

CERTAIN COMBINATORIAL TOPICS IN GROUP THEORY

C. K. GUPTA

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In memory of Narain Gupta

ABSTRACT. This article is intended to be a survey on some combinatorial topics in group theory. The bibliography at the end is neither claimed to be exhaustive, nor is it necessarily connected with a reference in the text. I include it as I see it revolves around the concepts which are discussed in the text.

1. Introduction

Automorphisms of free groups.

Let $F = F_n = \langle x_1, x_2, \dots, x_n \rangle$, $n \geq 2$, be a free group of rank n and let $\text{Aut}(F)$ denote the group of all automorphisms of F . Here we begin with **Elementary Nielsen (transformations) automorphisms**

$$\{x_i \longrightarrow x_{i\sigma}^{\pm 1}, \sigma \text{ permutation of } \{1, 2, \dots, n\}\};$$

$$\{x_i \longrightarrow x_i x_j^{\pm 1}, x_k \longrightarrow x_k, k \neq i\}; \quad \{x_i \longrightarrow x_j^{\pm 1} x_i, x_k \longrightarrow x_k, k \neq i\};$$

$$\{x_i \longrightarrow x_j^{-1} x_i x_j, x_k \longrightarrow x_k, k \neq i\}; \quad \{x_i \longrightarrow x_j x_i x_j^{-1}, x_k \longrightarrow x_k, k \neq i\}.$$

Every automorphism of the free group F is a power product of these elementary Nielsen transformations. Next, we state a significant result of Nielsen.

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Theorem 1.1 (Nielsen, [68]). $\text{Aut}(F)$ can be generated by the following four elementary automorphisms:

$$\begin{aligned}\tau_1 &= \{x_1 \longrightarrow x_1^{-1}, x_i \longrightarrow x_i, i \neq 1\}; \quad \tau_2 = \{x_1 \longrightarrow x_1x_2, x_i \longrightarrow x_i, i \neq 1\}; \\ \tau_3 &= \{x_1 \longrightarrow x_2, x_2 \longrightarrow x_1, x_i \longrightarrow x_i, i \neq 1, 2\}; \\ \tau_4 &= \{x_1 \longrightarrow x_2, x_2 \longrightarrow x_3, \dots, x_n \longrightarrow x_1\}.\end{aligned}$$

If $n \geq 4$, then $\text{Aut}(F)$ can, in fact, be generated by a set of two automorphisms.

Theorem 1.2 (B. H. Neumann, [67]). If $n = 4, 6, 8, \dots$, then $\text{Aut}(F) = \langle \tau_4, \varphi \rangle$, where

$$\begin{aligned}\varphi: x_1 &\longrightarrow x_2^{-1}, x_2 \longrightarrow x_1, x_3 \longrightarrow x_3, \dots, \\ x_{n-2} &\longrightarrow x_{n-2}, x_{n-1} \longrightarrow x_n x_{n-1}^{-1}, \quad x_n \longrightarrow x_{n-1}^{-1};\end{aligned}$$

if $n = 5, 7, 9, \dots$, then $\text{Aut}(F) = \langle \varphi, \chi \rangle$, where $\chi = \tau_4 \tau_{1i} \tau_{2i} \dots \tau_{ni}$, and $\tau_{ki}: x_k \longrightarrow x_k^{-1}, x_i \longrightarrow x_i, i \neq k$.

Let $F' = [F, F]$ denote the commutator subgroup of F . Consider the natural homomorphism $\alpha: \text{Aut}(F) \rightarrow \text{Aut}(F/F')$, the kernel of the homomorphism consists of all those automorphisms of F which induce the identity automorphism on F module F' . These are so-called IA-automorphisms of F . We denote by $\text{IA-Aut}(F)$ the subgroup of all IA-automorphisms of F . Elements of $\text{IA-Aut}(F)$ may be defined as

$$\alpha = \{x_1 \longrightarrow x_1 d_1, x_2 \longrightarrow x_2 d_2, \dots, x_n \longrightarrow x_n d_n\},$$

where $d_i \in F'$ are such that $\{x_1 d_1, x_2 d_2, \dots, x_n d_n\}$ is a basis of F .

Next, we mention an interesting result of Magnus.

Theorem 1.3 (W. Magnus, [57]). $\text{IA-Aut}(F_n)$, $n \geq 2$, is finitely generated. And

$$\begin{aligned}\text{IA-Aut}(F_n) &= \text{sgp}\{\alpha_{ijk}: x_i \longrightarrow x_i[x_j, x_k], x_t \longrightarrow x_t, t \neq i \\ &\quad \text{for all } i, j, k \mid j = i \text{ or } i \notin \{j, k\} \text{ and } j < k\}.\end{aligned}$$

(Here, we write the commutator $[x_j, x_k] = x_j^{-1} x_k^{-1} x_j x_k$.)

An inner automorphism of F is clearly an IA-automorphism. We denote by $\text{Inner-Aut}(F) = \text{Inn-Aut}(F)$ the subgroup of $\text{Aut}(F)$ which consists of all inner automorphisms of F . The centre of F is trivial, so $\text{Inn-Aut}(F) \cong F$ and the following inclusions of normal subgroups of $\text{Aut}(F)$ are now clear:

$$F \cong \text{Inn-Aut}(F) \leq \text{IA-Aut}(F) \leq \text{Aut}(F).$$

Nielsen [68] proved that if F is a free group of rank 2, then $\text{IA-Aut}(F) = \text{Inn-Aut}(F)$, see Lyndon and Schupp [56] for a proof. Formanek and Procesi [26] proved that $\text{Inn-Aut}(F_n)$, $n \geq 2$, is a unique normal free subgroup of rank n . Also, it is clear that $\text{Aut}(F_n/F'_n) \cong \text{GL}(n, \mathbb{Z})$, $n \geq 2$, the group of $n \times n$ invertible matrices over the integers. Here $F'_n = [F_n, F_n]$ is the commutator subgroup of F_n .

We recall the question: What IA-endomorphisms of the free group $F_n = F = \langle x_1, x_2, \dots, x_n \rangle$ ($\varphi: F_n \rightarrow F_n$) are automorphisms?

Nielsen [69] and Mal'tsev [60] have proved that if an endomorphism φ of the free group F_2 fixes the commutator of a pair of generators, then φ is an automorphism; Durnev [23] showed that if φ fixes the commutator modulo $F_2'' (= [F_2', F_2'])$, then φ defines an automorphism of F_2 ; Narain Gupta and Shpilrain [48] proved that if an endomorphism φ of F_2 fixes the commutator modulo the derived series $\delta_k(F_2)$, $k \geq 3$, then φ is not an automorphism. (Here, the Nielsen commutator test fails.)

2. Test words for automorphisms of free groups

Definition. A word $w \in F_n$ is a test word if any endomorphism fixing w is necessarily an automorphism.

Here we begin with some known results: Zieschang [101] showed that if φ is an endomorphism of the free group $F_n = F = \langle x_1, x_2, \dots, x_n \rangle$, $n \geq 2$, then φ is an automorphism if it fixes $x_1^k x_2^k \cdots x_n^k$ for some $k \geq 2$ or $[x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]$ if n is even; Rips [77] and Shpilrain [90] showed that if $\varphi([x_1, x_2, \dots, x_n]) = [x_1, x_2, \dots, x_n]$, $n \geq 2$, then φ is an automorphism; Dold [20] has found a series of test elements described in graph-theoretic methods; Edward Turner [97] proved that **test words** in F_n are those words that are not contained in any proper retracts of F_n . A retraction $\rho: G \rightarrow G$ is a homomorphism of a group G satisfying $\rho^2 = \rho$ and a retract is the image of a retraction, i.e., $R = \rho(G)$ is a retract.

Example. Let $\varphi: F_2 \rightarrow F_2$ be an endomorphism defined by

$$\varphi(x_1) = x_1^2 x_2 x_1^{-1} x_2^{-1}, \quad \varphi(x_2) = 1.$$

Then φ fixes $x_1^2 x_2 x_1^{-1} x_2^{-1}$ but $x_1^2 x_2 x_1^{-1} x_2^{-1}$ lies in a proper retract (namely, the image of φ). Therefore, $x_1^2 x_2 x_1^{-1} x_2^{-1}$ is not a test word, whereas one can check that $x_1 x_2 x_1^{-1} x_2^{-1}$ is a test word.

Test sets and test rank: Let $G = \langle x_1, \dots, x_n \rangle$ be an n -generator group. A set of elements $\{g_1, \dots, g_m\}$, $m \leq n$, is a test set of a group G if whenever $\varphi(g_i) = g_i$, $i = 1, \dots, m$ for some endomorphism φ of G , then φ must be an automorphism of G . The test rank of G is the minimal cardinality of a test set.

3. Test ranks of free nilpotent groups

Theorem 3.1 (C. K. Gupta, Roman'kov, Timoshenko [36]). *Let $N = N_{rc}$ be a free nilpotent group of rank $r \geq 2$, class $c \geq 2$. Then*

- (1) $\text{tr}(N) = 2$ for r odd and $c = 2$;
- (2) $\text{tr}(N) = 1$ in all other cases.
- (3) An element $g \in N_{2q,2}$ is a test element if and only if it can be written as

$$g = [x_1, x_2]^{t_1} \cdots [x_{2q-1}, x_{2q}]^{t_q}$$

in some basis x_1, x_2, \dots, x_{2q} , and t_1, \dots, t_q are non-zero integers.

Proof. First we show that **if a test element g exists in N_{r2} , then $g \in N'_{r2}$** , the commutator subgroup of N_{r2} . Suppose $g \notin N'_{r2}$, without loss of generality, assume $g = x_1^n u$, where

$$u = \prod_{i < j} [x_i, x_j]^{k_{ij}} \in N'_{r2}, \quad n \geq 1.$$

Let φ be an endomorphism of N_{r2} defined by

$$\begin{cases} x_1 \longrightarrow x_1 v, & v \in N'_{r2} \\ x_i \longrightarrow x_i^{n+1}, & i = 2, \dots, r. \end{cases}$$

Applying φ to g ,

$$\begin{aligned} \varphi(g) &= \varphi(x_1^n u) = (x_1 v)^n \prod_{1 < j} [x_1, x_j]^{k_{1j}(n+1)} \prod_{1 < i < j < n} [x_1, x_j]^{k_{ij}(n+1)^2} \\ &= x_1^n v^n u w^n \end{aligned}$$

for some $w \in N'_{r2}$.

$$\begin{aligned} \varphi(g) &= \varphi(x_1^n u) = x_1^n v^n u w^n \\ &= x_1^n (w^{-1})^n u w^n \quad \text{let } v = w^{-1} \\ &= x_1^n u, \end{aligned}$$

$u \in N'_{r2}$ and N_{r2} is nilpotent of class 2. Hence $\varphi(g) = g$. Obviously, φ is not an automorphism of N_{r2} . Thus, g is not a test element. So, we can assume that $g \in N' = N'_{rc}$. For instance, the set of elements $\{x_1, [x_2, x_3] \cdots [x_{r-1}, x_r]\}$ is a test set. Let

$$\alpha: \begin{cases} x_1 \longrightarrow y_1 \\ \dots \\ x_r \longrightarrow y_r \end{cases}$$

be an endomorphism of N , $y_i = x_1^{\alpha_{i1}} \cdots x_r^{\alpha_{ir}} c_i$, $\alpha_{ij} \in \mathbb{Z}$, $c_i \in N'$. Let $A = (\alpha_{ij})$. An endomorphism α of N is an automorphism if α induces an automorphism on N/N' [66]. It turns out that the matrix $A = (\alpha_{ij})$ is invertible over \mathbb{Z} . Thus, α is an automorphism.

For instance, r odd, $c \geq 3$, N_{rc} contains a test element $g = [[x_2, x_3] \cdots [x_{r-1}, x_r], x_1]$. If $r = 2q$ even, $c \geq 3$, then $g = [x_1, x_2]^{t_1} \cdots [x_{2q-1}, x_{2q}]^{t_q}$ is a test element in N . For details, see our paper. \square

4. Test rank of a direct product $F_r \times A_n$

$[F_r = \langle x_1, x_2, \dots, x_r \rangle$ free group of rank r ; $A_n = \langle a_1, a_2, \dots, a_n \rangle$ free abelian group of rank n .]

It follows from the works of Zieschang, Rips, Rosenberger and Turner: A free group F_r has test rank 1; a free abelian group A_n has test rank n . Fine, Rosenberger, Spellman, Stille [25] have shown that for the given integers n and k , there exists a group of rank n with test rank k .

Theorem 4.1 (Fine, Rosenberger, Spellman, Stille [25]). *Test rank of $G = G_{rn} = F_r \times A_n$ equals $n + 1$, for $r \geq 2$ and $n \geq 1$.*

Something is wrong with the statement of the theorem. We have now proved the following.

Theorem 4.2 (C. K. Gupta, Roman'kov, Timoshenko [36]). *The group $G = F_r \times A_n = G_{rn}$ has test rank n , where F_r : free group of rank $r \geq 2$, A_n : free abelian group of rank $n \geq 1$.*

Proof. Let x_1, \dots, x_r be a basis of F_r . Shpilrain [90] has shown that the commutator

$$w_1 = [\dots [x_1, x_2], \dots, x_r]$$

is a test element in F_r . He started with Nielsen's test element $[x_1, x_2]$ of F_2 , and his proof used induction on r . Also, as proved by Fine, Rosenberger, Spellman, Stille [25], every commutator $[x_1^m, x_2]$, $m \geq 2$, is also a test element in F_2 . We started with $[x_1^m, x_2]$ for $m = 2$, and using similar arguments as Shpilrain, we derive that the commutator

$$w_2 = [\dots [x_1^2, x_2], \dots, x_r]$$

is a test element in F_r . So, we have obtained w_1, w_2 two non-commuting test elements in F_r . Here, we have shown that the set of elements $g_1 = w_1 a_1, g_2 = w_2 a_2, \dots, g_n = w_2 a_n$ is a test set for G [here a_1, \dots, a_n is a basis of A_n]. That concludes the proof. \square

5. Invertibility criterion for endomorphisms, test ranks of metabelian products of abelian groups

Here we begin with a definition.

Definition. An endomorphism of a group G is an IA-endomorphism if it induces the identity map on G/G' .

Let $G = \prod *A_i$ be a free product of some non-trivial abelian groups A_1, \dots, A_n . Let $A = G/G''$ be the metabelian product of groups A_1, \dots, A_n . Denote $A = \prod_{i=1}^n *_{\mathfrak{A}^2} A_i$.

P. Ushakov studied IA-automorphisms of metabelian products of abelian groups. He considered an embedding of the IA-Aut A group in a group of matrices over some ring for metabelian products A of abelian torsion-free groups. He showed a stronger version of Bachmuth embedding. It follows from Ushakov's results: an IA-endomorphism φ is an IA-Aut A if and only if the determinant of the matrix $(D_i a_j)$ over the ring $\mathbb{Z}(A/A')$ can be written as

$$\det(D_i a_j) = \pm g(a_1 - 1) \cdots (a_n - 1),$$

where $g \in A/A'$, B_i a basis of A_i and $a_i \in B_i$.

Here, we give necessary and sufficient conditions for an endomorphism of a metabelian product of free abelian groups to be an automorphism.

Theorem 5.1 (C. K. Gupta and E. I. Timoshenko [44]). Let $A = \prod_{i=1}^n {}^*_{\mathfrak{A}^2} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks; B_i : a basis of A_i ; a_i some fixed element of B_i ; φ an endomorphism of A and $\varphi(a_i) = y_i$. A_1, \dots, A_m non-cyclic groups ($m \leq n$). Then $\varphi \in \text{Aut}(A)$ if and only if

- (1) the images of elements $\varphi(b)$, $b \in \bigcup_{i=1}^n B_i$ in the group A/A' generate this group;
- (2) $\det(D_i y_j) = J_{n \times n}$ over $\mathbb{Z}(A/A')$ can be written as

$$\det J = \pm g(y_i - 1) \cdots (y_m - 1)(a_{m+1} - 1) \cdots (a_n - 1)$$

for some $g \in A/A'$.

Next we begin with a definition.

Definition. Let G be a group and $g_1, \dots, g_n \in G$. The set of elements $\{g_1, \dots, g_m\}$ has the **property** (*) if, whenever $\varphi(g_i) = g_i$, for $i = 1, \dots, m$, for some IA-endomorphism φ of G , then this φ is an automorphism of G .

Next, we state the following.

Theorem 5.2 (C. K. Gupta and E. I. Timoshenko [44]). Let $A = \prod_{i=1}^n {}^*_{\mathfrak{A}^2} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $n \geq 2$. Then

- (1) there is a system $\{g_1, \dots, g_{n-1}\}$ of elements of the group A with the property (*);
- (2) there exists no system of elements $\{g_1, \dots, g_{n-2}\}$ with the property (*);
- (3) a system of elements $\{g_1, \dots, g_{n-1}\}$ of the group A has the property (*) if and only if all elements belong to A' and are independent over the ring $\mathbb{Z}(A/A')$.

In the proof of (1), we demonstrate that the set of elements $[a_1, a_2], \dots, [a_1, a_n]$ has the property (*), where B_i is a basis of the group A_i , and a_i some fixed element of B_i , $i = 1, \dots, n$. Let $\varphi \in \text{IA-End}(A)$ and let $\varphi([a_1, a_i]) = [a_1, a_i]$ for $i = 2, \dots, n$. Then it can be shown that the IA-endomorphism φ induces the identity map on the commutator subgroup A' . From this we conclude that φ is actually an automorphism of A . For the details of (2) and (3), see our paper.

Next, we have obtained the following:

Theorem 5.3 (C. K. Gupta and E. I. Timoshenko [44]). Let $A = \prod_{i=1}^n {}^*_{\mathfrak{A}^2} A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $n \geq 2$. Then the test rank of the group A equals $n - 1$.

6. Test ranks for some free polynilpotent groups

Theorem 6.1 (C. K. Gupta and E. I. Timoshenko [45]). Let $F = \langle x_1, \dots, x_r \rangle$ be a free group. Let $\mathbb{Z}(F/R)$ be a domain, having trivial units only, $R \leq F'$. If $\{\widetilde{g}_1, \dots, \widetilde{g}_m\} \in F/R'$ is a test collection,

then for every solution $\overline{\lambda}_1, \dots, \overline{\lambda}_m \in \mathbb{Z}(F/R)$ of the system of equations

$$\begin{aligned} \partial_1 \widetilde{g}_1 \cdot \overline{\lambda}_1 + \dots + \partial_r \widetilde{g}_1 \cdot \overline{\lambda}_r &= 0 \\ &\dots \\ \partial_1 \widetilde{g}_m \cdot \overline{\lambda}_1 + \dots + \partial_r \widetilde{g}_m \cdot \overline{\lambda}_r &= 0 \end{aligned}$$

and for any $\tilde{c} \in R/R'$, the following condition should hold:

$$\partial_1 \tilde{c} \cdot \overline{\lambda}_1 + \dots + \partial_r \tilde{c} \cdot \overline{\lambda}_r = 0.$$

Proposition. Let F be a free group with basis x_1, \dots, x_r , $\{1\} \neq R \trianglelefteq F$, $\tilde{c} \in R/R'$, $\mathbb{Z}(F/R)$ be a domain, $\overline{\lambda}_i \in \mathbb{Z}(F/R)$ for $i = 1, \dots, r$. Then the endomorphism

$$\varphi: \begin{cases} \widetilde{x}_1 \longrightarrow \tilde{c}^{\overline{\lambda}_1} \cdot \widetilde{x}_1 \\ \dots \\ \widetilde{x}_r \longrightarrow \tilde{c}^{\overline{\lambda}_r} \cdot \widetilde{x}_r \end{cases}$$

of F/R' will be an automorphism if and only if

$$\partial_1 \tilde{c} \cdot \overline{\lambda}_1 + \dots + \partial_r \tilde{c} \cdot \overline{\lambda}_r + 1$$

is an invertible element of the ring $\mathbb{Z}(F/R)$.

Theorem 6.2 (C. K. Gupta and E. I. Timoshenko [45]). Let $F = \langle x_1, \dots, x_r \rangle$ be a free group. Let $\mathbb{Z}(F/R)$ be a domain with trivial units only, R a non-trivial verbal subgroup of F' , and $\text{tr}(F/R')$ be the test rank. Then $\text{tr}(F/R')$ equals $r - 1$, or r .

From these theorems, we obtain the following:

Corollary 1. For $r \geq 2$ and every collection (c_1, \dots, c_l) of classes, a free polynilpotent group $F_r(\mathbb{A}N_{c_1} \cdots N_{c_l})$ has its test rank equal to $r - 1$, or r .

Corollary 2. Let $r \geq 2$, $c \geq 2$ and F be a free group of rank r . Then the test rank for a group $F/[\gamma_c(F), \gamma_c(F)]$ equals to $r - 1$.

Here, we mention Shmel'kin's theorems.

Theorem 6.3. Let $F = \langle x_1, \dots, x_r \rangle$ be a free group of rank $r \geq 2$. Let R be a verbal subgroup of F and F/R a free polynilpotent group. Assume endomorphism φ of F/R' acts identically on the subgroup $A = R/R'$. Then φ is an inner automorphism of F/R' induced by some element of $A = R/R'$.

Theorem 6.4. Let $F = \langle x_1, x_2 \rangle$ be a free group of rank 2. Let $c \geq 2$, $t \geq 1$, $R = \gamma_c(F)$ and $\tilde{F} = F/R'$. Assume $\widetilde{y}_1, \widetilde{y}_2 \in \tilde{F}$ and $[\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_2] = [\widetilde{y}_1, \widetilde{y}_2, \dots, \widetilde{y}_2]$ of weight $c + t$ are equal. Then the map $\varphi: \{\widetilde{x}_i \rightarrow \widetilde{y}_i, i = 1, 2\}$ is an automorphism of \tilde{F} .

7. Generating elements of groups F/R'

Theorem 7.1 (C. K. Gupta and E. I. Timoshenko [46]). *Let g_1, \dots, g_m be elements of the group $F = A_1 * \dots * A_n$ and $R \trianglelefteq F$; $R \cap A_i = \{1\}$ for $i = 1, \dots, n$; $A = F/R$, $T = t_1\mathbb{Z}A + \dots + t_n\mathbb{Z}A$ is a right $\mathbb{Z}A$ -module with basis t_1, \dots, t_n ; $\tau_j = t_1D_1g_j + \dots + t_nD_ng_j$; $L = t_1\Delta_1\mathbb{Z}A + \dots + t_n\Delta_n\mathbb{Z}A$, where Δ_i is the augmentation ideal of $\mathbb{Z}A_i$. Then the elements g_1, \dots, g_m generate the group F/R' if and only if the elements τ_1, \dots, τ_m generate a right $\mathbb{Z}A$ -module L .*

Corollary. *Let F_m be a free group of rank m with basis x_1, \dots, x_m ; $g_1, \dots, g_n \in F_m$, $n \geq m$; R is a normal subgroup of F_m with $R \cap \text{gp}(x_i) = \{1\}$ for $i = 1, \dots, m$; $\mathbb{Z}(F_m/R)$ is a domain. Then the elements g_1, \dots, g_n generate the group F_m/R' if and only if for the matrix $J(g) = (\partial_i g_j)_{m \times n}$, there exists a matrix $B_{n \times m}$ over $\mathbb{Z}(F_m/R)$ for which $J(g) \cdot B = E_{m \times m}$ (identity matrix).*

The following is a stronger version of Birman's theorem.

Corollary. *Elements $g_1, \dots, g_n \in F_m$ ($n \geq m$) of a free group F_m of rank m generate F_m if and only if for the matrix $J(g) = (\partial_i g_j)_{m \times n}$ there exists a matrix $B_{n \times m}$ such that $J(g) \cdot B = E_{m \times m}$ (identity matrix).*

Next we obtain the following:

Theorem 7.2 (C. K. Gupta and E. I. Timoshenko [46]). *Let A_i ($i = 1, \dots, n$) be free abelian groups of ranks m_i ; $m = m_1 + \dots + m_n$; $r \leq m$, $F = A_1 * \dots * A_n$; D is a Cartesian subgroup of F ; $A = F/D$; $G = F/D'$; Δ_i is the augmentation ideal of the ring $\mathbb{Z}A_i$; $g_1, \dots, g_r \in G$; $T = t_1\mathbb{Z}A + \dots + t_n\mathbb{Z}A$ is a free $\mathbb{Z}A$ -module with basis t_1, \dots, t_n ; $L = t_1\Delta_1\mathbb{Z}A + \dots + t_n\Delta_n\mathbb{Z}A$; $\tau_j = t_1D_1g_j + \dots + t_nD_ng_j \in L$, $j = 1, \dots, r$. Then there exist elements $g_{r+1}, \dots, g_m \in G$ such that $g_1, \dots, g_r, g_{r+1}, \dots, g_m$ generate the group G if and only if there exist elements $\tau_{r+1}, \dots, \tau_m \in L$ such that the elements $\tau_1, \dots, \tau_r, \tau_{r+1}, \dots, \tau_m$ generate the $\mathbb{Z}A$ -module L .*

The above theorem collects earlier results of Timoshenko, Roman'kov, Narain Gupta–Noskov–C. K. Gupta.

Corollary. *Let S_n be a free metabelian group of rank n ; $g_1, \dots, g_r \in S_n$ ($r \leq n$). Then the elements g_1, \dots, g_r can be included in a basis of S_n if and only if the ideal generated by the $r \times r$ minors of the matrix $(\partial_i g_j)$ over the ring $\mathbb{Z}(S_n/S'_n)$ coincides with the whole ring.*

8. Test ranks for certain solvable groups

Theorem 8.1 (C. K. Gupta and E. I. Timoshenko [43]). *The test rank of solvable products of m non-trivial free abelian groups A_1, \dots, A_m of finite ranks is equal to $m - 1$ if we consider the variety \mathfrak{A}^n , $n \geq 2$, of solvable groups.*

This statement implies the following.

Corollary 8.2. *Let $G = \mathfrak{A}^2 \prod_{i=1}^m A_i$ be a metabelian product of torsion-free abelian groups of finite ranks, $m \geq 2$. Then the test rank of G equals $m - 1$.*

Corollary 8.3. *The test rank of a free solvable group $F_r(\mathfrak{A}^n)$, $r \geq 2$, $n \geq 2$, is $r - 1$.*

Corollary 2 answers a question by **Fine and Shpilrain** (unsolved problems in [62, problem 14.88]): **Does a free solvable group of rank 2 and class $n \geq 3$, possess test elements?** Here, we answer their question positively.

9. Torsion in factors of polynilpotent series

A well-known result of Karrass, Magnus, Solitar [51] **states:** The group is torsion-free if and only if the relation is not a proper power of any word w in the group. Earlier, an analog with the Karrass, Magnus, Solitar theorem: Romanovskii [84] proved that the factors of the derived subgroup series of the group G are torsion-free if and only if r is not a proper power of any element of F modulo $F^{(k+1)}$.

Here, we consider a more general situation: let $F = \langle x_1, \dots, x_n \rangle$ be a free group. We consider in F some polynilpotent series of subgroups

$$(9.1) \quad F = F_{11} \geq F_{12} \geq \dots \geq F_{1,m_1+1} = F_{21} \geq \dots \geq F_{2,m_2+1} = F_{31} \geq \dots$$

We define $F_{ij} = \gamma_j(F_{i1})$, $\gamma_j(F_{i1})$ is the j -th term of the lower central series of F_{i1} .

The group $F/F_{s,m_s+1}$ is a free group of the variety $\mathbf{N}_{m_1}\mathbf{N}_{m_2}\dots\mathbf{N}_{m_s}$, where \mathbf{N}_m is a variety of nilpotent groups of class $\leq m$. [The product variety $\mathbf{N}_{c_1}\mathbf{N}_{c_2}$ is the variety of all extensions of groups in \mathbf{N}_{c_1} by groups in \mathbf{N}_{c_2} .]

Let $G = \langle x_1, \dots, x_n \mid r \rangle$ be a group with a single defining relation. Denote by G_{ij} a canonical image of F_{ij} in G . We have a polynilpotent series in G :

$$(9.2) \quad \begin{aligned} G &= G_{11} \geq G_{12} \geq \dots \geq G_{1,m_1+1} \\ &= G_{21} \geq \dots \geq G_{2,m_2+1} = G_{31} \geq \dots \end{aligned}$$

We prove more generally the following.

Theorem 9.1 (C. K. Gupta and N. S. Romanovskii [39]). *Let $G = F/r^F$ be a group with a single defining relation, $r \in F_{km} \setminus F_{k,m+1}$ ($m \leq m_k$), F_{ij} the term of some polynilpotent series of the free group F . We show that the factors of the corresponding polynilpotent series of the group G are torsion-free if and only if r is not a proper power of any element of $F \pmod{F_{k,m+1}}$.*

Denote $\gamma_i(G)$ as the i -th term of the lower central series of a group G , i.e., $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

We also give a description of the lower central series of a group $F/[R, R]$, when F/R is a nilpotent group with torsion-free lower central factors. Earlier, Narain Gupta, Frank Levin, and C. K. Gupta [31] gave a description of the lower central series of a group $F/[R, R]$, when F/R is a free nilpotent group. We make use of the Magnus Embedding in the proofs. We recall:

Let $F = \langle x_i \mid i \in I \rangle$ be a free group, $A = F/R$, a_i canonical image of x_i in A . Consider a right free $\mathbb{Z}A$ -module T with a basis $\{t_i \mid i \in I\}$, a group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$ and a group homomorphism

$$\varphi: F \longrightarrow \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} \quad \text{defined as} \quad x_i \longrightarrow \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix}.$$

Kernel of $\varphi = [R, R]$. Here φ defines an embedding of the group $F/[R, R]$ into the group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$ **known as the Magnus embedding**. $R\varphi$ consists of matrices of the type $\begin{pmatrix} 1 & 0 \\ t_1 u_1 + \dots + t_n u_n & 1 \end{pmatrix}$ for which $(a_1 - 1)u_1 + \dots + (a_n - 1)u_n = 0$. It follows from the construction, replace the ring $\mathbb{Z}A$ by $(\mathbb{Z}/m\mathbb{Z})A$, then $\ker \varphi = R^m[R, R]$.

10. Word problem for certain polynilpotent groups

We write $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m; \mathfrak{M} \rangle$, where a group G is presented in a variety \mathfrak{M} by generating elements x_1, \dots, x_n and by defining relations r_1, \dots, r_m if it is a quotient group of a free group in the variety \mathfrak{M} with basis $\{x_1, \dots, x_n\}$ w.r.t. a normal subgroup generated by r_1, \dots, r_m . In other words, all relations between x_1, \dots, x_n in G are consequences of the given defining relations and identities defining \mathfrak{M} .

For $k \geq 3$, there are examples of groups that are finitely presented in a variety \mathbf{A}^k of soluble groups (here, \mathbf{A} the variety of Abelian groups) that have an algorithmically undecidable word problem [72, 53]. \mathbf{N}_k : the variety of nilpotent groups of class at most k . Similar examples for varieties $\mathbf{N}_k \mathbf{A}$, with $k \geq 3$, are given by Epanchintsev and Kukin [24]. On the other hand, the word problem is decidable for groups that are finitely presented in varieties \mathbf{N}_k (Mal'tsev, [60]), \mathbf{A}^2 (Romanovskii, [84]), and $\mathbf{N}_2 \mathbf{A}$ (Kharlampovich, [52]).

A question of Kargapolov (in [62, Problem 3.16]): Decide whether the word problem is decidable for groups with one defining relation in \mathbf{A}^k , $k \geq 3$, remaining open.

We define an element r of a free soluble group F of derived length n is called **primitive** if $r \in F^{(i)} \setminus F^{(i+1)}$, in a series of derived subgroups, implies that r modulo $F^{(i+1)}$ is not a proper power of another element. Here we deal with a more general case—of a polynilpotent variety $\mathfrak{M} = \mathbf{N}_{m_s} \cdots \mathbf{N}_{m_1}$. We prove the following.

Theorem 10.1 (C. K. Gupta and N. S. Romanovskii [38]). *Polynilpotent groups with a single primitive defining relation have a decidable word problem.*

In the proof, essential use is made of earlier results [39] from our paper: where factors of a polynilpotent series in a group G are torsion-free.

11. Property of being equationally Noetherian

A group A is equationally Noetherian if, for any n , every system of equations in x_1, \dots, x_n with coefficients from A is equivalent to some finite subsystem of a given system. The condition of being equationally Noetherian is equivalent to being Noetherian for the Zariski topology defined on a set

A^n , where as the subbase of a system of closed sets we take algebraic sets, that is, sets of solutions to systems of equations over the group A . More details about this fact can be found in papers by G. Baumslag, Myasnikov, Remeslennikov [5, 6, 7].

Every group representable by matrices over a commutative Noetherian unitary ring is equationally Noetherian (G. Baumslag, Myasnikov, Remeslennikov [6]). They conclude: free groups are equationally Noetherian. Every abelian group is also equationally Noetherian. R. Bryant [9] proved that a finitely generated group, which is an extension of an abelian by a nilpotent group, has this property.

We give two simple examples of groups which are not equationally Noetherian. The first example is nilpotent group of class 2 (not finitely generated); the second example is centre-by-metabelian group and 2-generated. It is worth mentioning another result of Baumslag, Myasnikov, Roman'kov [7] which states as follows: a wreath product of any non-abelian group and any infinite group is not an equationally Noetherian group.

Let \mathfrak{B} be a class of groups A which are soluble, equationally Noetherian, and have a central series as

$$A = A_1 \geq A_2 \geq \cdots \geq A_n \geq \cdots,$$

for which $\bigcap A_n = 1$ and all factors A_n/A_{n+1} are torsion-free groups.

Here, we have proved the following.

Theorem 11.1 (C. K. Gupta and N. S. Romanovskii [37]). *Let D be a direct product of finitely many cyclic groups of infinite or prime orders, with $A \in \mathfrak{B}$. Then the wreath product $D \wr A$ is an equationally Noetherian group.*

Here, we have obtained the following two interesting corollaries.

Corollary 11.2 (C. K. Gupta and N. S. Romanovskii [37]). *Let a group $A \in \mathfrak{B}$ be representable as a factor group F/R , where F is a free group of finite rank. Then $F/[R, R] \in \mathfrak{B}$.*

Corollary 11.3. *Free soluble groups of arbitrary derived lengths and ranks are equationally Noetherian.*

Some remarks.

- (a) The class of equationally Noetherian groups is closed w.r.t. subgroups and finite direct products.
- (b) A group is equationally Noetherian if all of its countable subgroups have this property.

In [37], Romanovskii and I give two simple examples of groups which fail to be equationally Noetherian.

Example 1. Let a group A in a variety of nilpotent groups of class 2 be given by the following generators and relations:

$$\begin{aligned} A = \langle a_1, a_2, \dots, b_1, b_2, \dots \mid [b_1, a_1] = 1, [b_2, a_1] = [b_2, a_2] = 1, \dots, \\ [b_n, a_1] = [b_n, a_2] = \cdots = [b_n, a_n] = 1, [b_{n+1}, a_1] = [b_{n+1}, a_2] = \cdots \\ = [b_{n+1}, a_n] = [b_{n+1}, a_{n+1}] = 1, \dots \rangle. \end{aligned}$$

Consider a system of equations in one variable such as

$$[x, a_1] = 1, \quad [x, a_2] = 1, \quad [x, a_3] = 1, \quad \dots;$$

this system is not equivalent to any one of its finite subsystems of the form

$$[x, a_1] = 1, \quad [x, a_2] = 1, \quad \dots, \quad [x, a_n] = 1.$$

Since $x = b_n$ is a solution to the subsystem, but $[b_n, a_{n+1}] \neq 1$. Therefore, the group A is not equationally Noetherian.

Example 2. Consider a free centre-by-metabelian group of rank 2. Its derived subgroup is a free nilpotent group of class 2 of countable rank (C. K. Gupta [29]). We identify the basis of the derived subgroup with a set $\{a_1, a_2, \dots, b_1, b_2, \dots\}$, imposing on elements certain relations which the group A in the previous example enjoyed. We obtain a 2-generated centre-by-metabelian group C , whose derived subgroup is equal to A . Thus the group C is not equationally Noetherian, because its subgroup A has this property.

12. Automorphisms of free groups of countable infinite rank

$F = F_\infty = \langle x_1, x_2, \dots \rangle$: **Methods of Nielsen and Whitehead do not reduce an infinite set of generators of F in a finite number of steps.** D. H. Wagner extended Nielsen transformations to apply to infinite subsets of a group.

Wagner's τ -transformations: elementary simultaneous transformations, where generators of $\tau(x)$ have bounded lengths.

Question (D. Solitar): **Is the group of all automorphisms of F that are bounded relative to the given basis $\{x_i \mid i \in I\}$ generated by generalized elementary simultaneous Nielsen transformations?**

The following natural generalizations of the elementary Nielsen transformations:

- (i) automorphisms permuting the x_i ;
- (ii) automorphisms inverting any subset of the x_i 's and leaving the remainder fixed;
- (iii) automorphisms of the form: given any partition I_1 and I_2 (in I)

$$\{x_{i_1} \longrightarrow x_{i_1} x_{i_2} \quad i_1 \in I_1, i_2 \in I_2, \quad x_{i_2} \longrightarrow x_{i_2} \quad \text{for all } i_2 \in I_2\};$$

(iv)

$$\{x_{i_1} \longrightarrow x_{i_1} x_{i_2}^{\pm 1} \quad \text{or} \quad x_{i_1} \longrightarrow x_{i_2}^{\pm 1} x_{i_1}, \quad i_2 \in I_2\}.$$

We call automorphisms of these four types generalized Nielsen elementary transformations.

An automorphism $\varphi \in \text{Aut}(F_\infty)$ is bounded if the lengths of the words $x_i \varphi$ and $x_i \varphi^{-1}$ are bounded: there is an n such that $|x_i \varphi|, |x_i \varphi^{-1}| \leq n$ for all $i \in I$. (See also R. Cohen [18], Burns & Pi [15].) The problem of Solitar still remains open.

Theorem 12.1 (C. K. Gupta and W. Holubowski [35]). *We described a generating set for the group $\text{Aut } F_\infty$. We show: the group of all automorphisms (modulo the IA-automorphisms) is generated by strings and the lower triangular automorphisms. Some new subgroups of $\text{Aut } F_\infty$ are presented.*

13. Automorphisms of rooted trees

Here we prove the following:

Theorem 13.1 (C. K. Gupta, N. D. Gupta, and A. S. Oliynyk [33]). *Let a finite number of finite groups be given. Let n be the largest order of their composition factors. We prove explicitly that the group of finite state automorphisms of rooted n -tree contains subgroups isomorphic to the free product of given groups.*

14. Universal theories for partially commutative groups

Here Γ is a finite graph without loops, whose vertex set $\{x_1, \dots, x_n\}$ is denoted by X , and the set of edges (x_i, x_j) by E . For a graph Γ , a partially commutative group given by

$$G(c, \Gamma) = \langle x_1, \dots, x_n \mid x_i x_j = x_j x_i \iff (x_i, x_j) \in E; \quad \mathfrak{A}^2 \wedge \mathfrak{N}_c \rangle.$$

$\mathfrak{A}^2 \wedge \mathfrak{N}_c$: class of nilpotent metabelian groups.

We define certain transformations of a defining graph: we prove that these transformations do not change the universal theory of a partially commutative nilpotent metabelian group for each defining graph.

Theorem 14.1 (C. K. Gupta and E. I. Timoshenko [40]). *We give necessary and sufficient conditions for two partially commutative nilpotent metabelian groups defined by trees to be universally equivalent.*

Theorem 14.2 (C. K. Gupta and E. I. Timoshenko [41]). *Two partially commutative metabelian groups defined by cycles are universally equivalent if and only if the cycles are isomorphic.*

Definition. Two groups G and H are said to be universally equivalent if their universal theories coincide. That is, any \exists -formula is true on one of these groups if and only if it is true on the other.

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C. K. Gupta

Department of Mathematics, University of Manitoba, Winnipeg R3T 2N2, Canada

Email: cgupta@cc.umanitoba.ca