



COCHARACTERS OF UPPER TRIANGULAR MATRICES

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ABSTRACT. We survey some recent results on cocharacters of upper triangular matrices. In particular, we deal both with ordinary and graded cocharacter sequence; we list the principal combinatorial results; we show different techniques in order to solve similar problems.

1. Introduction

The theory of algebras with polynomial identities (or PI-algebras) has been largely investigated since the previous century. Moreover, we have that the set of polynomial identities of any associative algebra A , i.e. $T(A)$, is a two sided ideal of $F\langle X \rangle$, called the T-ideal of A . It is well known that if F is a field of characteristic 0, all the polynomial identities of an algebra A come from the multilinear ones. If we set V_n be the set of polynomials that are linear in the variables x_1, \dots, x_n , (multilinear), we can form for any $n \in \mathbb{N}$, the factor space

$$V_n(A) = V_n / (V_n \cap T(A)).$$

We call n -th codimension of A the dimension of $V_n(A)$ and we denote it by the symbol $c_n(A)$. Notice that $V_n(A)$ is an S_n -module so it affords a character called n -th cocharacter of A and we shall denote it by $\chi_n(A)$. Actually is more efficient to study $V_n(A)$ than $V_n \cap T(A)$. In fact, as a vector space, $V_n \cap T(A)$ grows factorially; on the other hand in [43], Regev, proved that if A is a PI-algebra, its codimension sequence is exponentially bounded.

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Suppose A is a PI-algebra, then Giambruno and Zaicev proved that the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}$$

always exists and is a non-negative integer called the PI-exponent of A or, in symbol, $\exp(A)$. If we use the language of varieties, we say that the variety generated by the algebra A is the class

$$\mathcal{V} = \mathcal{V}(A) = \{B \text{ associative algebra} \mid T(A) \subseteq T(B)\}.$$

We say that a variety of algebra \mathcal{V} is minimal with respect to its exponent if and only if for any proper subvariety \mathcal{U} of \mathcal{V} one has that $\exp(\mathcal{U}) < \exp(\mathcal{V})$. We say that a PI-algebra is minimal if it generates a minimal variety. If S is any commutative ring with 1, we denote by $UT_n(S)$ the set of upper triangular matrices with entries from S . The T-ideals of the algebras $UT_n(F)$ and $UT_n(E)$ have an interesting property: they are examples of maximal T-ideals of a given exponent of the codimension sequences (and the corresponding varieties of algebras are minimal varieties of this exponent).

In [23] and [24] Genov and in [33] Latyshev, they proved that every algebra satisfying the identities of $UT_n(F)$ has a finite basis of its polynomial identities. Later this result was generalized by Latyshev [32] and Popov [53] for PI-algebras satisfying the identity

$$[x_1, x_2, x_3] \cdots [x_{3n-2}, x_{3n-1}, x_{3n}]$$

which generates the T-ideal $T(UT_n(E)) = T(E)^n$ of the algebra $UT_n(E)$ of upper triangular matrices with entries from the infinite dimensional Grassmann algebra E . A famous theorem by Kemer says that if A is a PI-algebra over F , its T-ideal is finitely generated. For a long time, until Kemer developed his structure theory, the theorems of Genov, Latyshev and Popov covered all known examples of classes of PI-algebras with the finite basis property. By the way, the explicit set of generators for the T-ideal is well known only for a small number of algebras like the 2×2 matrix algebra $M_2(F)$ (Razmyzlov [40], Drensky [17]), the infinite dimensional Grassmann algebra E (Krakovsky and Regev [31]), the tensor square of the infinite dimensional Grassmann algebra $E \otimes E$ (Popov [39]), $UT_n(F)$ and $UT_n(E)$.

The explicit form of the multiplicities of $\chi_n(A)$ is also known for few algebras only, among them the Grassmann algebra E (Olsson and Regev [38]), the 2×2 matrix algebra $M_2(F)$ (Formanek [22], Drensky [14]), the algebra $UT_2(F)$ of the 2×2 upper triangular matrices (Mishchenko, Regev and Zaicev [37]), the tensor square $E \otimes E$ of the Grassmann algebra (Popov [39], Carini and Di Vincenzo [5]).

We consider now the set of indeterminates $X_k := \{x_1, \dots, x_k\}$. The algebra

$$F_k(A) := F\langle X_k \rangle / (F\langle X_k \rangle \cap T(A))$$

is called the relatively free algebra of rank k in the variety of algebras generated by the algebra A . The Hilbert series $H(F_k(A), T_k(A))$ of $F_k(A)$ in the indeterminates $T_k = \{t_1, \dots, t_k\}$ is a symmetric function and looks like

$$\sum_{\lambda} m_{\lambda}(A) S_{\lambda}(T_k), \quad \lambda = (\lambda_1, \dots, \lambda_k).$$

By a result of Berele [2] and Drensky [18, 17], the multiplicities $m_\lambda(A)$ are the same as in the cocharacter sequence $\{\chi_n(A)\}_{n \in \mathbb{N}}$. Hence, in principle, if we know the Hilbert series $H(F_k(A), T_k(A))$, we can find the multiplicities $m_\lambda(A)$ in $\chi_n(A)$ for those λ which are partitions in not more than k parts. When A is a finite dimensional algebra, the multiplicities $m_\lambda(A)$ are equal to zero for partitions $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_k = 0$, for $k > \dim(A)$, see [42]. Hence all $m_\lambda(A)$ can be recovered from $H(F_k(A), T_k)$ for k sufficiently large.

As we said above, the explicit knowledge of the generators of the T-ideals of PI-algebras is given for few algebras. In order to understand better the T-ideal of algebras, a useful tool consists in the study of some “weaker” polynomial identities. In fact, suppose that A is a G -graded algebra, where G is any group. Consider the disjoint union $X = \bigcup_{g \in G} X^g$, where for any $g \in G$, X^g is a countable set of indeterminates, then consider the free algebra $F\langle X \rangle$ generated by X . Similarly to the ordinary case, we can define the G -graded polynomial identities, the T_G -ideal, the G -graded codimension and cocharacter sequences, the G PI-exponent of a G -graded PI-algebra A . Under opportune hypothesis, some of the theorems for ordinary PI-algebras can be generalized for the graded case.

In this paper we survey some results about the cocharacter sequence of upper triangular matrices $UT_n(S)$. We recall that these objects generate minimal varieties in the case $S = F$ or $S = E$ and, from the PI-point of view, these are very interesting cases. The survey is divided into two parts. The first one (Sections 1,2,3,4 and 5) deals with ordinary cocharacter sequences and the second one (Sections 6,7 and 8) deals with graded cocharacter sequence. More precisely, we introduce the basic terminology and tools of the theory of PI-algebras, such as the cocharacter and codimension sequence. Then we give a small account about the representation theory of the symmetric group. In the third section we list the cocharacter sequence of some concrete PI-algebras. In the fourth section we start to investigate methods and techniques used for the computation of cocharacter sequences of upper triangular matrices. In particular, this section is entirely devoted to the work of Boumova and Drensky (see [4]). Here we draw up definitions and methods that brought the authors to their main result, i.e., an algorithm for the computation of cocharacters for $UT_n(F)$. In Section 5 we show some results of the author, which determine the exact Hilbert series for the algebra of 2×2 upper triangular matrices with entries from E and for its subalgebra \mathcal{L} that is one of the algebras that generate minimal varieties with exponent strictly larger than 2. As a consequence, we are able to determine the cocharacter sequence of those algebras.

The second part of the survey is dedicated to the graded sequence of cocharacters. Section 6 is introductory and may serve as a “background” for the results of Section 7, where we summarize the works of Di Vincenzo, Koshlukov and Valenti (see [13]) and Koshlukov and Valenti (see [30]) about the Y-proper graded identities for $UT_n(F)$ and we give a combinatorial tool for the computation of Y-proper graded cocharacters in the case $UT_n(F)$ is endowed with the grading of Vasilovsky (see [46]). The last section deals with the computation of \mathbb{Z}_2 -graded cocharacters of $UT_2(E)$. Notice that $UT_2(E)$ is isomorphic to $UT_2(S) \otimes E$, while in [10] Di Vincenzo and Nardoza described the $G \times \mathbb{Z}_2$ -graded cocharacters of $A \otimes E$ in terms of the G -graded cocharacters of A . The result of the last section are to be seen in light of the result of Di Vincenzo and Nardoza.

2. PI-algebras and related structures

All the fields we refer to are of characteristic 0 and all algebras are associative with unit.

Definition 2.1. Let F be a field and A be an F -algebra, and let $X = \{x_1, x_2, \dots\}$ be a countable set of variables. We denote by $f(x_1, \dots, x_n)$ the free associative algebra freely generated by X . We shall call polynomials the elements of $F\langle X \rangle$. We say that a polynomial $f(x_1, \dots, x_n) \in F\langle X \rangle$ is a polynomial identity for A (or A satisfies $f(x_1, \dots, x_n)$) if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. Moreover we say that A is a polynomial identity algebra (PI-algebra) if A satisfies a non trivial polynomial. We denote by $T(A)$ the set of all polynomial identities for A .

It is well known that $T(A)$ is a two-sided ideal also called T -ideal of A because it is invariant under all endomorphisms of $F\langle X \rangle$. For any $n \in \mathbb{N}$, we consider the vector space

$$V_n := \text{span}_F \langle x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} \mid \sigma \in S_n, x_i \in X \rangle.$$

The latter is called the space of multilinear polynomials of degree n . Since the characteristic of the ground field F is zero, a standard process of multilinearization shows that $T(A)$ is generated, as a T -ideal, by the subspaces $V_n \cap T(A)$. Actually, it is more efficient to study the factor space

$$V_n(A) := V_n / (V_n \cap T(A))$$

in fact, although $V_n \cap T(A)$ is huge as n goes to infinity, a result of Regev (see [43]) establishes that if A is a PI-algebra, $V_n(A)$ grows at most exponentially. An effective tool to the study of $V_n(A)$ is provided by the representation theory of the symmetric group. Indeed, one can notice that V_n is an S_n -module with respect to the natural left action (from now in advance, all S_n -modules are left), and $V_n \cap T(A)$ is an S_n -submodule too, hence the factor space $V_n(A)$ is also an S_n -module.

Definition 2.2. Let A be a PI-algebra and $n \in \mathbb{N}$, then we call n -th cocharacter of A , the character of the S_n -module $V_n(A)$ and we denote it by $\chi_n(A)$. We shall call n -th codimension of A , the dimension of the F -vector space $V_n(A)$ and we denote it by $c_n(A)$. Moreover, we say:

$(\chi_n(A))_{n \in \mathbb{N}}$ is the cocharacter sequence of A ;

$(c_n(A))_{n \in \mathbb{N}}$ is the codimension sequence of A .

The commutator of $a, b \in A$ is the Lie product $[a, b] := ab - ba$. One defines inductively higher (left-normed) commutators by setting

$$[a_1, \dots, a_n] := [[a_1, \dots, a_{n-1}], a_n],$$

for any $n \geq 2$. We denote by $B(X)$ the unitary subalgebra of $F\langle X \rangle$ generated by commutators, called the algebra of proper polynomials. Since our algebras are unitary, then $B(X) \cap T(A)$ generates the whole $T(A)$ as a T -ideal. We shall denote by $T^B(A)$ the set of proper polynomial identities of A and

by Γ_n the set of multilinear polynomials of V_n which are proper. It is not difficult to see that Γ_n is an S_n -submodule of V_n and the same holds for $\Gamma_n \cap T(A)$. Hence the factor module

$$\Gamma_n(A) = \Gamma_n / (\Gamma_n \cap T(A))$$

is an S_n -submodule of $V_n(A)$.

Definition 2.3. Let A be a PI-algebra and $n \in \mathbb{N}$, then we call n -th proper cocharacter of A , the character of the S_n -module $\Gamma_n(A)$ and we denote it by $\xi_n(A)$. We shall call n -th proper codimension of A , the dimension of the F -vector space $\Gamma_n(A)$ and we denote it by $\gamma_n(A)$.

2.1. Minimal varieties. We draw the principal results of the deep theory of Kemer and we point out the relation between verbally prime algebras and minimal varieties. We start off with the following definition:

Definition 2.4. The T -ideal S of $F\langle X \rangle$ is called T -semiprime or verbally semiprime if any T -ideal U such that $U^k \subseteq S$ for some k , lies in S , i.e. $U \subseteq S$. The T -ideal P is T -prime or verbally prime if the inclusion $U_1 U_2 \subseteq P$ for some T -ideals U_1 and U_2 implies $U_1 \subseteq P$ or $U_2 \subseteq P$.

The Grassmann algebra E of an infinite dimensional vector space with basis $\{e_1, e_2, \dots\}$ has a natural \mathbb{Z}_2 -grading $E = E^{(0)} \oplus E^{(1)}$, where

$$E^{(0)} := \text{span}\{1, e_{i_1} \cdots e_{i_{2k}} \mid 1 \leq i_1 < \cdots < i_{2k}, k \geq 0\},$$

$$E^{(1)} := \text{span}\{1, e_{i_1} \cdots e_{i_{2k+1}} \mid 1 \leq i_1 < \cdots < i_{2k+1}, k \geq 0\}.$$

Let p, q , where $p \geq q$, be positive integers and let $M_{p \times q}(E^{(1)})$ be the vector space of all $p \times q$ matrices with entries from $E^{(1)}$. The vector subspace of $M_{p+q}(E)$, where $M_n(E)$ is the $n \times n$ matrix algebra with entries from the Grassmann algebra,

$$M_{p,q}(E) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in M_p(E^{(0)}), b \in M_{p \times q}(E^{(1)}), c \in M_{q \times p}(E^{(1)}), d \in M_q(E^{(0)}) \right\}$$

is an algebra. The building blocks in the theory of Kemer are the polynomial identities of the matrix algebras, the Grassmann algebra and the algebras $M_{p,q}(E)$. In fact, we have the following theorem:

Theorem 2.5. (1) For every T -ideal U of $F\langle X \rangle$ there exist a T -semiprime T -ideal S and a positive integer k such that

$$S^k \subseteq U \subseteq S.$$

(2) Every T -semiprime T -ideal S is an intersection of a finite number of T -prime T -ideals Q_1, \dots, Q_m ,

$$S = Q_1 \cap \cdots \cap Q_m.$$

(3) A T -ideal P is T -prime if and only if P coincides with one of the following T -ideals:

$$T(M_n(F)), T(M_n(E)), T(M_{p,q}(E)), (0), F\langle X \rangle.$$

We introduce now the notion of variety of algebras.

Definition 2.6. Given a non-empty set $S \subseteq F\langle X \rangle$, the class of all algebras A such that A satisfies f for all $f \in S$ is called the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S .

A variety is called non-trivial if $S \neq \emptyset$ and \mathcal{V} is proper if it is non-trivial and contains a non-zero algebra. For example, the class of all commutative algebras forms a proper variety with $S = \{[x, y]\}$. Notice that if \mathcal{V} is the variety determined by the set S and $\langle S \rangle_T$ is the T -ideal of $F\langle X \rangle$ generated by S , then $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle_T)$ and $\langle S \rangle_T = \bigcap_{A \in \mathcal{V}} T(A)$. Let us write $\langle S \rangle_T = T(\mathcal{V})$. Thus to each variety corresponds a T -ideal of $F\langle X \rangle$; the converse is also true (see [25], Theorem 1.2.5).

The work of Giambruno and Zaicev [26] has contributed to clarify why the notion of PI-exponent is crucial for a classification of the T -ideals in terms of growth of the sequence of their codimension sequence. In [29], [28] Giambruno and Zaicev proved that:

Theorem 2.7. If A is any PI-algebra, then there exists the limit

$$\exp(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)},$$

and it is a non-negative integer.

This limit is called the PI-exponent of A , or equivalently the PI-exponent of the variety generated by A . We shall use the symbol $\exp(A)$ to indicate the PI-exponent of A . Analogously, if \mathcal{V} is the variety associated to a certain T -ideal S , we shall use $\exp(\mathcal{V})$ to indicate the PI-exponent of the variety generated by S .

Definition 2.8. A variety \mathcal{V} (or the corresponding T -ideal $I = T(\mathcal{V})$) is minimal of exponent d if $\exp(\mathcal{V}) = d$ and $\exp(\mathcal{U}) < d$ for all proper subvarieties \mathcal{U} of \mathcal{V} .

We should remark that the extremal varieties introduced in [16] are the minimal varieties in our definition. Drensky also conjectured that:

Conjecture 2.9. A variety is minimal if and only if its T -ideal of identities is a product of verbally prime T -ideals, i.e., of T -ideals of the verbally prime algebras.

Partial results were obtained by Drensky in [15] and [16] and by Stoyanova-Venkova in [45]. In [26], Giambruno and Zaicev solved into affirmative the conjecture of Drensky showing how important is the study of minimal varieties. In particular, we have:

Theorem 2.10. Let \mathcal{V} be a variety of algebras over a field F of characteristic zero such that $\exp(\mathcal{V}) \geq 2$. Then the following properties are equivalent:

- (1) \mathcal{V} is a minimal variety of exponent d .
- (2) $T(\mathcal{V})$ is a product of verbally prime T -ideals
- (3) $\mathcal{V} = \mathcal{V}(G(A))$, for some minimal superalgebra A s.t. $\dim_F A_{ss} = d$,

where A_{ss} is the maximal semisimple subalgebra of A and $G(A)$ is the Grassmann envelope of A .

3. Representation of the symmetric group and cocharacters

3.1. Cocharacters of concrete algebras. We shall describe some applications of the representation theory of the symmetric and the general linear group to the study of PI-algebras. We consider the following definition:

We recall that a partition of the non-negative integer n is a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_r)$ such that

$$\lambda_1 \geq \dots \geq \lambda_r > 0 \text{ and } \lambda_1 + \dots + \lambda_r = n.$$

In this case we shall write

$$\lambda \vdash n.$$

We assume that two partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ are equal if $r = s$ and

$$\lambda_1 = \mu_1, \dots, \lambda_r = \mu_r.$$

When $\lambda = (\lambda_1, \dots, \lambda_{k_1+\dots+k_p})$ and

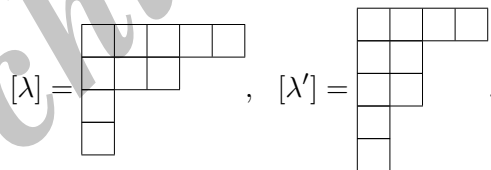
$$\lambda_1 = \dots = \lambda_{k_1} = \mu_1, \dots, \lambda_{k_1+\dots+k_{p-1}+1} = \dots = \lambda_{k_1+\dots+k_p} = \mu_p,$$

we accept the notation

$$\lambda = (\mu_1^{k_1}, \dots, \mu_p^{k_p}).$$

Definition 3.1. Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, we associate to λ the skew tableau $[\lambda]$ having r rows and the i -th row contains λ_i squares. We call $[\lambda]$ the Young diagram of λ .

Moreover we shall indicate by the symbol λ' the conjugate partition of λ , i.e., the partition obtained by $[\lambda]$ transposing its rows as in the figure below:



By the Theorem of Maschke, if G is a finite group, every finite dimensional representation is completely reducible, i.e., the group algebra FG is semisimple and isomorphic to the direct sum of matrix algebras with entries from division algebras. Moreover, every finite dimensional left G -module is a direct sum of irreducible G -modules that are isomorphic to a minimal left ideal of FG . If $G = S_n$, the symmetric group of order n , the left irreducible S_n -modules (and their related characters) may be described in terms of partitions and Young diagrams. In fact, we have the following result:

Theorem 3.2. Let F be a field of characteristic zero and $n \geq 1$. Then there is a one-to-one correspondence between irreducible S_n -characters and partitions of n . Let $\{\chi_\lambda | \lambda \vdash n\}$ be a complete set of irreducible characters of S_n and let $d_\lambda = \chi_\lambda(1)$ be the degree of χ_λ , $\lambda \vdash n$. Then

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda,$$

where $I_\lambda = e_\lambda FS_n$ and $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ is up to a scalar, the unit element of I_λ .

In light of the previous result, if we decompose the n -th cocharacter of any PI-algebra A into irreducible, we obtain:

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda.$$

The explicit form of the multiplicities m_λ of $\chi_n(A)$ is known for few algebras only, among them the Grassmann algebra E (Olsson and Regev [38]), the 2×2 matrix algebra $M_2(F)$ (Formanek [22], Drensky [17]), the algebra $UT_2(F)$ of the 2×2 upper triangular matrices (Mishchenko, Regev and Zaicev [37]), the tensor square $E \otimes E$ of the Grassmann algebra (Popov [39], Carini and Di Vincenzo [5]). We list the explicit form of the cocharacters for the algebras above.

Theorem 3.3. *For any $n \geq 1$, we have:*

$$\chi_n(F) = \chi_{(n)}.$$

Theorem 3.4. *For any $n \geq 0$, we have:*

$$\chi_n(UT_2(F)) = \sum_{i=1}^3 m_{\lambda^{(i)}} S_{\lambda^{(i)}},$$

where

$$\lambda^{(1)} = (n), \lambda^{(2)} = (k_1, k_2), \lambda^{(3)} = (k_1, k_2, 1)$$

and

- (1) $m_{\lambda^{(1)}} = 1$,
- (2) $m_{\lambda^{(2)}} = k_1 - k_2 + 1$,
- (3) $m_{\lambda^{(3)}} = k_1 - k_2 + 1$.

Theorem 3.5. *Let E be the infinite dimensional Grassmann algebra. Then for any $n \geq 1$, we have:*

$$\chi_n(E) = \sum_{k=1}^n \chi_{(k, 1^{n-k})}.$$

Theorem 3.6. *For any $n \geq 0$, we have:*

$$\chi_n(E \otimes E) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where

- (1) $m_{(n)} = 1$;
- (2) $m_{(\lambda_1, 1)} = \lambda_1 - 1$;
- (3) $m_{(\lambda_1, 1^l)} = 2\lambda_1 - 1$ if $l \geq 2$;
- (4) $m_{(\lambda_1, \lambda_2, 2^k, 1^l)} = 4(\lambda_1 - \lambda_2 + 1)$ if $\lambda_1 \geq \lambda_2 \geq 2$ and $l \geq 1, k \geq 0$;
- (5) $m_{(\lambda_1, \lambda_2, 2^k)} = 3(\lambda_1 - \lambda_2 + 1)$ if $\lambda_1 \geq \lambda_2 \geq 2$ and $k \geq 1$;
- (6) $m_{(\lambda_1, \lambda_2)} = 2(\lambda_1 - \lambda_2 + 1)$ if $\lambda_1 \geq \lambda_2 \geq 2$.

Theorem 3.7. For any $n \geq 0$, we have:

$$\chi_n(M_2(F)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and

- (1) $m_{(n)} = 1$;
- (2) $m_{(\lambda_1, \lambda_2)} = (\lambda_1 - \lambda_2 + 1)\lambda_2$, if $\lambda_2 > 0$;
- (3) $m_{(\lambda_1, 1, 1, \lambda_4)} = \lambda_1(2 - \lambda_4) - 1$;
- (4) $m_\lambda = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)$ for all other partitions.

The following result of Drensky relates the ordinary cocharacters of a PI-algebra A with the proper cocharacters of A (Drensky, [14], Theorem 12.5.4):

Theorem 3.8. Let A be a PI-algebra and $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A) \chi_\lambda$ its n -th cocharacter. Let $\xi_p(A) = \sum_{\nu \vdash p} k_\nu(A) \chi_\nu$ its p -th proper cocharacter, then

$$m_\lambda(A) = \sum_{\nu \in S} k_\nu(A),$$

where $S = \{\nu = (\nu_1, \dots, \nu_n) \mid \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \dots \geq \lambda_n \geq \nu_n\}$.

The next proposition relates the ordinary codimension sequence of a PI-algebra A with the proper codimension sequence of A (Drensky, [14], Theorem 4.3.12) :

Theorem 3.9. Let A be a PI-algebra and $c_n(A)$ its n -th codimension. Let $\gamma_p(A)$ its p -th proper codimension, then

$$c_n(A) = \sum_{p=0}^n \binom{n}{p} \gamma_p(A).$$

3.2. Hilbert series of PI-algebras. We point out the fact that Hilbert series of PI-algebras and the sequence of cocharacters of PI-algebras are strictly related. We also survey some known results about Hilbert series in general. We recall that a F -algebra A is \mathbb{N}^m -graded if

$$A = \bigoplus_{(n_1, \dots, n_m) \in \mathbb{N}^m} A^{(n_1, \dots, n_m)},$$

and

$$A^{(n_1, \dots, n_m)} \cdot A^{(n_1, \dots, n_m)} \subseteq A^{(n_1, \dots, n_m)}.$$

Definition 3.10. Let $A = \sum_{n \in \mathbb{N}} A^{(n_1, \dots, n_m)}$ be a \mathbb{N}^m -graded algebra and suppose that $\dim_F A^{(n_1, \dots, n_m)} < \infty$. The formal power series

$$H(A, t_1, \dots, t_m) = \sum \dim_F A^{(n_1, \dots, n_m)} t_1^{n_1} \dots t_m^{n_m}$$

is called the Hilbert series of A in the variables t_1, \dots, t_m .

If A is a PI-algebra we consider the Hilbert series of a particular quotient space:

Definition 3.11. Let A be a PI-algebra over F . It is well known that $T(A)$ is a multihomogeneous ideal of $F\langle X \rangle$. Then, if $\bar{t} := (t_1, \dots, t_m)$, we denote by

$$H(A, \bar{t}) := H(F\langle x_1, x_2, \dots, x_m \rangle / (F\langle x_1, x_2, \dots, x_m \rangle \cap T(A)), \bar{t})$$

the Hilbert series of the relatively free algebra in m variables and we call $H(A, \bar{t})$ the Hilbert series of A .

Hilbert series is related with usual operations between graded vector spaces. In fact:

Proposition 3.12. Let A, B be \mathbb{N}^m -graded algebras and U be a \mathbb{N}^m -graded ideal of A . Then

- $H(A \oplus B, t_1, \dots, t_m) = H(A, t_1, \dots, t_m) + H(B, t_1, \dots, t_m)$
- $H(A \otimes B, t_1, \dots, t_m) = H(A, t_1, \dots, t_m) \cdot H(B, t_1, \dots, t_m)$
- $H(A, t_1, \dots, t_m) = H(A/U, t_1, \dots, t_m) + H(U, t_1, \dots, t_m)$.

Formanek in [21] gave a formula for the Hilbert series of the product of two T -ideals as a function of the Hilbert series of the factors. In fact, we have the following theorem:

Theorem 3.13. Let U and V be multihomogeneous ideals of the free algebra $F\langle x_1, \dots, x_d \rangle$. Then the Hilbert series of UV , U and V are related by the equation

$$H(U, t_1, \dots, t_d)H(V, t_1, \dots, t_d) = H(UV, t_1, \dots, t_d)H(F\langle x_1, \dots, x_d \rangle, t_1, \dots, t_d).$$

Corollary 3.14. Let A, B and C be PI-algebras over an infinite field F such that $T(A) = T(B)T(C)$. Then the Hilbert series of the relatively free algebras of A, B and C satisfy the equation

$$H(F_d(A), t_1, \dots, t_d) = H(F_d(B), t_1, \dots, t_d) + H(F_d(C), t_1, \dots, t_d) \\ + (t_1 + \dots + t_d - 1)H(F_d(B), t_1, \dots, t_d)H(F_d(C), t_1, \dots, t_d).$$

Notice that PI-algebras A having the property $T(A) = T(B)T(C)$ for some PI-algebras B, C are called algebras with factorable T -ideals. For instance, $UT_n(F)$ and $UT_n(E)$ are algebras with factorable T -ideals (see the Introduction and Theorem 4.1). In [3] Berele and Regev translated the result of the previous corollary in terms of cocharacters. In particular, if $\chi_n(A_1)$ and $\chi_n(A_2)$ are, respectively, the n -th cocharacter of the F -algebras A_1 and A_2 , then the n -th cocharacter related to the T -ideal $T(A) = T(A_1)T(A_2)$ is:

$$\chi_n(A_1) + \chi_n(A_2) + \chi_{(1)} \widehat{\otimes} \sum_{k=0}^{n-1} \chi_k(A_1) \widehat{\otimes} \chi_{n-1-k}(A_2) \\ - \sum_{k=0}^n \chi_k(A_1) \widehat{\otimes} \chi_{n-k}(A_2),$$

where $\widehat{\otimes}$ denotes the outer tensor product. We may also define the so called proper Hilbert series of a PI-algebra.

Definition 3.15. Let A be a PI-algebra over F , then, if $\bar{t} := (t_1, \dots, t_m)$, we denote by

$$H^B(A, \bar{t}) := H(B(x_1, \dots, x_m) / (B(x_1, \dots, x_m) \cap T(A)), \bar{t})$$

the proper Hilbert series of A in m variables.

Indeed, Hilbert and proper Hilbert series of PI-algebras are related. In fact, we have the following:

Proposition 3.16. Let A be a PI-algebra, then

$$H(A, \bar{t}) = \prod_{i=1}^m \frac{1}{1-t_i} \cdot H^B(A, \bar{t}).$$

A joint use of Proposition 3.16 and Corollary 3.14 gives us the possibility to state the following formula that is the analog of the formula by Berele and Regev for proper cocharacters of a PI-algebra A such as $T^B(A) = T^B(A_1)T^B(A_2)$. In particular we have:

$$\begin{aligned} \xi_n(A_1) + \xi_n(A_2) + \xi_{(1)} \widehat{\otimes} \sum_{k=0}^{n-1} \xi_k(A_1) \widehat{\otimes} \xi_{n-1-k}(A_2) \\ - \sum_{k=0}^n \xi_k(A_1) \widehat{\otimes} \xi_{n-k}(A_2). \end{aligned}$$

We consider now the following definition:

Definition 3.17. Let T be a tableau of shape λ (Young diagram of $[\lambda]$) filled in with natural numbers $\{1, \dots, k\}$ and let d_i be the multiplicity of i in T . A tableau is said to be semistandard if the entries weakly increase along each row and strictly increase down each column. Let

$$S_\lambda := \sum_{T_\lambda \text{ semistandard}} t^{T_\lambda},$$

where $t^{T_\lambda} = t_1^{d_1} t_2^{d_2} \dots t_k^{d_k}$. We say S_λ is the Schur function of λ in the variables t_1, \dots, t_k .

Hilbert series is strictly connected to the sequence of cocharacters of PI-algebras. Before recalling the rule (see the work of Berele [3] and the work of Drensky [17] or [18]), we say that for any $k, l, n \in \mathbb{N}$ we define by the symbol $H(k, l; n)$ the set of all partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ of n such that $\lambda_{k+1} \leq l$.

Theorem 3.18. Let A be a PI-algebra and let $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A) \chi_\lambda$ its n -th cocharacter. If $H(A, t_1, \dots, t_k)$ is the Hilbert series of A , then

$$H(A, t_1, \dots, t_k) = \sum_{n \geq 0} \sum_{\lambda \in H(k, 0, n)} m_\lambda(A) S_\lambda(t_1, \dots, t_k).$$

Moreover, the converse is also true. In fact, the multiplicities $m_\lambda(A)$ can be uniquely determined by the Hilbert series $H(A, t_1, \dots, t_k)$ for all λ which are partitions in $\leq k$ parts.

Hence, if we know the Hilbert series $H(A, t_1, \dots, t_k)$, we can find the multiplicities $m_\lambda(A)$ in $\chi_n(A)$ for those λ which are partitions in not more than k parts.

We complete this section citing the following famous result of Amitsur and Regev (see [1]):

Theorem 3.19. *Let A be a PI-algebra, then the Young diagrams corresponding to the irreducible characters participating in the cocharacter sequence of A are in a hook, i.e. there exist integers k, l such that*

$$\chi_n(A) = \sum_{\lambda \in H(k,l)} m_\lambda \chi_\lambda, \quad n = 0, 1, 2, \dots,$$

where $H(k, l) = \bigcup_{n \geq 0} H(k, l; n)$.

4. Cocharacters of $UT_n(F)$

In this section we want to draw up the main points of the work of Boumova and Drensky (see [4]). In this paper they give an easy algorithm which calculates the generating function of the cocharacters of $UT_n(F)$, i.e., the algebra of upper triangular matrices with entries from the field $F = \mathbb{C}$.

The generators of the T-ideal of polynomial identities of the algebra of upper triangular matrices with entries from F , have been found by Maltsev in [36]. More precisely,

Theorem 4.1. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then $T(UT_n(F))$ is generated as a T-ideal by the identity*

$$[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}].$$

The previous result also says that $T(UT_n(F)) = (T(F))^n$, then, in the light of the result of Giamb Bruno and Zaicev about Conjecture 2.9 and the fact that $T(F)$ is a T-prime ideal, $UT_n(F)$ generates a minimal variety of PI-exponent n . As we said above, in the literature we still do not have an explicit knowledge of the sequence of cocharacters of $UT_n(F)$ for $n \geq 3$.

From now on, we fix the ground field to be the complex field. Let $k \in \mathbb{N}$ be a non-zero integer and consider the algebra $F[[T_k]] := F[[t_1, \dots, t_k]]$ of formal power series in k commuting indeterminates. Let $F[[T_k]]^{S_k}$ be the subalgebra of symmetric functions, then every symmetric function $f(T_k)$ in the variables t_1, \dots, t_k can be presented as a linear combination of Schur functions, i.e.,

$$f = \sum_{\lambda} m_{\lambda} S_{\lambda}(T_k).$$

For details on the theory of Schur functions see the book by Macdonald ([35]). If $f \in F[[T_k]]^{S_k}$, we may associate to f a special series. For this purpose, let us consider the following:

Definition 4.2. *Let $f \in F[[T_k]]^{S_k}$ and consider the series*

$$M(f; T_k) = \sum_{\lambda} m_{\lambda} T_k^{\lambda} = \sum_{\lambda} m_{\lambda} t_1^{\lambda_1} \cdots t_k^{\lambda_k} \in F[[T_k]].$$

We shall call $M(f; T_k)$ the multiplicities series of f in the set of indeterminates T_k .

It is also convenient to consider the subalgebra $F[[V_k]]$ of $F[[T_k]]$ of the formal power series in the new set of variables $V_k = \{v_1, \dots, v_k\}$, where $v_1 = t_1$, $v_2 = t_1 t_2$, \dots , $v_k = t_1 \cdots t_k$. Then we may also rewrite the multiplicity series $M(f; T_k)$ in the following way:

$$M'(f; V_k) = \sum_{\lambda} m_{\lambda} v_1^{\lambda_1 - \lambda_2} \cdots v_k^{\lambda_k}.$$

We also call $M'(f; V_k)$ the multiplicity series of f . Notice that mapping

$$M' : F[[T_k]]^{S_k} \rightarrow F[[V_k]]$$

such that $M' : f(T_k) \mapsto M'(f; V_k)$ one has a bijection.

Definition 4.3. *If A is a PI-algebra, we define the multiplicity series of A as*

$$M(A; T_k) = M(A; t_1, \dots, t_k) = \sum_{\lambda} m_{\lambda}(A) T_k^{\lambda} = \sum_{\lambda} m_{\lambda}(A) t_1^{\lambda_1} \dots t_k^{\lambda_k}.$$

Similarly we define the series $M'(A; V_k)$. Before going further, we want to spend a few words about the product of Schur functions. It is well known (see [44], Chapter 4) that the product of the Schur functions

$$S_{\lambda}(t_1, \dots, t_k) S_{\mu}(t_1, \dots, t_k)$$

corresponds in a natural way to the tensor product of the irreducible modules corresponding to the partitions λ, μ and with abuse of notations we will always write

$$(\lambda \otimes \mu)^{\uparrow S_n},$$

where $\lambda \vdash l, \mu \vdash m$ and $l + m = n$. Then in the computation of products of Schur functions we are allowed to use the combinatorial tool of the Littlewood-Richardson rule. In most of the cases we shall use a very partial case of this rule, when one of the partitions is (m) or (1^m) , i.e., the Young rule. We recall that if $[\mu]$ is a subdiagram of $[\lambda]$, the skew-diagram $[\lambda/\mu]$ is obtained by λ delating the squares of μ . Translated in the language of Schur functions, the Young rule is stated as:

Proposition 4.4. *Let $m, k \in \mathbb{N}$ be non-zero integer and consider the set T_k of commuting indeterminates, then*

$$S_{(m)}(T_k) S_{\mu}(T_k) = \sum_{\lambda} S_{\lambda}(T_k),$$

where the summation runs over all partitions λ such that the skew Young diagram $[\lambda/\mu]$ is a horizontal strip of size m .

In [41] Regev introduced the notion of Young-derived sequences of S_n -characters. More precisely:

Definition 4.5. *Let $\{\zeta_n\}_{n \in \mathbb{N}}$ be a sequence of S_n -characters, then we say that the sequence is Young-derived if it is obtained from another sequence $\{\chi_n\}_{n \in \mathbb{N}}$ of S_n -characters by applying the Young rule.*

The previous definition may be paraphrased in terms of symmetric functions. In particular, if $f(T_k) = \sum_{\lambda} S_{\lambda}(T_k) \in F[[T_k]]^{S_k}$, we say that it is Young-derived from $g(T_k) = \sum_{\mu} S_{\mu}(T_k) \in F[[T_k]]^{S_k}$ if and only if the multiplicities m_{λ} and p_{μ} are related with the condition:

$$m_{(\lambda_1, \dots, \lambda_k)} = \sum p_{(\mu_1, \dots, \mu_k)}.$$

We suggest to compare this result with the Theorem 3.8 of Drensky.

We note that $\sum_{m \geq 0} S_{(m)}(T_k) = \prod_{i=1}^k \frac{1}{1-t_i}$. Then it turns out that f is the Young-derived of g if

$$f(T_k) = g(T_k) \prod_{i=1}^k \frac{1}{1-t_i}.$$

With this in mind, we consider the following definition:

Definition 4.6. Let us consider the operator $Y : F[[V_k]] \rightarrow F[[V_k]]$ such that

$$Y(M(g); T_k) = M(f; T_k) = M\left(g(T_k) \prod_{i=1}^k \frac{1}{1-t_i}; T_k\right).$$

We call Y the Young operator.

The following result of Drensky and Genov [19] translates Young-derived sequences in terms of multiplicities series.

Proposition 4.7. Let $f(T_k)$ be the Young-derived of the symmetric function $g(T_k)$. Then

$$\begin{aligned} Y(M(g); T_k) &= M(f; T_k) = M\left(g(T_k) \prod_{i=1}^k \frac{1}{1-t_i}; T_k\right) \\ &= \prod_{i=1}^k \frac{1}{1-t_i} \sum (-t_2)^{\epsilon_2} \cdots (-t_k)^{\epsilon_k} M(g; t_1 t_2^{\epsilon_2}, t_2^{1-\epsilon_2} t_3^{\epsilon_3}, \dots, t_{k-1}^{1-\epsilon_{k-1}} t_k^{\epsilon_k} t_k^{1-\epsilon_k}), \end{aligned}$$

where the summation runs on all $\epsilon_2, \dots, \epsilon_k \in \{0, 1\}$.

Now we have the main result by Boumova and Drensky ([4]):

Theorem 4.8. The Hilbert series $H(UT_n(F), T_k)$ of the algebra of upper triangular matrices with entries in F is:

$$\begin{aligned} H(UT_n(F), T_k) &= \frac{1}{t_1 + \dots + t_k - 1} \left(\left(1 + (t_1 + \dots + t_k - 1) \prod_{i=1}^k \frac{1}{1-t_i} \right)^n - 1 \right) \\ &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^k \frac{1}{1-t_i} \right)^j (t_1 + \dots + t_k - 1)^{j-1}. \end{aligned}$$

In the light of Proposition 4.7, Theorem 4.8 may be restated in order to give an easy algorithm to compute the multiplicities series of $UT_n(F)$. Here we recall the result:

Corollary 4.9. Let Y be the linear operator which sends the multiplicity series of the symmetric function $g(T_k)$ to the multiplicity series of its Young-derived:

$$Y(M(g); T_k) = M\left(g(T_k) \prod_{i=1}^k \frac{1}{1-t_i}; T_k\right).$$

Then the multiplicity series of $UT_n(F)$ is

$$M(UT_n(F); T_k) = \sum_{j=1}^k \sum_{q=0}^{j-1} \sum_{\lambda \vdash q} (-1)^{j-q-1} \binom{k}{j} \binom{j-1}{q} d_\lambda Y^j(T_k^\lambda),$$

where d_λ is the degree of the irreducible S_q -character χ_λ and $T_k^\lambda = t_1^{\lambda_1} \cdots t_k^{\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_k)$.

Always in [4] the authors described the partitions λ with $m_\lambda(UT_n(F)) \neq 0$ and the explicit form of the multiplicities for the partitions of “maximal” possible shape.

5. Two algebras generating minimal varieties of exponent 3 and 4

In this section we list the results concerning the Hilbert series, the proper Hilbert series and cocharacters of two algebras generating minimal varieties of PI-exponent 3 and 4. We start with a classical result of Lewin (see [34]) that is the key tool in what follows.

Let I and J be two T -ideals. Consider the quotient algebras $F\langle X \rangle/I$, $F\langle X \rangle/J$ and let U be a $F\langle X \rangle/I$ - $F\langle X \rangle/J$ -bimodule. We define:

$$R = \begin{pmatrix} F\langle X \rangle/I & U \\ 0 & F\langle X \rangle/J \end{pmatrix}.$$

Fix u_i a countable set of elements of U . Then $\varphi : x_i \rightarrow a_i$ defines an algebra homomorphism, where:

$$a_i = \begin{pmatrix} x_i + I & u_i \\ 0 & x_i + J \end{pmatrix}.$$

If $f(x_1, \dots, x_n) \in F\langle X \rangle$ one has that $f(x_1, \dots, x_n) \rightarrow f(a_1, \dots, a_n)$, where:

$$f(a_1, \dots, a_n) = \begin{pmatrix} f(x_1, \dots, x_n) + I & \delta(f) \\ 0 & f(x_1, \dots, x_n) + J \end{pmatrix}$$

and $\delta(f)$ is an element of U . Then $IJ \subseteq \ker(\varphi) = I \cap J \cap \ker(\delta(f))$ and

$$\delta : f(x_1, \dots, x_n) \rightarrow U$$

is an F -derivation.

Theorem 5.1. *If $\{u_i\}$ is a free generating set of the bimodule U then for the homomorphism φ defined by $\{u_i\}$, we have $\ker(\varphi) = IJ$.*

Corollary 5.2. *If the bimodule U contains a countable free generating set $\{u_i\}$, then $T(R) = IJ$.*

Let A, B be F -algebras, suppose they are both PI-algebras and let U be an $A - B$ -bimodule, then we can consider

$$R = \begin{pmatrix} A & U \\ 0 & B \end{pmatrix}$$

that is still an F -algebra and a PI-algebra such that $T(R) \supseteq T(A)T(B)$. Suppose now that $T(R) = T(A)T(B)$, then we may use Corollary 3.14 in order to compute its Hilbert series. We consider the following algebras:

$$\mathcal{L} = \begin{pmatrix} E^{(0)} & E \\ 0 & E \end{pmatrix}, \quad UT_2(E) = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}.$$

In [27] Giambruno and Zaicev listed five algebras such that a variety has PI-exponent strictly greater than 2 if and only if some of the five algebras from the list belong to the variety. One of

these five algebras is \mathcal{L} . The variety generated by the algebra \mathcal{L} was studied by Stoyanova-Venkova in [45] where she described its proper cocharacter sequence which, in principle, determines the ordinary cocharacter sequence. In [11], Di Vincenzo, Drensky and Nardozza described asymptotically its proper subvarieties. We also remark that \mathcal{L} is a proper subalgebra of $UT_2(E)$. In what follows we present the results of the author ([6]) concerning the Hilbert series of \mathcal{L} and $UT_2(E)$. Notice also that in light of Theorem 5.1 and Corollary 5.2, we have

$$T(\mathcal{L}) = T(E^{(0)})T(E) = T(F)T(E)$$

and

$$T(UT_2(E)) = T(E)T(E).$$

Hence we are able to use Corollary 3.14 in order to compute the Hilbert series and we may compare such results with the formula of Berele and Regev we showed in the second section of this paper. We start off with the algebra \mathcal{L} .

Theorem 5.3.

$$H(\mathcal{L}, \bar{t}) = \sum m_\lambda S_\lambda,$$

where $\lambda = (k_1, k_2, 1^l)$ or $\lambda = (k_1, k_2, 2, 1^l)$. If $\lambda = (k_1, k_2, 1^l)$ then:

- (1) $m_\lambda = 4(k_1 - k_2 + 1)$ if $k_2 \geq 2$ and $l \geq 1$
- (2) $m_\lambda = 2k - 1$ if $\lambda = (k, 1^l)$ and $l \geq 2$,
- (3) $m_\lambda = 2(k_1 - k_2 + 1)$ if $\lambda = (k_1, k_2)$,
- (4) $m_\lambda = k$ if $\lambda = (k, 1)$,
- (5) $m_\lambda = 1$ if $\lambda = (k)$.

If $\lambda = (k_1, k_2, 2, 1^l)$, then:

$$m_\lambda = 2(k_1 - k_2 + 1).$$

By Theorem 3.18 we are able to determine the explicitly the cocharacter sequence of \mathcal{L} . Here we have an example:

Example 5.4.

$$\chi_1(\mathcal{L}) = (1),$$

$$\chi_2(\mathcal{L}) = (2) + (1^2),$$

$$\chi_3(\mathcal{L}) = (3) + 2(2, 1) + (1^3),$$

$$\chi_4(\mathcal{L}) = (4) + 3(3, 1) + 2(2^2) + 3(2, 1^2) + (1^4),$$

$$\chi_5(\mathcal{L}) = (5) + 4(4, 1) + 4(3, 2) + 5(3, 1^2) + 4(2^2, 1) + 3(2, 1^3) + (1^5),$$

$$\chi_6(\mathcal{L}) = (6) + 5(5, 1) + 6(4, 2) + 7(4, 1^2) + 8(3, 2, 1) + 2(3^2) + 5(3, 1^3) + 2(2^3) + 4(2^2, 1^2) + 3(2, 1^4) + (1^6).$$

We consider now $UT_2(E)$, then as we said above, $T(UT_2(E)) = T(E)T(E)$ and we are allowed to use the formula by Berele and Regev. By the way, the main combinatorial problem with that formula is that we still do not have a good algorithm for the computation of the outer tensor product $\chi_k(E) \widehat{\otimes} \chi_{n-k}(E)$. We avoid this problem computing the proper Hilbert series of $UT_2(E)$. In fact, it is well known (see for example Chapters 4 and 12 of [14],) that for any $n \geq 1$:

$$\xi_n(E) = \begin{cases} \chi_\emptyset & \text{if } n \text{ is odd;} \\ \chi_{(1^n)} & \text{if } n \text{ is even.} \end{cases}$$

By the formula for proper cocharacters of a factorable PI-algebra, this means that we have just to apply several times the Young rule in order to compute the proper cocharacter sequence of $UT_2(E)$. After that, Theorem 3.8 will give us the complete sequence of ordinary cocharacters applying once more the Young rule. In what follows we list the results:

Theorem 5.5.

$$H^B(UT_2(E), \bar{t}) = \sum m_\lambda S_\lambda,$$

where

$\lambda = (k, 2^m, 1^l)$ or $\lambda = (k, 3, 2^m, 1^l)$. If $\lambda = (k, 2^m, 1^l)$, then:

- (1) $m_\lambda = 2(l + 1)$ if $k \geq 3, m \geq 1$,
- (2) $m_\lambda = l + 1$ if $m \geq 2$,
- (3) $m_\lambda = l$ if $k \geq 3, m = 0$,
- (4) $m_\lambda = \begin{cases} 1 & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd} \end{cases}$ if $m = k = 0$,
- (5) $m_\lambda = \begin{cases} \frac{l}{2} & \text{if } l \text{ is even} \\ \frac{l+1}{2} & \text{if } l \text{ is odd} \end{cases}$ if $k = 2, m = 0$.

If $\lambda = (k, 3, 2^m, 1^l)$, then:

$$m_\lambda = l + 1.$$

Theorem 5.6.

$$H(UT_2(E), \bar{t}) = \sum m_\lambda S_\lambda,$$

where $\lambda = (k_1, k_2, 2^m, 1^l)$ or $\lambda = (k_1, k_2, 3, 2^m, 1^l)$.

If $\lambda = (k_1, k_2, 2^m, 1^l)$, then:

- (1) $m_\lambda = 12(k_1 - k_2 + 1)(l + 1)$ if $k_1 \geq k_2 \geq 3, m \geq 1$,
- (2) $m_\lambda = 4(k_1 - k_2 + 1)(2l + 1)$ if $k_1 \geq k_2 \geq 3, m = 0$,
- (3) $m_\lambda = 8(k_1 - 2)(l + 1) + 4(l + 1)$ if $k_1 \geq k_2 = 2, m \geq 1$,
- (4) $m_\lambda = 3(k_1 - 2)(2l + 1) + 3l + 2$ if $k_1 \geq k_2 = 2, m = 0$,
- (5) $m_\lambda = (k_1 - 2)(2l - 1) + l + 1$ if $k_1 \geq 2, k_2 = 0, m = 0, l \geq 1$,
- (6) $m_\lambda = 1$ if $\lambda = (1^l)$ or $\lambda = (k)$.

If $\lambda = (k_1, k_2, 2^m, 1^l)$, then:

$$m_\lambda = 4(k_1 - k_2 + 1)(l + 1) \text{ if } k_2 \geq 3, m \geq 1.$$

By Theorem 3.18 we are able to determine explicitly the cocharacter sequence of $UT_2(E)$. Here we have an example:

Example 5.7.

$$\begin{aligned}\chi_1(R) &= (1), \\ \chi_2(R) &= (2) + (1^2), \\ \chi_3(R) &= (3) + 2(2, 1) + (1^3), \\ \chi_4(R) &= (4) + 3(3, 1) + 2(2^2) + 3(2, 1^2) + (1^4), \\ \chi_5(R) &= (5) + 4(4, 1) + 5(3, 2) + 6(3, 1^2) + 5(2^2, 1) + 4(2, 1^3) + (1^5), \\ \chi_6(R) &= (6) + 5(5, 1) + 8(4, 2) + 9(4, 1^2) + 14(3, 2, 1) + 4(3^2) + 9(3, 1^3) + (2^3) + 8(2^2, 1^2) + 5(2, 1^4) + (1^6).\end{aligned}$$

6. Graded identities and related graded structures

As long as we said above, the generators of the T-ideal of PI-algebras is well known only for some classes of associative PI-algebras. Keeping this in mind, we are allowed to study something "more simple", i.e., we try to find polynomial identities such that their indeterminates are in somewhat sense specialized. This is the case of graded identities of graded algebras. In what follows we shall give a small account about the graded structures that we shall use later on.

Definition 6.1. Let G be a group and A an algebra over a field F . We say that the algebra A is G -graded if A can be written as the direct sum of subspaces $A = \bigoplus_{g \in G} A^{(g)}$ such that for all $g, h \in G$, one has that $A^{(g)}A^{(h)} \subseteq A^{(gh)}$.

Example 6.2. Let A be any algebra and G be a group, then A inherits a G -grading by setting $A = A^{(1_G)}$ and $A^{(g)} = 0$ for any $g \neq 1_G$. We shall call this grading the trivial grading.

Example 6.3. Let $A = F\langle X \rangle$, then A has a natural \mathbb{Z} -grading by setting

$$A = \bigoplus_{n \in \mathbb{Z}} A^{(n)},$$

where $A^{(n)} = 0$ if $n < 0$, and $A^{(n)}$ is the linear span of all monomials of total degree n , in case $n \geq 0$.

Example 6.4. Let us consider $M_n(F)$ and let $B = \{e_{1,1}, e_{1,2}, \dots, e_{n,n}\}$ be a basis of $M_n(F)$ where the matrices $e_{i,j}$ are the unit matrices. Consider the following map

$$\|\cdot\| : B \rightarrow \mathbb{Z}_n$$

such that $\|e_{i,j}\| = [j - i]_n$. The map φ induces a \mathbb{Z}_n -grading on $M_n(F)$ introduced by Vasilovsky in [46]. More precisely, $M_n(F)^{(0)}$ consists of matrices of the form

$$\begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{n,n} \end{pmatrix},$$

where $a_{1,1}, a_{2,2}, \dots, a_{n,n} \in F$, and for $0 \leq t \leq n - 1$, $M_n(F)^{(t)}$ consists of the matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & a_{1,t+1} & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & a_{2,t+2} & & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & a_{n-t,n} \\ a_{n-t+1,1} & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & a_{n,t} & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $a_{1,t+1}, a_{2,t+2}, \dots, a_{n-t,n}, a_{n-t+1,1}, a_{n-t+2,2}, \dots, a_{n,t} \in F$. Indeed the grading of Vasilovsky may be extended in a natural way to $UT_n(F)$.

Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets and consider

$$X = \bigcup_{g \in G} X^g.$$

We shall denote by $F\langle X \rangle$ the free associative algebra freely generated by the set X . An indeterminate $x \in X$ is said to be of homogeneous G -degree g , written $\|x\| = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G -degree of a monomial $m = x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined to be

$$\|m\| = \|x_{i_1}\| \cdot \|x_{i_2}\| \cdots \|x_{i_k}\|.$$

For every $g \in G$, we denote by $F\langle X \rangle^{(g)}$ the subspace of $F\langle X \rangle$ spanned by all the monomials having homogeneous G -degree g . Notice that $F\langle X \rangle^{(g)} F\langle X \rangle^{(g')} \subseteq F\langle X \rangle^{(gg')}$ for all $g, g' \in G$. Thus

$$F\langle X \rangle = \bigoplus_{g \in G} F\langle X \rangle^{(g)}$$

proves $F\langle X \rangle$ to be a G -graded algebra. The elements of the G -graded algebra $F\langle X \rangle$ are referred to as G -graded polynomials or, simply, graded polynomials.

Definition 6.5. If A is a G -graded algebra, a G -graded polynomial $f(x_1, \dots, x_n)$ is said to be a graded polynomial identity for A if

$$f(a_1, a_2, \dots, a_n) = 0$$

for all $a_1, a_2, \dots, a_n \in \bigcup_{g \in G} A^g$ such that $a_k \in A^{\|x_k\|}$, $k = 1, \dots, n$. Moreover, we denote by $T_G(A)$ the ideal of G -graded polynomial identities for A .

As long as for the ordinary case, the theory of G -graded PI-algebras passes through the representation theory of the symmetric group. More precisely for any $n \in \mathbb{N}$, we study the following S_n -modules:

$$V_n^G = \text{span}\langle x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \cdots x_{\sigma(n)}^{g_n} \mid g_i \in G, \sigma \in S_n \rangle.$$

The elements in V_n^G will be called the multilinear polynomials of degree n of $F\langle X \rangle$ and it turns out that V_n^G is a left S_n -module under the natural action of the symmetric group S_n . In analogy to the

ordinary case, we consider the following:

$$V_n^G(A) := V_n^G / (V_n^G \cap T_G(A)).$$

It turns out that $V_n^G(A)$ is a S_n -module, too.

Definition 6.6. We call n -th graded cocharacter of A , the character of the S_n -module $V_n^G(A)$ and we denote it by $\chi_n^G(A)$. We call n -th graded codimension of A , the dimension of $V_n^G(A)$ and we denote it by $c_n^G(A)$. We say that

$(\chi_n^G(A))_{n \in \mathbb{N}}$ is the G -graded cocharacter sequence of A ,

$(c_n^G(A))_{n \in \mathbb{N}}$ is the G -graded codimension sequence of A .

Assume that $|G| = r$ and that $G = \{g_1, \dots, g_r\}$. Now, for $l_{g_1}, \dots, l_{g_r} \in \mathbb{N}$ let us consider the blended components of the multilinear polynomials in the indeterminates labelled as follows: $x_1^{g_1}, \dots, x_{l_{g_1}}^{g_1}$, then $x_{l_{g_1}+1}^{g_2}, \dots, x_{l_{g_1}+l_{g_2}}^{g_2}$ and so on. We denote this linear space by $V_{l_{g_1}, \dots, l_{g_r}}^G$. Of course, this is a left $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ -module. We shall denote by $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$ the character of the module

$$V_{l_{g_1}, \dots, l_{g_r}}^G(A) := V_{l_{g_1}, \dots, l_{g_r}}^G / (V_{l_{g_1}, \dots, l_{g_r}}^G \cap T_G(A))$$

and by $c_{l_{g_1}, \dots, l_{g_r}}^G(A)$ its dimension.

Since the ground field F is infinite, a standard Vandermonde-argument yields that a polynomial f is a G -graded polynomial identity for A if and only if its homogeneous components (with respect to the ordinary \mathbb{N} -grading), are identities as well. Moreover, since $\text{char}(F) = 0$, the well known multilinearization process shows that the T_G -ideal of a G -graded algebra A is determined by its multilinear polynomials, i.e., by the various $V_{l_{g_1}, \dots, l_{g_r}}^G(A)$ for $n_i \in \mathbb{N}$. We remark that, given the cocharacter $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$, the graded cocharacter $\chi_n^G(A)$ is known as well. More precisely, the following is due to Di Vincenzo (see [9], Theorem 2):

Theorem 6.7. Let A be a G -graded algebra with graded cocharacter sequences $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$. Then

$$\chi_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \chi_{l_{g_1}, \dots, l_{g_r}}^G(A)^{\uparrow S_n}.$$

Moreover

$$c_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \binom{n}{l_{g_1}, \dots, l_{g_r}} c_{l_{g_1}, \dots, l_{g_r}}^G(A).$$

Suppose we are dealing with a G -graded set of indeterminates and consider the free algebra $F\langle Y \cup Z \rangle$ (where Y is the set of all indeterminates of G -degree 1_G and Z is the set of all the remaining indeterminates). We consider the following definition (see [20], Section 2; [12], Section 2):

Definition 6.8. Let us consider the unitary F -subalgebra B^Y of $F\langle Y \cup Z \rangle$ generated by the elements of Z and by all non-trivial commutators. We call Y -proper polynomials the elements of B^Y .

Roughly speaking, a polynomial $f \in F\langle Y \cup Z \rangle$ is Y -proper if all the $y \in Y$ occurring in f appear in commutators only. Notice that if $f \in F\langle Z \rangle$, then f is Y -proper. It is well known (see, for instance, Lemma 1 Section 2 in [12]) that all graded polynomial identities of a superalgebra A follow from the Y -proper ones. This means that the set $T_{\mathbb{Z}_2}(A) \cap B$ generates the whole $T_{\mathbb{Z}_2}(A)$ as a $T_{\mathbb{Z}_2}$ -ideal. Similarly, for any group G , all the G -graded polynomial identities of a G -graded algebra A follow from the Y -proper ones. This means that the set $T_G(A) \cap B$ generates the whole $T_G(A)$ as a T_G -ideal.

Let us denote $B^Y(A) := B^Y / (T_G(A) \cap B^Y)$. We shall denote by Γ_n^G the set of multilinear polynomials of V_n^G that are Y -proper. It is not difficult to see that Γ_n^G is a left S_n -submodule of V_n^G and the same holds for $\Gamma_n^G \cap T_G(A)$. Hence the factor module

$$\Gamma_n^G(A) := \Gamma_n^G / (\Gamma_n^G \cap T_G(A))$$

is a S_n -submodule of $V_n^G(A)$.

Definition 6.9. We call n -th Y -proper graded cocharacter of A , the character of the S_n -module $\Gamma_n^G(A)$ and we denote it by $\xi_n^G(A)$. We call n -th Y -proper graded codimension of A , the dimension of the dimension of $\Gamma_n^G(A)$ and we denote it by $\gamma_n^G(A)$. We say that

$$\begin{aligned} (\xi_n^G(A))_{n \in \mathbb{N}} & \text{ is the } G\text{-graded proper cocharacter sequence of } A \\ (\gamma_n^G(A))_{n \in \mathbb{N}} & \text{ is the } G\text{-graded proper codimension sequence of } A \end{aligned}$$

We shall denote by $\Gamma_{m_1, \dots, m_r}^G$ the set of multilinear polynomials of V_{m_1, \dots, m_r}^G such that $m = \sum_{i=1}^{r-1} m_i$ that are Y -proper. It is not difficult to see that $\Gamma_{m_1, \dots, m_r}^G$ is a left $S_{m_1} \times \dots \times S_{m_r}$ -submodule of V_{m_1, \dots, m_r}^G and the same holds for $\Gamma_{m_1, \dots, m_r}^G \cap T_G(A)$. Hence the factor module

$$\Gamma_{m_1, \dots, m_r}^G(A) := \Gamma_{m_1, \dots, m_r}^G / (\Gamma_{m_1, \dots, m_r}^G \cap T_G(A))$$

is an $S_{m_1} \times \dots \times S_{m_r}$ -submodule of $V_{m_1, \dots, m_r}(A)^G$. We denote the $S_{m_1} \times \dots \times S_{m_r}$ -character of the factor module $\Gamma_{m_1, \dots, m_r}^G / (\Gamma_{m_1, \dots, m_r}^G \cap T_G(A))$ by $\xi_{m_1, \dots, m_r}^G(A)$, and by $\gamma_{m_1, \dots, m_r}(A)$ its dimension over F . Indeed when we refer to A without any ambiguity, we shall use γ_{m_1, \dots, m_r} instead of $\gamma_{m_1, \dots, m_r}(A)$.

Following word by word the proof of Di Vincenzo in [9], we have the analog of Theorem 6.7 for the graded proper cocharacters and codimensions:

Theorem 6.10. Let A be a G -graded algebra with graded proper cocharacter sequences $\xi_{l_{g_1}, \dots, l_{g_r}}^G(A)$. Then

$$\xi_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \xi_{l_{g_1}, \dots, l_{g_r}}^G(A)^{\uparrow S_n}.$$

Moreover

$$\gamma_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \binom{n}{l_{g_1}, \dots, l_{g_r}} \gamma_{l_{g_1}, \dots, l_{g_r}}^G(A)$$

7. Y-proper graded cocharacters of $UT_m(F)$

In this section we show a combinatorial method (see [8]) in order to compute the exact value of the Y -proper graded cocharacter sequence of $UT_m(F)$ endowed with the grading induced by that of Vasilovsky and of small size. In particular, we are able to describe explicitly the Y -proper graded cocharacter and codimension sequence of the algebras $UT_2(F)$, $UT_3(F)$. In [8], the author and Cirrito computed the values of multiplicities for specific and significant partitions in the Y -proper graded cocharacter sequence of $UT_4(F)$.

Definition 7.1. We shall call normal any Y -commutator c of Z -degree at most 1 such that $c = z$ or $c = [z, y_{i_1}, \dots, y_{i_t}]$ for some $t \geq 1$.

Moreover we say that:

- a normal commutator $[y_{i_1}, \dots, y_{i_t}]$ of Z -degree 0 is semistandard if the indices i_1, \dots, i_p satisfy the inequalities $i_1 > i_2 \leq \dots \leq i_p$.
- a normal commutator $[z, y_{i_1}, \dots, y_{i_t}]$ of Z -degree 1 is semistandard if the indices i_1, \dots, i_p satisfy the inequalities $i_1 \leq i_2 \leq \dots \leq i_p$.

Let us consider $UT_m(F)$ endowed with an elementary G -grading. Consider the following definition (see [13]):

Definition 7.2. Let $\tilde{\mu} = (\mu_1, \dots, \mu_k)$ be an element of G^k . We say that $\tilde{\mu}$ is a good sequence with respect to the G -grading if there exists a sequence of k matrix units (r_1, \dots, r_k) in the Jacobson radical of $UT_m(F)$ such that the product $r_1 \cdots r_k$ is non-zero and also the homogeneous degree of r_i is μ_i for all $i = 1, \dots, k$. In this case we say that $\tilde{\mu}$ is good, otherwise $\tilde{\mu}$ is called bad sequence.

In [13] Di Vincenzo, Koshlukov and Valenti obtained the following description of the Y -proper polynomials in the relatively free graded algebra $F\langle X \rangle / (F\langle X \rangle \cap T_G(UT_m(F)))$:

Proposition 7.3. A linear basis for the Y -proper polynomials in the relatively free graded algebra $F\langle X \rangle / (F\langle X \rangle \cap T_G(UT_m(F)))$ consists of 1 and of the polynomials $c_1 \cdots c_k$ where each polynomial c_i is a semistandard commutator and the sequence $\tilde{\mu}_c = (\|c_1\|, \dots, \|c_k\|)$ is good.

Notice that Koshlukov and Valenti in [30], found a multilinear basis of the relatively free algebra $F\langle X \rangle / (F\langle X \rangle \cap T_{\mathbb{Z}_m}(UT_m(F)))$. In particular, we have the following description of the $T_{\mathbb{Z}_m}$ -ideal of $UT_m(F)$, where $UT_m(F)$ is endowed with the grading of Vasilovsky:

Theorem 7.4. The $T_{\mathbb{Z}_m}$ -ideal of $UT_m(F)$ is generated by the following set

$$\{[x_1^0, x_2^0], x_1^i x_2^j \mid i + j > m\}$$

Now we start the study of the Y -proper graded cocharacters of $UT_m(F)$. We have the following lemma that we shall use later on:

Lemma 7.5. *Let $m \geq 2$ and consider $UT_m(F)$ with its Vasilovsky \mathbb{Z}_m -grading. Then we have that for any $n \in \mathbb{N}$ and for any $\sigma \in S_n$,*

$$[z, y_{\sigma(1)}, \dots, y_{\sigma(n)}] \equiv_{T_{\mathbb{Z}_m}(UT_m(F))} [z, y_1, \dots, y_n].$$

Proof. It suffices to show that

$$[z, y_1, \dots, y_a, y_{a+1}, \dots, y_n] \equiv_{T_{\mathbb{Z}_m}(UT_m(F))} [z, y_1, \dots, y_{a+1}, y_a, \dots, y_n].$$

We have that

$$\begin{aligned} & [z, y_1, \dots, y_a, y_{a+1}, \dots, y_m] \\ & \equiv [z, y_1, \dots, y_{a+1}, y_a, \dots, y_m] + [z, y_1, \dots, [y_a, y_{a+1}], \dots, y_m] \end{aligned}$$

but $[y_a, y_{a+1}]$ is a \mathbb{Z}_m -graded identities for $UT_m(F)$, hence

$$[z, y_1, \dots, [y_a, y_{a+1}], \dots, y_m] \equiv 0$$

and we are done. □

Given a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of elements of \mathbb{Z}_m , we can associate to α the n -th uple of multiplicities

$$\mu(\alpha) = (\mu_1, \dots, \mu_m)$$

such that for any $i = 1, \dots, m$,

$$\mu_i = \text{number of } \alpha_j \text{ such that } \alpha_j = [i - 1]_m.$$

We observe that distinct sequences are allowed to have the same m -th uple of multiplicities. We have that

$$\alpha \text{ is a good sequence if and only if } \mu_1 = 0 \text{ and } \sum_{j=2}^m \mu_j(j - 1) \leq m - 1.$$

Fix now $\tilde{l} = (l_1, \dots, l_m)$ such that $\sum_{j=2}^m l_j(j - 1) \leq m - 1$ and let

$$S_{\tilde{l}} = \{\alpha = (\alpha_1, \dots, \alpha_k) \mid \mu(\alpha) = (0, l_2, \dots, l_m)\}.$$

In what follows we use the following notation: if G is a group and F a field, we denote by FG , the F -group algebra of G . Moreover if M is a FG -module and $m \in M$, we denote by FGm the action of FG on m . We also suggest that in what follows we implicitly use the fact that two representations afford the same G -character if they represent isomorphic FG -modules. With this in mind, we have the following:

Theorem 7.6. *Let $m \in \mathbb{N}$ and consider $UT_m(F)$ with its \mathbb{Z}_m -Vasilovsky grading. Then for any $l_1, \dots, l_m \in \mathbb{N}$, such that $\sum_{j=2}^m l_j(j - 1) \leq m - 1$*

$$\begin{aligned} & \xi_{l_1, \dots, l_m}^{\mathbb{Z}_m}(UT_m(F)) = \\ & = \sum_{(\alpha_1, \dots, \alpha_k) \in S_{\tilde{l}}} \sum_{s_1 + \dots + s_k = l_1} \left(\underbrace{\quad}_{s_1} \otimes \dots \otimes \underbrace{\quad}_{s_k} \right)^{\uparrow S_{l_1}} \end{aligned}$$

$$\otimes(\square \otimes \dots \otimes \square)^{\uparrow S_{l_2}} \otimes \dots \otimes (\square \otimes \dots \otimes \square)^{\uparrow S_{l_m}}.$$

Proof. Let $(\alpha_1, \dots, \alpha_k) \in S_i$ such that $\mu(\alpha) = (0, l_2, \dots, l_m)$. Let $l_1 \in \mathbb{N}$ such that $\sum_{i=1}^m l_i = n$. Put $A = UT_m(F)$ and let $f \in \Gamma_{l_1, \dots, l_m} \subseteq \Gamma_n^{\mathbb{Z}_m}(A)$. If $H = S_{l_1} \times \dots \times S_{l_m}$, then $FHf^{\uparrow S_n}$ has dimension $|S_n : H| \dim_F FHf$. On the other hand, $FS_n f$ is generated by

$$[z^{\alpha_i}, y_{i_1}, \dots, y_{i_{s_1}}] \cdots [z^{\alpha_j}, y_{j_1}, \dots, y_{j_{s_k}}],$$

where the indices of the y 's are ordered in the light of Lemma 7.5 and $s_1 + \dots + s_k = l_1$. The latter polynomials are linearly independent, hence $\dim_F FS_n f = |S_n : H| \dim_F F[H]f$. Now the proof follows since the S_n -action in both of the situations is the same and the action on the y 's is the trivial one. □

The next two propositions are consequences of the previous theorem and may be showed as examples.

Proposition 7.7. *Let $n \geq 2$, then*

$$\xi_n^{\mathbb{Z}_2}(UT_2(F)) = (n) + (n-1, 1)$$

and

$$\gamma_n^{\mathbb{Z}_2}(UT_2(F)) = n.$$

Proposition 7.8. *Let $n \geq 2$, then*

$$\gamma_n^{\mathbb{Z}_3}(UT_3(F)) = 2n + 2^{n-2}n(n-1).$$

Moreover,

$$\xi_n^{\mathbb{Z}_3}(UT_3(F)) = \sum_{\lambda \vdash n} m_\lambda \lambda,$$

where

$$m_\lambda = \begin{cases} n+1 & \text{if } \lambda = (n) \\ 3(n-1) & \text{if } \lambda = (n-1, 1) \\ 4(a-b+1) & \text{if } \lambda = (a, b) \text{ and } b \geq 2 \\ 3a-1 & \text{if } \lambda = (a, 1^2) \\ 4(a-b+1) & \text{if } \lambda = (a, b, 1) \text{ and } b \geq 2 \\ a-b+1 & \text{if } \lambda = (a, b, 2) \\ a-b+1 & \text{if } \lambda = (a, b, 1^2) \end{cases}$$

8. \mathbb{Z}_2 -graded cocharacters of $UT_2(E)$

We conclude this survey with the completion of the results obtained in section 3 about the algebra $UT_2(E)$. In particular, we want to show the \mathbb{Z}_2 -graded cocharacter sequence of the latter algebra as a consequence of a result of Di Vincenzo and Nardoza ([10]). The results of this Section may be found in [6].

First of all we note that the algebra $R = UT_2(E)$ has a natural structure of \mathbb{Z}_2 -graded algebra, i.e., $R = R^{(0)} \oplus R^{(1)}$, where

$$R^{(0)} = \begin{pmatrix} E^{(0)} & E^{(0)} \\ 0 & E^{(0)} \end{pmatrix}$$

and

$$R^{(1)} = \begin{pmatrix} E^{(1)} & E^{(1)} \\ 0 & E^{(1)} \end{pmatrix},$$

and $E^{(0)}, E^{(1)}$ are, respectively, the 0 and the 1 part of the natural \mathbb{Z}_2 -grading of E . As a \mathbb{Z}_2 -graded algebra, R is naturally isomorphic to $UT_2(F) \otimes E$. In general, let A be a G -graded algebra and consider the following subgroups of S_n :

$$H = S_{p_1+q_1} \times \cdots \times S_{p_r+q_r},$$

$$\bar{H} = S_{p_1} \times S_{q_1} \times \cdots \times S_{p_r} \times S_{q_r}.$$

Notice that \bar{H} is a subgroup of H , therefore the following two spaces are \bar{H} -modules, too:

$$V_{p_1, q_1, \dots, p_r, q_r}^{G \times \mathbb{Z}_2} \cap T^{G \times \mathbb{Z}_2}(A \otimes E),$$

$$V_{p_1+q_1, \dots, p_r+q_r}^G \cap T^G(A).$$

We denote by $\langle \lambda, \mu \rangle$ the partition $(\lambda_1, \mu_1, \dots, \lambda_r, \mu_r)$, where $\lambda_i \vdash p_i$ and $\mu_i \vdash q_i$. Then the relation between the graded cocharacter sequences of A and $A \otimes E$ is given by the following result by Di Vincenzo and Nardoza (see [10]):

Theorem 8.1. *Let*

$$(\chi_{p_1+q_1, \dots, p_r+q_r}^G(A))_{\downarrow \bar{H}} = \sum m_{\langle \lambda, \mu \rangle} \langle \lambda, \mu \rangle$$

be the cocharacter sequence of the \bar{H} -module

$$(V_{p_1+q_1, \dots, p_r+q_r}^G \cap T^G(A))_{\downarrow \bar{H}}.$$

Then the cocharacter sequence of the \bar{H} -module

$$(V_{p_1, q_1, \dots, p_r, q_r}^{G \times \mathbb{Z}_2} \cap T^{G \times \mathbb{Z}_2}(A \otimes E))$$

is

$$(\chi_{p_1, q_1, \dots, p_r, q_r}^{G \times \mathbb{Z}_2}(A \otimes E))_{\downarrow \bar{H}} = \sum m_{\langle \lambda, \mu' \rangle} \langle \lambda, \mu' \rangle,$$

where

$$\langle \lambda, \mu' \rangle = \lambda_1 \otimes \mu'_1 \otimes \cdots \otimes \lambda_r \otimes \mu'_r$$

and μ'_i is the conjugate partition of μ_i .

As a consequence of the previous result, we have the following:

Corollary 8.2. Let $k, l \in \mathbb{N}$ such that $k + l = n$ and consider $H = S_k \times S_l$. If $(\chi_n(UT_2(F)))_{\downarrow H} = \sum m_{\lambda, \mu} \lambda \otimes \mu$, then

$$\chi_n^{\mathbb{Z}_2}(R) = \sum_{k+l=n} \sum_{\substack{\lambda \vdash k \\ \mu \vdash l}} m_{\lambda, \mu} \lambda \otimes \mu'$$

In the light of Corollary 8.2, it suffices to know $(\chi_n(UT_2(F)))_{\downarrow H} = \sum m_{\lambda, \mu} \lambda \otimes \mu$, then we have immediately the n -th \mathbb{Z}_2 -graded cocharacter of $UT_2(E)$.

Theorem 8.3. Let $k, l \in \mathbb{N}$ such that $k + l = n$ and consider $H = S_k \times S_l$. For any $n \geq 0$, $(\chi_n(S))_{\downarrow H} = \sum m_{\lambda, \mu} \lambda \otimes \mu$, where

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, \lambda_3), & \lambda_3 &\leq 1 \\ \mu &= (\mu_1, \mu_2, \mu_3), & \mu_3 &\leq 1. \end{aligned}$$

More precisely,

$$\begin{aligned} (n)_{\downarrow H} &= (k) \otimes (l) \\ (k_1, k_2)_{\downarrow H} &= \begin{cases} (k) \otimes (l) \\ (\lambda_1, \lambda_2) \otimes (l) \\ (\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2) \end{cases} \\ (k_1, k_2, 1)_{\downarrow H} &= \begin{cases} (\lambda_1, \lambda_2) \otimes (l) \\ 2((\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2)) & \text{if } \mu_1 - 1 \geq \mu_2 \\ (\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2) & \text{if } \mu_1 - 1 < \mu_2 \\ (\lambda_1, \lambda_2, 1) \otimes (l) \\ (\lambda_1, \lambda_2, 1) \otimes (\mu_1, \mu_2). \end{cases} \end{aligned}$$

Proof. By Branching rule (see [44], Chapter 2) we have that when we restrict the representation $\nu = \sum (k_1, k_2, k_3)$ of S_n , with $k_3 \leq 1$, to its subgroup $H = S_k \times S_l$, then its H -irreducible components are $\lambda \otimes \mu$, where $\lambda = (k'_1, k'_2, k'_3)$ and $\mu = (l_1, l_2, l_3)$ are such that ν appears in the tensor product $(\lambda \otimes \mu)^{\uparrow S_n}$. By Frobenius multiplicity law (see [44]), the multiplicity of $\lambda \otimes \mu$ in the previous decomposition equals the multiplicity of ν in the induced representation $(\lambda \otimes \mu)^{\uparrow S_n}$. We will argue for the irreducible cocharacters of $\chi_n(F)$, i.e., (n) , (k_1, k_2) where $k_2 \geq 1$ and $(k_1, k_2, 1)$. In the first case, it is easy to see that $(n)_{\downarrow H} = m_{\lambda, \mu}((l) \otimes (k))$, where $\lambda = (k)$ and $\mu = (l)$. By the Littlewood-Richardson Rule, the multiplicity of (n) in the induced representation $(\lambda \otimes \mu)^{\uparrow S_n}$ is 1. Let $\nu = (k_1, k_2)$. Then

$$(k_1, k_2)_{\downarrow H} = \begin{cases} m_{\lambda, \mu}((k) \otimes (l)) & a) \\ m_{\lambda, \mu}((\lambda_1, \lambda_2) \otimes (l)) & b) \\ m_{\lambda, \mu}((\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2)) & c). \end{cases}$$

The case *a)* has been already treated. Consider the case *b)*. Here, the only way to obtain ν is adding $k_2 - \lambda_2$ $\boxed{1}$ to λ_2 so the multiplicity of ν in $(\lambda_1, \lambda_2) \otimes (l)$ is 1. Even in the case *c)*, the only possible

way to obtain ν is adding all the boxes $\boxed{2}$ to λ_2 so the multiplicity of ν in $(\lambda_1, \lambda_2) \otimes (l)$ is 1. Finally, let $\nu = (k_1, k_2, 1)$. Then

$$(k_1, k_2, 1)_{\downarrow H} = \begin{cases} m_{\lambda, \mu}((\lambda_1, \lambda_2) \otimes (l)) & a) \\ m_{\lambda, \mu}((\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2)) & b) \\ m_{\lambda, \mu}((\lambda_1, \lambda_2, 1) \otimes (l)) & c) \\ m_{\lambda, \mu}((\lambda_1, \lambda_2, 1) \otimes (\mu_1, \mu_2)) & d). \end{cases}$$

Firstly, we note that the cases $a), c), d)$ are similar to those computed for $(k_1, k_2)_{\downarrow H}$. Thus we have to argue only for the case $b)$. Suppose $\mu_1 - 1 \geq \mu_2$, then we have to add a final box $\boxed{1}$ or $\boxed{2}$ to λ if $\mu_1 - 1 \geq \mu_2$, finally we have to add all the remaining $\boxed{2}$ in the only possible way. If $\mu_1 - 1 < \mu_2$, we have to add only the final box $\boxed{2}$ to λ , finally we have to add all the remaining $\boxed{2}$ in the only possible way and we are done. \square

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