pair (v, i) , where i represents the position of the lamplighter, and v represents a configuration of the lamps.

Theorem 4.1. *Let X be a connected d -regular graph on n vertices and let H_n be the n -dimensional cube. The replacement product $H_n \textcircled{R} X$ coincides with the Lamplighter graph $X \wr L_2$, where L_2 is the graph in Fig. 12. In particular, the simple random walk on $H_n \textcircled{R} X$ is the “Walk or switch” Lamplighter random walk on X .*

Proof. The proof follows from the definition of the replacement construction. To see this, we identify the vertices of X with the set $[n]$. As usual, we assume that, for each vertex $i \in [n]$, the edges incident to i are labelled by a color $h \in [d]$ near i , and that any two distinct edges issuing from i have a different color. Moreover, we label the edges of H_n as follows: if an edge connects two n -tuples u and v of \mathbb{Z}_2^n which differ in the i -th coordinate, then we label it by i both near u and v .

Consider now a vertex $(v, i) \in H_n \textcircled{R} X$, so that v is a n -tuple over the alphabet $\{0, 1\}$ and i is a vertex of X . This vertex will have degree $d + 1$ in $H_n \textcircled{R} X$. By construction, d edges emanating from (v, i) connect it with the vertices $(v, i_1), \dots, (v, i_d)$, where i_1, \dots, i_d are the neighbors of i in X . These connections correspond to the “walk” case, i.e., the lamplighter walks from a vertex i to a neighbor vertex i' , without changing the state of any lamp. The remaining $(d + 1)$ -st edge issuing from (v, i) connects (v, i) with (v', j) , where v and v' are adjacent vertices in H_n , joined by an edge labelled by i near v and by j near v' . It follows from our construction that $i = j$ and $v' = v + \mathbf{e}_i$. As a consequence, moving from (v, i) to (v', i) corresponds to the “switch” case, i.e., the lamplighter does not move, but he changes the state of the lamp at the vertex i where he stands. □

The replacement product in Example 3.14 corresponds to the “Walk or switch” model of the Lamplighter random walk on the triangle.

We introduce now a new model of Lamplighter random walk on X , that we call the “Walk-switch-walk” model: the lamplighter starts from a vertex $x \in X$, then he moves to a vertex x' neighbor of x in X , he changes the state of the lamp at x' , and finally takes a second a step to a vertex x'' which is a neighbor of x' in X , so that it may be $x = x''$ or $x \neq x''$.

Theorem 4.2. *Let X and H_n be as in Theorem 4.1. The simple random walk on the zig-zag product $H_n \circledast X$ is the “Walk-switch-walk” Lamplighter random walk model on X .*

Proof. The proof follows from the definition of the zig-zag construction. We have already said that an edge in $H_n \circledast X$ is produced by three consecutive steps of type zig (within a first cloud), jump from this cloud to a second cloud, zag (within this second cloud). It is clear that the zig and the zag steps correspond to the “walk” phases, i.e., the lamplighter moves from x to x' and then from x' to x'' , whereas the jump step corresponds to the “switch” phase, i.e., the lamplighter does not move but changes the state of the lamp at the vertex x' where he arrived after the first step. \square

The zig-zag product in Example 3.17 corresponds to the “Walk-switch-walk” model of the Lamplighter random walk on the triangle.

4.3. Wreath product of graphs and groups. In this section we investigate connections between the Lamplighter random walk, wreath product of graphs and wreath product of groups.

First of all, we want to remark that the notation $X \wr L$ is used to denote the Lamplighter graph, since this construction coincides with the classical notion of *wreath product* of the graphs X and L , that we recall in Definition 4.3 (see, for instance, [15, 16], where two different generalizations of such a product are introduced).

Definition 4.3. *Let A and B be two finite graphs. The wreath product $A \wr B$ is the graph with vertex set $B^A \times A = \{(f, a) | f : A \rightarrow B, a \in A\}$, where two vertices (f_1, a_1) and (f_2, a_2) are connected by an edge if:*

- (1) *(edges of the first type) either $a_1 = a_2 = a$ and $f_1(x) = f_2(x)$ for every $x \neq a$, and $f_1(a) \sim f_2(a)$ in B ;*
- (2) *(edges of the second type) or $f_1(a) = f_2(a)$, for every $a \in A$, and $a_1 \sim a_2$ in A .*

On the other hand, the Lamplighter graph represents a graph-analogue of the well known *wreath product of groups*, recalled in the following definition.

Definition 4.4. *Let A and B be two finite groups. The set $B^A = \{f : A \rightarrow B\}$ can be endowed with a group structure with respect to the pointwise multiplication: $(f_1 f_2)(a) = f_1(a) f_2(a)$. The wreath product $A \wr B$ is the semidirect product $B^A \rtimes A$, where A acts on B^A by shifts, i.e., if $f \in B^A$, one has*

$$f^a(x) = f(a^{-1}x), \quad \text{for all } a, x \in A.$$

It is known (see, for instance, [15]) that the wreath product of the Cayley graphs of two groups gives the Cayley graph of the wreath product of the group, with a suitable choice of the generating sets.

Finally, in order to better understand the relation between wreath product, replacement product and other product constructions involving graphs (Remark 4.6), we recall the definition of *Cartesian product* of graphs (see, for instance, [31]) and of *composition*, or *lexicographic product*, introduced in [20]. See also [30] or [13], where this product is called *wreath product* of graphs. We will refer to that construction as the lexicographic product, in order to avoid confusion with the wreath product described in Definition 4.3.

Definition 4.5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs.

- The Cartesian product $G_1 \square G_2$ is the graph with vertex set $V_1 \times V_2$, where two vertices (v_1, v_2) and (w_1, w_2) are adjacent if and only if:
 - (1) either $v_1 = w_1$ and $v_2 \sim w_2$ in G_2 ;
 - (2) or $v_2 = w_2$ and $v_1 \sim w_1$ in G_1 .
- The lexicographic product $G_1 \circ G_2$ is the graph with vertex set $V_1 \times V_2$, where two vertices (v_1, v_2) and (w_1, w_2) are adjacent if and only if:
 - (1) $v_1 \sim w_1$ in G_1 ;
 - (2) $v_1 = w_1$ and $v_2 \sim w_2$ in G_2 .

Remark 4.6. Let A and B be two finite graphs. The wreath product $A \wr B$ of Definition 4.3 is isomorphic to a subgraph of the lexicographic product $A \circ B^A$, if we define the graph B^A as the graph whose vertex set is given by $B^A = \{f : A \rightarrow B\}$ and where two functions f and g are adjacent if there exists $a \in A$ such that $f \equiv g$ in $A \setminus \{a\}$. More precisely, edges of the first type in Definition 4.3 constitute a subset of the edge set of type (2) in the definition of the lexicographic product since, in the lexicographic case, the functions are allowed to differ in any vertex. On the other hand, the edges of the second type are a subset of the edge set of type (1) since, in the lexicographic case, there are no conditions on the functions f and g .

Note also that the replacement product $A \textcircled{R} B$ is isomorphic to a subgraph of $A \circ B$: more precisely, edges of type (2) in the definition of the lexicographic product correspond exactly to edges joining vertices within a single cloud in $A \textcircled{R} B$, whereas edges connecting two different clouds constitute a proper subset of the edge set of type (1).

Finally, observe that the Cartesian and the lexicographic products of Definition 4.5 can be regarded as particular cases of more general constructions introduced and studied in [9, 10, 11] in the setting of finite Markov chains.

Acknowledgments

I would like to express my deepest gratitude to Fabio Scarabotti and Tullio Ceccherini-Silberstein for enlightening discussions and for their continuous encouragement. Part of this work was developed during my stay at the Technische Universität of Graz, and I want to thank Wolfgang Woess and Franz Lehner for several useful discussions. This research was partially supported by the European Science Foundation (Research Project RGLIS 4915).

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