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# CERTAIN COMBINATORIAL TOPICS IN GROUP THEORY

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*In memory of Narain Gupta*

ABSTRACT. This article is intended to be a survey on some combinatorial topics in group theory. The bibliography at the end is neither claimed to be exhaustive, nor is it necessarily connected with a reference in the text. I include it as I see it revolves around the concepts which are discussed in the text.



## 1. Introduction

**Automorphisms of free groups.**

Let $F = F_n = \langle x_1, x_2, \ldots, x_n \rangle$, $n \geq 2$, be a free group of rank $n$ and let $\mathrm{Aut}(F)$ denote the group of all automorphisms of $F$. Here we begin with **Elementary Nielsen (transformations) automorphisms**

$$\{x_i \longrightarrow x_{i_\sigma}^{\pm 1},\ \sigma \text{ permutation of } \{1, 2, \ldots, n\}\};$$
$$\{x_i \longrightarrow x_i x_j^{\pm 1},\ x_k \longrightarrow x_k,\ k \neq i\}; \quad \{x_i \longrightarrow x_j^{\pm 1} x_i,\ x_k \longrightarrow x_k,\ k \neq i\};$$
$$\{x_i \longrightarrow x_j^{-1} x_i x_j,\ x_k \longrightarrow x_k,\ k \neq i\}; \quad \{x_i \longrightarrow x_j x_i x_j^{-1},\ x_k \longrightarrow x_k,\ k \neq i\}.$$

Every automorphism of the free group $F$ is a power product of these elementary Nielsen transformations. Next, we state a significant result of Nielsen.

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**Theorem 1.1** (**Nielsen,** [68]). Aut($F$) *can be generated by the following four elementary automorphisms:*

$$\tau_1 = \{x_1 \longrightarrow x_1^{-1},\ x_i \longrightarrow x_i,\ i \neq 1\}; \quad \tau_2 = \{x_1 \longrightarrow x_1x_2,\ x_i \longrightarrow x_i,\ i \neq 1\};$$
$$\tau_3 = \{x_1 \longrightarrow x_2,\ x_2 \longrightarrow x_1,\ x_i \longrightarrow x_i,\ i \neq 1,2\};$$
$$\tau_4 = \{x_1 \longrightarrow x_2,\ x_2 \longrightarrow x_3, \ldots, x_n \longrightarrow x_1\}.$$

If $n \geq 4$, then Aut($F$) can, in fact, be generated by a set of two automorphisms.

**Theorem 1.2** (**B. H. Neumann,** [67]). *If* $n = 4, 6, 8, \ldots,$ *then* $\mathrm{Aut}(F) = \langle \tau_4, \varphi \rangle$, *where*

$$\varphi\colon x_1 \longrightarrow x_2^{-1}, x_2 \longrightarrow x_1, x_3 \longrightarrow x_3, \ldots,$$
$$x_{n-2} \longrightarrow x_{n-2}, x_{n-1} \longrightarrow x_n x_{n-1}^{-1}, \quad x_n \longrightarrow x_{n-1}^{-1};$$

*if* $n = 5, 7, 9, \ldots,$ *then* $\mathrm{Aut}(F) = \langle \varphi, \chi \rangle$, *where* $\chi = \tau_4\tau_{1i}\tau_{2i}\cdots\tau_{ni}$, *and* $\tau_{ki}\colon x_k \to x_k^{-1}, x_i \to x_i,\ i \neq k$.

Let $F' = [F, F]$ denote the commutator subgroup of $F$. Consider the natural homomorphism $\alpha\colon \mathrm{Aut}(F) \to \mathrm{Aut}(F/F')$, the kernel of the homomorphism consists of all those automorphisms of $F$ which induce the identity automorphism on $F$ module $F'$. These are so-called IA-automorphisms of $F$. We denote by IA-Aut($F$) the subgroup of all IA-automorphisms of $F$. Elements of IA-Aut($F$) may be defined as

$$\alpha = \{x_1 \longrightarrow x_1d_1, x_2 \longrightarrow x_2d_2, \ldots, x_n \longrightarrow x_nd_n\},$$

where $d_i \in F'$ are such that $\{x_1d_1, x_2d_2, \ldots, x_nd_n\}$ is a basis of $F$.

Next, we mention an interesting result of Magnus.

**Theorem 1.3** (**W. Magnus,** [57]). IA-Aut($F_n$), $n \geq 2$, *is finitely generated. And*

$$\text{IA-Aut}(F_n) = \mathrm{sgp}\big\{\alpha_{ijk}\colon x_i \longrightarrow x_i[x_j, x_k], x_t \longrightarrow x_t, t \neq i$$
$$\textit{for all } i, j, k \mid j = i \textit{ or } i \notin \{j, k\} \textit{ and } j < k\big\}.$$

(Here, we write the commutator $[x_j, x_k] = x_j^{-1}x_k^{-1}x_jx_k$.)

An inner automorphism of $F$ is clearly an IA-automorphism. We denote by Inner-Aut($F$) = Inn-Aut($F$) the subgroup of Aut($F$) which consists of all inner automorphisms of $F$. The centre of $F$ is trivial, so Inn-Aut($F$) $\cong F$ and the following inclusions of normal subgroups of Aut($F$) are now clear:

$$F \cong \text{Inn-Aut}(F) \leq \text{IA-Aut}(F) \leq \mathrm{Aut}(F).$$

Nielsen [68] proved that if $F$ is a free group of rank 2, then IA-Aut($F$) = Inn-Aut($F$), see Lyndon and Schupp [56] for a proof. Formanek and Procesi [26] proved that Inn-Aut($F_n$), $n \geq 2$, is a unique normal free subgroup of rank $n$. Also, it is clear that $\mathrm{Aut}(F_n/F_n') \cong \mathrm{GL}(n, \mathbb{Z})$, $n \geq 2$, the group of $n \times n$ invertible matrices over the integers. Here $F_n' = [F_n, F_n]$ is the commutator subgroup of $F_n$.

We recall the question: What IA-endomorphisms of the free group $F_n = F = \langle x_1, x_2, \ldots, x_n \rangle$ ($\varphi\colon F_n \to F_n$) are automorphisms?

Nielsen [69] and Mal'tsev [60] have proved that if an endomorphism $\varphi$ of the free group $F_2$ fixes the commutator of a pair of generators, then $\varphi$ is an automorphism; Durnev [23] showed that if $\varphi$ fixes the commutator modulo $F_2''$ $(= [F_2', F_2'])$, then $\varphi$ defines an automorphism of $F_2$; Narain Gupta and Shpilrain [48] proved that if an endomorphism $\varphi$ of $F_2$ fixes the commutator modulo the derived series $\delta_k(F_2)$, $k \geq 3$, then $\varphi$ is not an automorphism. (Here, the Nielsen commutator test fails.)

## 2. Test words for automorphisms of free groups

**Definition.** A word $w \in F_n$ is a test word if any endomorphism fixing $w$ is necessarily an automorphism.

Here we begin with some known results: Zieschang [101] showed that if $\varphi$ is an endomorphism of the free group $F_n = F = \langle x_1, x_2, \ldots, x_n \rangle$, $n \geq 2$, then $\varphi$ is an automorphism if it fixes $x_1^k x_2^k \cdots x_n^k$ for some $k \geq 2$ or $[x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]$ if $n$ is even; Rips [77] and Shpilrain [90] showed that if $\varphi([x_1, x_2, \ldots, x_n]) = [x_1, x_2, \ldots, x_n]$, $n \geq 2$, then $\varphi$ is an automorphism; Dold [20] has found a series of test elements described in graph-theoretic methods; Edward Turner [97] proved that **test words in** $F_n$ are those words that are not contained in any proper retracts of $F_n$. A retraction $\rho\colon G \to G$ is a homomorphism of a group $G$ satisfying $\rho^2 = \rho$ and a retract is the image of a retraction, *i.e.*, $R = \rho(G)$ is a retract.

**Example.** Let $\varphi\colon F_2 \to F_2$ be an endomorphism defined by

$$\varphi(x_1) = x_1^2 x_2 x_1^{-1} x_2^{-1}, \quad \varphi(x_2) = 1.$$

Then $\varphi$ fixes $x_1^2 x_2 x_1^{-1} x_2^{-1}$ but $x_1^2 x_2 x_1^{-1} x_2^{-1}$ lies in a proper retract (namely, the image of $\varphi$). Therefore, $x_1^2 x_2 x_1^{-1} x_2^{-1}$ is not a test word, whereas one can check that $x_1 x_2 x_1^{-1} x_2^{-1}$ is a test word.

Test sets and test rank: Let $G = \langle x_1, \ldots, x_n \rangle$ be an $n$-generator group. A set of elements $\{g_1, \ldots, g_m\}$, $m \leq n$, is a test set of a group $G$ if whenever $\varphi(g_i) = g_i$, $i = 1, \ldots, m$ for some endomorphism $\varphi$ of $G$, then $\varphi$ must be an automorphism of $G$. The test rank of $G$ is the minimal cardinality of a test set.

## 3. Test ranks of free nilpotent groups

**Theorem 3.1** (**C. K. Gupta, Roman'kov, Timoshenko** [36]). *Let $N = N_{rc}$ be a free nilpotent group of rank $r \geq 2$, class $c \geq 2$. Then*

(1) $\mathrm{tr}(N) = 2$ *for $r$ odd and $c = 2$;*
(2) $\mathrm{tr}(N) = 1$ *in all other cases.*
(3) *An element $g \in N_{2q,2}$ is a test element if and only if it can be written as*

$$g = [x_1, x_2]^{t_1} \cdots [x_{2q-1}, x_{2q}]^{t_q}$$

*in some basis $x_1, x_2, \ldots, x_{2q}$, and $t_1, \ldots, t_q$ are non-zero integers.*

*Proof.* First we show that **if a test element** $g$ **exists in** $N_{r2}$**, then** $g \in N'_{r2}$**,** the commutator subgroup of $N_{r2}$. Suppose $g \notin N'_{r2}$, without loss of generality, assume $g = x_1^n u$, where

$$u = \prod_{i<j} [x_i, x_j]^{k_{ij}} \in N'_{r2}, \quad n \geq 1.$$

Let $\varphi$ be an endomorphism of $N_{r2}$ defined by

$$\begin{cases} x_1 \longrightarrow x_1 v, & v \in N'_{r2} \\ x_i \longrightarrow x_i^{n+1}, & i = 2, \ldots, r. \end{cases}$$

Applying $\varphi$ to $g$,

$$\begin{aligned} \varphi(g) = \varphi(x_1^n u) &= (x_1 v)^n \prod_{1<j} [x_1, x_j]^{k_{ij}(n+1)} \prod_{1<i<j<n} [x_1, x_j]^{k_{ij}(n+1)^2} \\ &= x_1^n v^n u w^n \end{aligned}$$

for some $w \in N'_{r2}$.

$$\begin{aligned} \varphi(g) = \varphi(x_1^n u) &= x_1^n v^n u w^n \\ &= x_1^n (w^{-1})^n u w^n \quad \text{let } v = w^{-1} \\ &= x_1^n u, \end{aligned}$$

$u \in N'_{r2}$ and $N_{r2}$ is nilpotent of class 2. Hence $\varphi(g) = g$. Obviously, $\varphi$ is not an automorphism of $N_{r2}$. Thus, $g$ is not a test element. So, we can assume that $g \in N' = N'_{rc}$. For instance, the set of elements $\{x_1, [x_2, x_3] \cdots [x_{r-1}, x_r]\}$ is a test set. Let

$$\alpha \colon \begin{cases} x_1 \longrightarrow y_1 \\ \cdots \\ x_r \longrightarrow y_r \end{cases}$$

be an endomorphism of $N$, $y_i = x_1^{\alpha_{i1}} \cdots x_r^{\alpha_{ir}} c_i$, $\alpha_{ij} \in \mathbb{Z}$, $c_i \in N'$. Let $A = (\alpha_{ij})$. An endomorphism $\alpha$ of $N$ is an automorphism if $\alpha$ induces an automorphism on $N/N'$ [66]. It turns out that the matrix $A = (\alpha_{ij})$ is invertible over $\mathbb{Z}$. Thus, $\alpha$ is an automorphism.

For instance, $r$ odd, $c \geq 3$, $N_{rc}$ contains a test element $g = \big[[x_2, x_3] \cdots [x_{r-1}, x_r], x_1\big]$. If $r = 2q$ even, $c \geq 3$, then $g = [x_1, x_2]^{t_1} \cdots [x_{2q-1}, x_{2q}]^{t_q}$ is a test element in $N$. For details, see our paper.

□

## 4. Test rank of a direct product $F_r \times A_n$

[$F_r = \langle x_1, x_2, \ldots, x_r \rangle$ free group of rank $r$; $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ free abelian group of rank $n$.]

It follows from the works of Zieschang, Rips, Rosenberger and Turner: A free group $F_r$ has test rank 1; a free abelian group $A_n$ has test rank $n$. Fine, Rosenberger, Spellman, Stille [25] have shown that for the given integers $n$ and $k$, there exists a group of rank $n$ with test rank $k$.

**Theorem 4.1** (**Fine, Rosenberger, Spellman, Stille** [25]). *Test rank of* $G = G_{rn} = F_r \times A_n$ *equals* $n+1$, *for* $r \geq 2$ *and* $n \geq 1$.

Something is wrong with the statement of the theorem. We have now proved the following.

**Theorem 4.2** (**C. K. Gupta, Roman'kov, Timoshenko** [36]). *The group* $G = F_r \times A_n = G_{rn}$ *has test rank* $n$, *where* $F_r$: *free group of rank* $r \geq 2$, $A_n$: *free abelian group of rank* $n \geq 1$.

*Proof.* Let $x_1, \ldots, x_r$ be a basis of $F_r$. Shpilrain [90] has shown that the commutator

$$w_1 = [\cdots[x_1, x_2], \ldots, x_r]$$

is a test element in $F_r$. He started with Nielsen's test element $[x_1, x_2]$ of $F_2$, and his proof used induction on $r$. Also, as proved by Fine, Rosenberger, Spellman, Stille [25], every commutator $[x_1^m, x_2]$, $m \geq 2$, is also a test element in $F_2$. We started with $[x_1^m, x_2]$ for $m = 2$, and using similar arguments as Shpilrain, we derive that the commutator

$$w_2 = [\cdots[x_1^2, x_2], \ldots, x_r]$$

is a test element in $F_r$. So, we have obtained $w_1, w_2$ two non-commuting test elements in $F_r$. Here, we have shown that the set of elements $g_1 = w_1 a_1$, $g_2 = w_2 a_2, \ldots, g_n = w_2 a_n$ is a test set for $G$ [here $a_1, \ldots, a_n$ is a basis of $A_n$]. That concludes the proof. □

## 5. Invertibility criterion for endomorphisms, test ranks of metabelian products of abelian groups

Here we begin with a definition.

**Definition.** An endomorphism of a group $G$ is an IA-endomorphism if it induces the identity map on $G/G'$.

Let $G = \prod * A_i$ be a free product of some non-trivial abelian groups $A_1, \ldots, A_n$. Let $A = G/G''$ be the metabelian product of groups $A_1, \ldots, A_n$. Denote $A = \prod_{i=1}^n *_{\mathfrak{A}^2} A_i$.

P. Ushakov studied IA-automorphisms of metabelian products of abelian groups. He considered an embedding of the IA-Aut $A$ group in a group of matrices over some ring for metabelian products $A$ of abelian torsion-free groups. He showed a stronger version of Bachmuth embedding. It follows from Ushakov's results: an IA-endomorphism $\varphi$ is an IA-Aut $A$ if and only if the determinant of the matrix $(D_i a_j)$ over the ring $\mathbb{Z}(A/A')$ can be written as

$$\det(D_i a_j) = \pm g(a_1 - 1) \cdots (a_n - 1),$$

where $g \in A/A'$, $B_i$ a basis of $A_i$ and $a_i \in B_i$.

**Here, we give necessary and sufficient conditions for an endomorphism of a metabelian product of free abelian groups to be an automorphism.**

**Theorem 5.1** (**C. K. Gupta and E. I. Timoshenko** [44]). *Let* $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$ *be a metabelian product of torsion-free abelian groups of finite ranks;* $B_i$*: a basis of* $A_i$*;* $a_i$ *some fixed element of* $B_i$*;* $\varphi$ *an endomorphism of* $A$ *and* $\varphi(a_i) = y_i$*.* $A_1, \dots, A_m$ *non-cyclic groups* ($m \le n$)*. Then* $\varphi \in \mathrm{Aut}(A)$ *if and only if*

(1) *the images of elements* $\varphi(b)$, $b \in \bigcup_{i=1}^{n} B_i$ *in the group* $A/A'$ *generate this group;*
(2) $\det(D_i y_j) = J_{n\times n}$ *over* $\mathbb{Z}(A/A')$ *can be written as*

$$\det J = \pm g(y_i - 1)\cdots(y_m - 1)(a_{m+1} - 1)\cdots(a_n - 1)$$

*for some* $g \in A/A'$.

Next we begin with a definition.

**Definition.** Let $G$ be a group and $g_1, \dots, g_n \in G$. The set of elements $\{g_1, \dots, g_m\}$ has the **property** $(*)$ if, whenever $\varphi(g_i) = g_i$, for $i = 1, \dots, m$, for some IA-endomorphism $\varphi$ of $G$, then this $\varphi$ is an automorphism of $G$.

Next, we state the following.

**Theorem 5.2** (**C. K. Gupta and E. I. Timoshenko** [44]). *Let* $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$ *be a metabelian product of torsion-free abelian groups of finite ranks,* $n \ge 2$*. Then*

(1) *there is a system* $\{g_1, \dots, g_{n-1}\}$ *of elements of the group* $A$ *with the property* $(*)$*;*
(2) *there exists no system of elements* $\{g_1, \dots, g_{n-2}\}$ *with the property* $(*)$*;*
(3) *a system of elements* $\{g_1, \dots, g_{n-1}\}$ *of the group* $A$ *has the property* $(*)$ *if and only if all elements belong to* $A'$ *and are independent over the ring* $\mathbb{Z}(A/A')$*.*

In the proof of (1), we demonstrate that the set of elements $[a_1, a_2], \dots, [a_1, a_n]$ has the property $(*)$, where $B_i$ is a basis of the group $A_i$, and $a_i$ some fixed element of $B_i$, $i = 1, \dots, n$. Let $\varphi \in \text{IA-End}(A)$ and let $\varphi([a_1, a_i]) = [a_1, a_i]$ for $i = 2, \dots, n$. Then it can be shown that the IA-endomorphism $\varphi$ induces the identity map on the commutator subgroup $A'$. From this we conclude that $\varphi$ is actually an automorphism of $A$. For the details of (2) and (3), see our paper.

Next, we have obtained the following:

**Theorem 5.3** (**C. K. Gupta and E. I. Timoshenko** [44]). *Let* $A = \prod_{i=1}^{n} *_{\mathfrak{A}^2} A_i$ *be a metabelian product of torsion-free abelian groups of finite ranks,* $n \ge 2$*. Then the test rank of the group* $A$ *equals* $n - 1$.

## 6. Test ranks for some free polynilpotent groups

**Theorem 6.1** (**C. K. Gupta and E. I. Timoshenko** [45]). *Let* $F = \langle x_1, \dots, x_r \rangle$ *be a free group. Let* $\mathbb{Z}(F/R)$ *be a domain, having trivial units only,* $R \le F'$*. If* $\{\widetilde{g_1}, \dots, \widetilde{g_m}\} \in F/R'$ *is a test collection,* *then for every solution* $\overline{\lambda_1}, \ldots, \overline{\lambda_m} \in \mathbb{Z}(F/R)$ *of the system of equations*

$$\partial_1 \widetilde{g_1} \cdot \overline{\lambda_1} + \cdots + \partial_r \widetilde{g_1} \cdot \overline{\lambda_r} = 0$$
$$\cdots$$
$$\partial_1 \widetilde{g_m} \cdot \overline{\lambda_1} + \cdots + \partial_r \widetilde{g_m} \cdot \overline{\lambda_r} = 0$$

*and for any* $\tilde{c} \in R/R'$*, the following condition should hold:*

$$\partial_1 \tilde{c} \cdot \overline{\lambda_1} + \cdots + \partial_r \tilde{c} \cdot \overline{\lambda_r} = 0.$$

**Proposition.** *Let* $F$ *be a free group with basis* $x_1, \ldots, x_r$*,* $\{1\} \neq R \trianglelefteq F$*,* $\tilde{c} \in R/R'$*,* $\mathbb{Z}(F/R)$ *be a domain,* $\overline{\lambda_i} \in \mathbb{Z}(F/R)$ *for* $i = 1, \ldots, r$*. Then the endomorphism*

$$\varphi\colon \begin{cases} \widetilde{x_1} \longrightarrow \tilde{c}^{\overline{\lambda_1}} \cdot \widetilde{x_1} \\ \cdots \\ \widetilde{x_r} \longrightarrow \tilde{c}^{\overline{\lambda_r}} \cdot \widetilde{x_r} \end{cases}$$

*of* $F/R'$ *will be an automorphism if and only if*

$$\partial_1 \tilde{c} \cdot \overline{\lambda_1} + \cdots + \partial_r \tilde{c} \cdot \overline{\lambda_r} + 1$$

*is an invertible element of the ring* $\mathbb{Z}(F/R)$*.*

**Theorem 6.2** (**C. K. Gupta and E. I. Timoshenko** [45])**.** *Let* $F = \langle x_1, \ldots, x_r \rangle$ *be a free group. Let* $\mathbb{Z}(F/R)$ *be a domain with trivial units only,* $R$ *a non-trivial verbal subgroup of* $F'$*, and* $\mathrm{tr}(F/R')$ *be the test rank. Then* $\mathrm{tr}(F/R')$ *equals* $r - 1$*, or* $r$*.*

From these theorems, we obtain the following:

**Corollary 1.** *For* $r \geq 2$ *and every collection* $(c_1, \ldots, c_l)$ *of classes, a free polynilpotent group* $F_r(\mathbb{A}\mathbb{N}_{c_1} \cdots \mathbb{N}_{c_l})$ *has its test rank equal to* $r - 1$*, or* $r$*.*

**Corollary 2.** *Let* $r \geq 2$*,* $c \geq 2$ *and* $F$ *be a free group of rank* $r$*. Then the test rank for a group* $F/[\gamma_c(F), \gamma_c(F)]$ *equals to* $r - 1$*.*

**Here, we mention Shmel'kin's theorems.**

**Theorem 6.3.** *Let* $F = \langle x_1, \ldots, x_r \rangle$ *be a free group of rank* $r \geq 2$*. Let* $R$ *be a verbal subgroup of* $F$ *and* $F/R$ *a free polynilpotent group. Assume endomorphism* $\varphi$ *of* $F/R'$ *acts identically on the subgroup* $A = R/R'$*. Then* $\varphi$ *is an inner automorphism of* $F/R'$ *induced by some element of* $A = R/R'$*.*

**Theorem 6.4.** *Let* $F = \langle x_1, x_2 \rangle$ *be a free group of rank* 2*. Let* $c \geq 2$*,* $t \geq 1$*,* $R = \gamma_c(F)$ *and* $\widetilde{F} = F/R'$*. Assume* $\widetilde{y_1}, \widetilde{y_2} \in \widetilde{F}$ *and* $[\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_2}] = [\widetilde{y_1}, \widetilde{y_2}, \ldots, \widetilde{y_2}]$ *of weight* $c + t$ *are equal. Then the map* $\varphi\colon \{\widetilde{x_i} \to \widetilde{y_i}, i = 1, 2\}$ *is an automorphism of* $\widetilde{F}$*.*

## 7. Generating elements of groups $F/R'$

**Theorem 7.1** (**C. K. Gupta and E. I. Timoshenko** [46]). *Let $g_1, \dots, g_m$ be elements of the group $F = A_1 * \cdots * A_n$ and $R \trianglelefteq F$; $R \cap A_i = \{1\}$ for $i = 1, \dots, n$; $A = F/R$, $T = t_1\mathbb{Z}A + \cdots + t_n\mathbb{Z}A$ is a right $\mathbb{Z}A$-module with basis $t_1, \dots, t_n$; $\tau_j = t_1D_1g_j + \cdots + t_nD_ng_j$; $L = t_1\triangle_1\mathbb{Z}A + \cdots + t_n\triangle_n\mathbb{Z}A$, where $\triangle_i$ is the augmentation ideal of $\mathbb{Z}A_i$. Then the elements $g_1, \dots, g_m$ generate the group $F/R'$ if and only if the elements $\tau_1, \dots, \tau_m$ generate a right $\mathbb{Z}A$-module $L$.*

**Corollary.** *Let $F_m$ be a free group of rank $m$ with basis $x_1, \dots, x_m$; $g_1, \dots, g_n \in F_m$, $n \geq m$; $R$ is a normal subgroup of $F_m$ with $R \cap \mathrm{gp}(x_i) = \{1\}$ for $i = 1, \dots, m$; $\mathbb{Z}(F_m/R)$ is a domain. Then the elements $g_1, \dots, g_n$ generate the group $F_m/R'$ if and only if for the matrix $J(g) = (\partial_i g_j)_{m\times n}$, there exists a matrix $B_{n\times m}$ over $\mathbb{Z}(F_m/R)$ for which $J(g) \cdot B = E_{m\times m}$ (identity matrix).*

The following is a stronger version of Birman's theorem.

**Corollary.** *Elements $g_1, \dots, g_n \in F_m$ $(n \geq m)$ of a free group $F_m$ of rank $m$ generate $F_m$ if and only if for the matrix $J(g) = (\partial_i g_j)_{m\times n}$ there exists a matrix $B_{n\times m}$ such that $J(g) \cdot B = E_{m\times m}$ (identity matrix).*

Next we obtain the following:

**Theorem 7.2** (**C. K. Gupta and E. I. Timoshenko** [46]). *Let $A_i$ $(i = 1, \dots, n)$ be free abelian groups of ranks $m_i$; $m = m_1 + \cdots + m_n$; $r \leq m$, $F = A_1 * \cdots * A_n$; $D$ is a Cartesian subgroup of $F$; $A = F/D$; $G = F/D'$; $\triangle_i$ is the augmentation ideal of the ring $\mathbb{Z}A_i$; $g_1, \dots, g_r \in G$; $T = t_1\mathbb{Z}A + \cdots + t_n\mathbb{Z}A$ is a free $\mathbb{Z}A$-module with basis $t_1, \dots, t_n$; $L = t_1\triangle_1\mathbb{Z}A + \cdots + t_n\triangle_n\mathbb{Z}A$; $\tau_j = t_1D_1g_j + \cdots + t_nD_ng_j \in L$, $j = 1, \dots, r$. Then there exist elements $g_{r+1}, \dots, g_m \in G$ such that $g_1, \dots, g_r, g_{r+1}, \dots, g_m$ generate the group $G$ if and only if there exist elements $\tau_{r+1}, \dots, \tau_m \in L$ such that the elements $\tau_1, \dots, \tau_r, \tau_{r+1}, \dots, \tau_m$ generate the $\mathbb{Z}A$-module $L$.*

The above theorem collects earlier results of Timoshenko, Roman'kov, Narain Gupta–Noskov–C. K. Gupta.

**Corollary.** *Let $S_n$ be a free metabelian group of rank $n$; $g_1, \dots, g_r \in S_n$ $(r \leq n)$. Then the elements $g_1, \dots, g_r$ can be included in a basis of $S_n$ if and only if the ideal generated by the $r \times r$ minors of the matrix $(\partial_i g_j)$ over the ring $\mathbb{Z}(S_n/S_n')$ coincides with the whole ring.*

## 8. Test ranks for certain solvable groups

**Theorem 8.1** (**C. K. Gupta and E. I. Timoshenko** [43]). *The test rank of solvable products of $m$ non-trivial free abelian groups $A_1, \dots, A_m$ of finite ranks is equal to $m - 1$ if we consider the variety $\mathfrak{A}^n$, $n \geq 2$, of solvable groups.*

This statement implies the following.

**Corollary 8.2.** *Let $G = \mathfrak{A}^2 \prod_{i=1}^m A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $m \geq 2$. Then the test rank of $G$ equals $m - 1$.*

**Corollary 8.3.** *The test rank of a free solvable group $F_r(\mathfrak{A}^n)$, $r \geq 2$, $n \geq 2$, is $r-1$.*

**Corollary 2 answers a question by Fine and Shpilrain** (unsolved problems in [62, problem 14.88]): **Does a free solvable group of rank 2 and class $n \geq 3$, possess test elements?** Here, we answer their question positively.

## 9. Torsion in factors of polynilpotent series

**A well-known result of Karrass, Magnus, Solitar** [51] **states:** The group is torsion-free if and only if the relation is not a proper power of any word $w$ in the group. Earlier, an analog with the Karrass, Magnus, Solitar theorem: Romanovskii [84] proved that the factors of the derived subgroup series of the group $G$ are torsion-free if and only if $r$ is not a proper power of any element of $F$ modulo $F^{(k+1)}$.

Here, we consider a more general situation: let $F = \langle x_1, \ldots, x_n \rangle$ be a free group. We consider in $F$ some polynilpotent series of subgroups

$$F = F_{11} \geq F_{12} \geq \cdots \geq F_{1,m_1+1} = F_{21} \geq \cdots \geq F_{2,m_2+1} = F_{31} \geq \cdots . \tag{9.1}$$

We define $F_{ij} = \gamma_j(F_{i1})$, $\gamma_j(F_{i1})$ is the $j$-th term of the lower central series of $F_{i1}$.

The group $F/F_{s,m_s+1}$ is a free group of the variety $\mathbf{N}_{m_1}\mathbf{N}_{m_2}\cdots\mathbf{N}_{m_s}$, where $\mathbf{N}_m$ is a variety of nilpotent groups of class $\leq m$. [The product variety $\mathbf{N}_{c_1}\mathbf{N}_{c_2}$ is the variety of all extensions of groups in $\mathbf{N}_{c_1}$ by groups in $\mathbf{N}_{c_2}$.]

Let $G = \langle x_1, \ldots, x_n \mid r \rangle$ be a group with a single defining relation. Denote by $G_{ij}$ a canonical image of $F_{ij}$ in $G$. We have a polynilpotent series in $G$:

$$\begin{aligned} G = G_{11} &\geq G_{12} \geq \cdots \geq G_{1,m_1+1} \\ &= G_{21} \geq \cdots \geq G_{2,m_2+1} = G_{31} \geq \cdots . \end{aligned} \tag{9.2}$$

We prove more generally the following.

**Theorem 9.1** (**C. K. Gupta and N. S. Romanovskii** [39]). *Let $G = F/r^F$ be a group with a single defining relation, $r \in F_{km} \setminus F_{k,m+1}$ $(m \leq m_k)$, $F_{ij}$ the term of some polynilpotent series of the free group $F$. We show that the factors of the corresponding polynilpotent series of the group $G$ are torsion-free if and only if $r$ is not a proper power of any element of $F$ $\bmod F_{k,m+1}$.*

Denote $\gamma_i(G)$ as the $i$-th term of the lower central series of a group $G$, *i.e.*, $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

We also give a description of the lower central series of a group $F/[R,R]$, when $F/R$ is a nilpotent group with torsion-free lower central factors. Earlier, Narain Gupta, Frank Levin, and C. K. Gupta [31] gave a description of the lower central series of a group $F/[R,R]$, when $F/R$ is a free nilpotent group. We make use of the Magnus Embedding in the proofs. We recall:

Let $F = \langle x_i \mid i \in I \rangle$ be a free group, $A = F/R$, $a_i$ canonical image of $x_i$ in $A$. Consider a right free $\mathbb{Z}A$-module $T$ with a basis $\{t_i \mid i \in I\}$, a group of matrices $\left(\begin{smallmatrix} A & 0 \\ T & 1 \end{smallmatrix}\right)$ and a group homomorphism

$$\varphi\colon F \longrightarrow \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} \quad \text{defined as} \quad x_i \longrightarrow \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix}.$$

Kernel of $\varphi = [R, R]$. Here $\varphi$ defines an embedding of the group $F/[R, R]$ into the group of matrices $\left(\begin{smallmatrix} A & 0 \\ T & 1 \end{smallmatrix}\right)$ **known as the Magnus embedding**. $R\varphi$ consists of matrices of the type $\left(\begin{smallmatrix} 1 & 0 \\ t_1u_1+\cdots+t_nu_n & 1 \end{smallmatrix}\right)$ for which $(a_1 - 1)u_1 + \cdots + (a_n = 1)u_n = 0$. It follows from the construction, replace the ring $\mathbb{Z}A$ by $(\mathbb{Z}/m\mathbb{Z})A$, then $\ker\varphi = R^m[R, R]$.

## 10. Word problem for certain polynilpotent groups

We write $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m; \mathfrak{M} \rangle$, where a group $G$ is presented in a variety $\mathfrak{M}$ by generating elements $x_1, \dots, x_n$ and by defining relations $r_1, \dots, r_m$ if it is a quotient group of a free group in the variety $\mathfrak{M}$ with basis $\{x_1, \dots, x_n\}$ w.r.t. a normal subgroup generated by $r_1, \dots, r_m$. In other words, all relations between $x_1, \dots, x_n$ in $G$ are consequences of the given defining relations and identities defining $\mathfrak{M}$.

For $k \geq 3$, there are examples of groups that are finitely presented in a variety $\mathbf{A}^k$ of soluble groups (here, $\mathbf{A}$ the variety of Abelian groups) that have an algorithmically undecidable word problem [72, 53]. $\mathbf{N}_k$: the variety of nilpotent groups of class at most $k$. Similar examples for varieties $\mathbf{N}_k\mathbf{A}$, with $k \geq 3$, are given by Epanchintsev and Kukin [24]. On the other hand, the word problem is decidable for groups that are finitely presented in varieties $\mathbf{N}_k$ (Mal'tsev, [60]), $\mathbb{A}^2$ (Romanovskii, [84]), and $\mathbb{N}_2\mathbf{A}$ (Kharlampovich, [52]).

**A question of Kargapolov (in** [62, Problem 3.16]**): Decide whether the word problem is decidable for groups with one defining relation in $\mathbb{A}^k$, $k \geq 3$, remaining open.**

**We define an element $r$ of a free soluble group $F$ of derived length $n$ is called primitive if $r \in F^{(i)} \setminus F^{(i+1)}$, in a series of derived subgroups, implies that $r$ modulo $F^{(i+1)}$ is not a proper power of another element.** Here we deal with a more general case—of a polynilpotent variety $\mathfrak{M} = \mathbf{N}_{ms} \cdots \mathbf{N}_{m1}$. We prove the following.

**Theorem 10.1** (**C. K. Gupta and N. S. Romanovskii** [38]). *Polynilpotent groups with a single primitive defining relation have a decidable word problem.*

**In the proof, essential use is made of earlier results** [39] **from our paper:** where factors of a polynilpotent series in a group $G$ are torsion-free.

## 11. Property of being equationally Noetherian

A group $A$ is equationally Noetherian if, for any $n$, every system of equations in $x_1, \dots, x_n$ with coefficients from $A$ is equivalent to some finite subsystem of a given system. The condition of being equationally Noetherian is equivalent to being Noetherian for the Zariski topology defined on a set

$A^n$, where as the subbase of a system of closed sets we take algebraic sets, that is, sets of solutions to systems of equations over the group $A$. More details about this fact can be found in papers by G. Baumslag, Myasnikov, Remeslennikov [5, 6, 7].

Every group representable by matrices over a commutative Noetherian unitary ring is equationally Noetherian (G. Baumslag, Myasnikov, Remeslennikov [6]). They conclude: free groups are equationally Noetherian. Every abelian group is also equationally Noetherian. R. Bryant [9] proved that a finitely generated group, which is an extension of an abelian by a nilpotent group, has this property.

We give two simple examples of groups which are not equationally Noetherian. The first example is nilpotent group of class 2 (not finitely generated); the second example is centre-by-metabelian group and 2-generated. It is worth mentioning another result of Baumslag, Myasnikov, Roman'kov [7] which states as follows: a wreath product of any non-abelian group and any infinite group is not an equationally Noetherian group.

Let $\mathfrak{B}$ be a class of groups $A$ which are soluble, equationally Noetherian, and have a central series as

$$A = A_1 \geq A_2 \geq \cdots \geq A_n \geq \cdots,$$

for which $\bigcap A_n = 1$ and all factors $A_n/A_{n+1}$ are torsion-free groups.

Here, we have proved the following.

**Theorem 11.1** (**C. K. Gupta and N. S. Romanovskii** [37]). *Let $D$ be a direct product of finitely many cyclic groups of infinite or prime orders, with $A \in \mathfrak{B}$. Then the wreath product $D \wr A$ is an equationally Noetherian group.*

Here, we have obtained the following two interesting corollaries.

**Corollary 11.2** (**C. K. Gupta and N. S. Romanovskii** [37]). *Let a group $A \in \mathfrak{B}$ be representable as a factor group $F/R$, where $F$ is a free group of finite rank. Then $F/[R,R] \in \mathfrak{B}$.*

**Corollary 11.3.** *Free soluble groups of arbitrary derived lengths and ranks are equationally Noetherian.*

Some remarks.

(a) The class of equationally Noetherian groups is closed w.r.t. subgroups and finite direct products.

(b) A group is equationally Noetherian if all of its countable subgroups have this property.

In [37], Romanovskii and I give two simple examples of groups which fail to be equationally Noetherian.

**Example 1.** Let a group $A$ in a variety of nilpotent groups of class 2 be given by the following generators and relations:

$$\begin{aligned} A = \langle a_1, a_2, \ldots, b_1, b_2, \cdots \mid [b_1, a_1] = 1, [b_2, a_1] = [b_2, a_2] = 1, \ldots, \\ [b_n, a_1] = [b_n, a_2] = \cdots = [b_n, a_n] = 1, [b_{n+1}, a_1] = [b_{n+1}, a_2] = \cdots \\ = [b_{n+1}, a_n] = [b_{n+1}, a_{n+1}] = 1, \ldots \rangle. \end{aligned}$$

Consider a system of equations in one variable such as

$$[x, a_1] = 1, \quad [x, a_2] = 1, \quad [x, a_3] = 1, \quad \dots;$$

this system is not equivalent to any one of its finite subsystems of the form

$$[x, a_1] = 1, \quad [x, a_2] = 1, \quad \dots, \quad [x, a_n] = 1.$$

Since $x = b_n$ is a solution to the subsystem, but $[b_n, a_{n+1}] \neq 1$. Therefore, the group $A$ is not equationally Noetherian.

**Example 2.** Consider a free centre-by-metabelian group of rank 2. Its derived subgroup is a free nilpotent group of class 2 of countable rank (C. K. Gupta [29]). We identify the basis of the derived subgroup with a set $\{a_1, a_2, \dots, b_1, b_2, \dots\}$, imposing on elements certain relations which the group $A$ in the previous example enjoyed. We obtain a 2-generated centre-by-metabelian group $C$, whose derived subgroup is equal to $A$. Thus the group $C$ is not equationally Noetherian, because its subgroup $A$ has this property.

## 12. Automorphisms of free groups of countable infinite rank

$F = F_\infty = \langle x_1, x_2, \dots \rangle$: **Methods of Nielsen and Whitehead do not reduce an infinite set of generators of $F$ in a finite number of steps.** D. H. Wagner extended Nielsen transformations to apply to infinite subsets of a group.

**Wagner's $\tau$-transformations: elementary simultaneous transformations, where generators of $\tau(x)$ have bounded lengths.**

Question (D. Solitar): **Is the group of all automorphisms of $F$ that are bounded relative to the given basis $\{x_i \mid i \in I\}$ generated by generalized elementary simultaneous Nielsen transformations?**

The following natural generalizations of the elementary Nielsen transformations:

(i) automorphisms permuting the $x_i$;

(ii) automorphisms inverting any subset of the $x_i$'s and leaving the remainder fixed;

(iii) automorphisms of the form: given any partition $I_1$ and $I_2$ (in $I$)

$$\{x_{i_1} \longrightarrow x_{i_1}x_{i_2} \quad i_1 \in I_1, i_2 \in I_2, \quad x_{i_2} \longrightarrow x_{i_2} \quad \text{for all } i_2 \in I_2\};$$

(iv)

$$\{x_{i_1} \longrightarrow x_{i_1}x_{i_2}^{\pm 1} \quad \text{or} \quad x_{i_1} \longrightarrow x_{i_2}^{\pm 1}x_{i_1}, \quad i_2 \in I_2\}.$$

**We call automorphisms of these four types generalized Nielsen elementary transformations.**

An automorphism $\varphi \in \mathrm{Aut}(F_\infty)$ is bounded if the lengths of the words $x_i\varphi$ and $x_i\varphi^{-1}$ are bounded: there is an $n$ such that $|x_i\varphi|, |x_i\varphi^{-1}| \leq n$ for all $i \in I$. (See also R. Cohen [18], Burns & Pi [15].) The problem of Solitar still remains open.

**Theorem 12.1 (C. K. Gupta and W. Holubowski [35]).** *We described a generating set for the group* $\operatorname{Aut} F_\infty$*. We show: the group of all automorphisms (modulo the* IA*-automorphisms) is generated by strings and the lower triangular automorphisms. Some new subgroups of* $\operatorname{Aut} F_\infty$ *are presented.*

## 13. Automorphisms of rooted trees

Here we prove the following:

**Theorem 13.1 (C. K. Gupta, N. D. Gupta, and A. S. Oliynyk [33]).** *Let a finite number of finite groups be given. Let* $n$ *be the largest order of their composition factors. We prove explicitly that the group of finite state automorphisms of rooted* $n$*-tree contains subgroups isomorphic to the free product of given groups.*

## 14. Universal theories for partially commutative groups

Here $\Gamma$ is a finite graph without loops, whose vertex set $\{x_1, \ldots, x_n\}$ is denoted by $X$, and the set of edges $(x_i, x_j)$ by $E$. For a graph $\Gamma$, a partially commutative group given by

$$G(c, \Gamma) = \langle x_1, \ldots, x_n \mid x_i x_j = x_j x_i \Longleftrightarrow (x_i, x_j) \in E; \quad \mathfrak{A}^2 \wedge \mathfrak{N}_c \rangle.$$

$\mathfrak{A}^2 \wedge \mathfrak{N}_c$: class of nilpotent metabelian groups.

We define certain transformations of a defining graph: we prove that these transformations do not change the universal theory of a partially commutative nilpotent metabelian group for each defining graph.

**Theorem 14.1 (C. K. Gupta and E. I. Timoshenko [40]).** *We give necessary and sufficient conditions for two partially commutative nilpotent metabelian groups defined by trees to be universally equivalent.*

**Theorem 14.2 (C. K. Gupta and E. I. Timoshenko [41]).** *Two partially commutative metabelian groups defined by cycles are universally equivalent if and only if the cycles are isomorphic.*

**Definition.** Two groups $G$ and $H$ are said to be universally equivalent if their universal theories coincide. That is, any $\exists$-formula is true on one of these groups if and only if it is true on the other.

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