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LINEAR ANALOGUES OF THEOREMS OF SCHUR, BAER AND HALL

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For Rostislav Grigorchuk on the occasion of his 60th birthday

Abstract. A celebrated result of I. Schur asserts that the derived subgroup of a group is finite provided the group modulo its center is finite, a result that has been the source of many investigations within the Theory of Groups. In this paper we exhibit a similar result to Schur's Theorem for vector spaces, acted upon by certain groups. The proof of this analogous result depends on the characteristic of the underlying field. We also give linear versions of corresponding theorems of R. Baer and P. Hall.

1. Introduction

One of the classic results in the Theory of Groups is a theorem due to I. Schur [11], which establishes a connection between the central factor group $G/\zeta(G)$ of an arbitrary group G and the derived subgroup $[G,G]$ of G. As a consequence, one has that *if $G/\zeta(G)$ is finite then $[G,G]$ is finite.* This form of Schur's theorem has been the source of many investigations studying the relationship between $G/\zeta(G)$ and $[G,G]$. The following natural question arises:

- For which classes of groups $\mathfrak{X}$ does $G/\zeta(G) \in \mathfrak{X}$ always imply that $[G,G] \in \mathfrak{X}$?

A class of groups $\mathfrak{X}$ is called a *Schur class* if it satisfies this property. Thus Schur's theorem asserts that the class of finite groups is a Schur class and a straightforward consequence of this is that the class of locally finite groups is also a Schur class. On the other hand the class of periodic groups is not a Schur class. S. I. Adian [1] has constructed an example of a non-abelian torsion-free group G in which $G/\zeta(G)$ has prime exponent. Other examples of Schur classes are the following.

- The class of all polycyclic–by–finite groups.

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- The class of all Chernikov groups (see [7, Theorem 3.9]).
- The class of soluble–by–finite minimax groups ([5]).

To conclude this brief history the following result was recently established in [8].

Theorem 1.1. *Let G be a locally generalized radical group. If $G/\zeta(G)$ has finite special rank r, then there exists a function κ such that $[G,G]$ has special rank at most $\kappa(r)$.*

We recall that a group G is said to have *finite special rank* $r(G) = r$ if every finitely generated subgroup of G can be generated by at most r elements and there exists a finitely generated subgroup K which requires at least r generators.

In this paper we prove a linear analogue of Schur's theorem and certain connected theorems. To do this, we first define the linear analogues of the centre and the derived subgroup of a group. We give this definition in the general case of modules over group rings. If G is a group, R is a ring and A is an RG–module then let

$$\zeta_{RG}(A) = \{a \in A \mid a(g-1) = 0 \text{ for each element } g \in G\} = C_A(G).$$

Clearly $\zeta_{RG}(A)$ is an RG–submodule of A called *the RG–centre of A*. On the other hand, if ωRG is the augmentation ideal of the group ring RG, the two-sided ideal generated by the elements $g-1$, where $g \in G$, then the submodule $A(\omega RG)$ is called *the derived submodule of A*. In this paper we let F denote a field and A a vector space over F. We let $GL(F,A)$ denote the group of all non-singular F-linear transformations of A under composition. The following question arises:

- Let G be a subgroup of $GL(F,A)$. Suppose that $\zeta_{FG}(A)$ has finite codimension. Is $A(\omega FG)$ finite dimensional?

We remark that $A(\omega FG)$ is not totally analogous to the derived subgroup. If A is an RG-module, then there is a natural semidirect product $K = A \rtimes G$ so that $[K,K] = [A,G][G,G]$. Therefore we can expect that the answer to the above question in general is negative and the following simple example shows this.

Let A have countably infinite dimension over F and let $\{a_n \mid n \geq 1\}$ be a basis of A. For $k \geq 1$ define an F-automorphism g_k of A by

$$a_n g_k = \begin{cases} a_1 + a_k, & \text{if } n = 1; \\ a_n, & \text{if } n > 1. \end{cases}$$

and let $G = \langle g_k | k \in \mathbb{N}\rangle$. Clearly $G = \mathrm{Dr}_{k\geq 1}\langle g_k\rangle$. If F has characteristic 0, then G is a free abelian group, and if F has characteristic $p > 0$, then G is an elementary abelian p-group. It follows that $\zeta_{FG}(A)$ is the subspace generated by $\{a_n \mid n > 1\}$, so that the codimension $\mathrm{codim}_F \zeta_{FG}(A) = 1$. However, $A(\omega FG)$ is also the subspace generated by $\{a_n \mid n > 1\}$, so that $A(\omega FG)$ is infinite dimensional. This example shows that a more pertinent question is the following:

- Let G be a subgroup of $GL(F,A)$. Suppose that $\zeta_{FG}(A)$ has finite codimension. For which groups G does $A(\omega FG)$ have finite dimension?

It is easy to see that if G is a finite group then the answer to this question is positive, which suggests that the best candidates to study are groups satisfying some finiteness conditions. The example above suggests the following choice of these conditions. Let p be a prime. We say that a group G has finite section p-rank $r_p(G) = r$ if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$. Similarly, we say that a group G has finite section 0-rank $sr_0(G) = r$ if, for every torsion-free abelian section U/V of G, we have $r_{\mathbb{Z}}(U/V) \le r$ and there is an abelian torsion-free section A/B such that $r_{\mathbb{Z}}(A/B) = r$. Here $r_{\mathbb{Z}}(A)$ is the $\mathbb{Z}$-rank of the abelian group A, the rank of A as a $\mathbb{Z}$-module.

The main result of this paper is the following theorem.

Theorem A. *Let G be a subgroup of $GL(F, A)$. Suppose that $codim_F\, \zeta_{FG}(A) = c$ is finite. Then the following assertions hold.*

(A) *If $char(F) = 0$ and $sr_0(G) = r$ is finite, then $A(\omega FG)$ has finite dimension; and*

(B) *If $char(F) = p > 0$ and $r_p(G) = r$ is finite, then $A(\omega FG)$ has finite dimension.*

Moreover there exists a function κ such that $dim_F A(\omega FG) \le \kappa(c, r)$.

This means, under the given hypothesis, that there is an FG-submodule B of (c, r)-bounded dimension over F such that $A = \zeta_{FG}(A) + B$. Conversely, if there is such a submodule B then clearly Theorem A follows. We shall also obtain generalizations of analogous theorems due to R. Baer and P. Hall. We state these results here and discuss the notation in Section 4. First we give the analogue of Baer's theorem:

Theorem B. *Let G be a subgroup of $GL(F, A)$ and suppose there exists a natural number k such that $codim_F\, \zeta^k_{FG}(A) = c$ is finite. Then the following assertions hold.*

(i) *If $char(F) = 0$ and $sr_0(G) = r$ is finite, then $\gamma^{k+1}_{FG}(A)$ has finite dimension;*

(ii) *If $char(F) = p > 0$ and $r_p(G) = r$ is finite, then $\gamma^{k+1}_{FG}(A)$ has finite dimension.*

Moreover there exists a function λ such that $dim_F\, \gamma^{k+1}_{FG}(A) \le \lambda(c, r, k)$.

The version of Hall's theorem that we present is as follows.

Theorem C. *Let G be a subgroup of $GL(F,\ A)$ and suppose that $dim_F\, \gamma^{k+1}_{FG}(A) = c$ is finite for some natural number k.*

(i) *If $char(F) = 0$ and $sr_0(G) = r$ is finite, then $\zeta^k_{FG}(A)$ has finite codimension;*

(ii) *If $char(F) = p > 0$ and $r_p(G) = r$ is finite, then $\zeta^k_{FG}(A)$ has finite codimension.*

Moreover there exists a function β such that $codim_F\, \zeta^k_{FG}(A) \le \beta(c, r, k)$.

2. **Preliminary Results**

We start by calculating some related dimensions.

If G is a group, R is a ring and A is an RG-module, then for each $a \in A, g_1, \dots, g_n \in G$ it is well-known that

$$A(\omega R\langle g_1, \cdots, g_n\rangle) = A(g_1 - 1) + \cdots + A(g_n - 1).$$

Consequently we have

$$\dim_F A(\omega F\langle g_1, \cdots, g_n\rangle) \leq \dim_F A(g_1 - 1) + \cdots + \dim_F A(g_n - 1).$$

If $g \in G$ then we let $C_A(g) = \{a \in A | a \cdot g = a\}$. This makes our first result very easy to establish. We phrase it in such a way as to reflect a certain duality.

Lemma 2.1. *Let G be a subgroup of $GL(F, A)$. Suppose that $g_1, \dots, g_n$ are elements of G. Then $\mathrm{codim}_F\, C_A(g_j) = \dim_F A(g_j - 1)$. Furthermore*

(i) *If $\mathrm{codim}_F\, C_A(g_j) = c_j$ is finite for each j with $1 \leq j \leq n$, then $\dim_F A(\omega F\langle g_1, \dots, g_n\rangle) \leq c_1 + \cdots + c_n$;*

(ii) *If $\dim_F A(g_j - 1) = c_j$ is finite for each j with $1 \leq j \leq n$, then $\mathrm{codim}_F\, C_A(\langle g_1, \dots, g_n\rangle) \leq c_1 + \cdots + c_n$.*

Proof. For each j and each $a \in A$, the function $\xi_j : A \longrightarrow A$ defined by $a \longmapsto a(g_j - 1)$ is F-linear. We have $\mathrm{Ker}(\xi_j) = C_A(g_j)$ and $\mathrm{Im}(\xi_j) = A(g_j - 1)$, so that

$$A/C_A(g_j) = A/\mathrm{Ker}(\xi_j) \cong_F \mathrm{Im}(\xi_j) = A(g_j - 1).$$

The first of our claims is now immediate.

Also, it follows that if $\mathrm{codim}_F\, C_A(g_j) = c_j$ then

$$\begin{aligned}\dim_F A(\omega F\langle g_1, \dots, g_n\rangle) &\leq \dim_F A(g_1 - 1) + \cdots + \dim_F A(g_n - 1)\\ &= c_1 + \cdots + c_n.\end{aligned}$$

On the other hand let $\dim_F A(g_j - 1) = c_j$. Clearly $C_A(\langle g_1 \dots, g_n\rangle) = \bigcap_{1 \leq j \leq n} C_A(g_j)$ and the embedding

$$A/C_A(\langle g_1 \dots, g_n\rangle) \hookrightarrow A/C_A(g_1) \oplus \cdots \oplus A/C_A(g_n)$$

implies that $\mathrm{codim}_F\, C_A(\langle g_1, \dots, g_n\rangle) \leq c_1 + \cdots + c_n$. □

Since $\zeta_{FG}(A) \leq C_A(g)$ for all $g \in G$ it is easy to establish:

Corollary 2.2. *Let $G \leq GL(F, A)$ and let $g_1, \dots, g_n$ be elements of G.*

(i) *If $\mathrm{codim}_F\, \zeta_{FG}(A) = c$ is finite then $\dim_F A(\omega F\langle g_1, \dots, g_n\rangle) \leq nc$;*

(ii) *If $\dim_F A(\omega FG) = c$ is finite then $\mathrm{codim}_F\, C_A(\langle g_1, \dots, g_n\rangle) \leq nc$*

With slightly more effort the following result can also be proved.

Corollary 2.3. *Let $G \leq GL(F, A)$. Suppose that $K \leq G$ and $r(K) \leq r$.*

(i) *If $\mathrm{codim}_F\, \zeta_{FG}(A) = c$ is finite then $\dim_F A(\omega FK) \leq rc$;*

(ii) *If $\dim_F A(\omega FG) = c$ is finite then $\mathrm{codim}_F\, C_A(K) \leq rc$.*

Proof. Let $\mathcal{L}$ be the local family consisting of all finitely generated subgroups of K. If $L \in \mathcal{L}$, then L contains elements $x_1, \dots, x_m$ such that $m \le r$ and $L = \langle x_1, \cdots, x_m \rangle$.

(i) Corollary 2.2 shows that $\dim_F A(\omega FL) \le mc \le rc$. Choose a subgroup $V \in \mathcal{L}$ such that $\dim_F A(\omega FV)$ is maximal. If L is an arbitrary element of $\mathcal{L}$ then there exists a subgroup $U \in \mathcal{L}$ such that $L, V \le U$. It follows that $A(\omega FL), A(\omega FV) \le A(\omega FU)$. However the choice of V implies that $\dim_F A(\omega FU) = \dim_F A(\omega FV)$, and hence $A(\omega FU) = A(\omega FV)$. It follows that $A(\omega FL) \le A(\omega FV)$ for each $L \in \mathcal{L}$. Thus $A(\omega FK) = A(\omega FV)$, and then $\dim_F A(\omega FK) \le rc$.

(ii) The proof of this part of the result follows in a very similar fashion. □

We now deepen the study of the groups considered. The next two lemmas are slightly more general than we need.

Lemma 2.4. *Let G be a subgroup of $GL(F, A)$ and suppose that B is a G-invariant section of A of finite dimension over F. Suppose that char$(F) = p > 0$ and that $C_G(B)$ is an elementary abelian p-group. If G contains no non-cyclic free subgroups, then G has normal subgroups $P \le K$ such that:*

(i) *P is a bounded p–subgroup;*
(ii) *K/P is abelian; and*
(iii) *G/K is locally finite.*

Proof. Let $N = C_G(B)$, a normal subgroup of G and let $\dim_F B = c$. Then G/N is isomorphic to some subgroup of $GL(c, F)$. We claim that G/N has no non-cyclic free subgroups. For, if S/N is a non-cyclic free subgroup, then it is well-known that $S = N \leftthreetimes V$ (see [9, §52]) and we obtain the contradiction $V \cong S/N$. In this case, G/N has a soluble normal subgroup R/N such that G/R is locally finite (see [12, Corollary 10.17]).

The hypotheses imply that R is also soluble. Since R/N is a soluble subgroup of $GL(c, F)$, it follows that there are normal subgroups $P_0 \le K_0$ such that P_0/N is a bounded nilpotent p-subgroup, K_0/P_0 is abelian and R/K_0 is finite of order at most $\mu(c)$, where μ is the Maltsev function (see [12, Theorem 3.6]). Since P_0/N is normal in R/N, $P/N = P_0^G/N$ is a bounded nilpotent p-subgroup, by [12, Theorem 9.1]. Since N is an elementary abelian p–subgroup, P is a bounded p–subgroup. Since $P_0 \le P$, K_0P/P is abelian. Now let

$$K = \bigcap_{g \in G} (K_0P)^g.$$

Since $K \le K_0P$, K/P is also abelian. We have $R/K_0^g = R^g/K_0^g \cong R/K_0$ and so R/K_0^g is finite of order at most $\mu(c)$. From Remak's theorem we obtain the embedding

$$R/K \hookrightarrow \mathrm{Cr}_{g \in G} R/K_0^g.$$

Then the order of the elements of $\mathrm{Cr}_{g \in G} R/K_0^g$ divides $\mu(c)$. Hence R/K is a bounded soluble group so it is locally finite and likewise G/K is locally finite. The result holds. □

We next obtain an analogous result for the characteristic 0 case.

Lemma 2.5. *Let G be a subgroup of $GL(F, A)$ and suppose that B is a G-invariant section of A of finite dimension.. Suppose that char$(F) = 0$ and that $C_G(B)$ is a torsion-free abelian group. If G contains no non-cyclic free subgroups, then G has normal subgroups $K \leq R$ satisfying the following conditions:*

(i) *K is a torsion-free abelian subgroup;*
(ii) *R has a series of normal subgroups $K = K_1 \leq K_2 \leq R$ such that K_2/K_1 is torsion-free nilpotent of nilpotency class at most $c-1$ and R/K_2 is abelian; and*
(iii) *G/R is finite.*

Proof. Let $K = C_G(B)$. Then G/K is isomorphic to some subgroup of $GL(c, F)$. As in the proof of Lemma 2.4 we may prove that G/K also contains no non-cyclic free subgroups. Then G/K contains a soluble normal subgroup S/K such that G/S is finite, by [12, Corollary 10.17]). Since K is torsion-free abelian it follows that S is soluble. Since S/K is a soluble subgroup of $GL(c, F)$ we see, using [12, Theorem 3.6], that S contains normal subgroups $H_2 \leq H_3$ containing K such that H_2/K is a torsion-free nilpotent group of nilpotency class at most $c-1$, H_3/H_2 is abelian and S/H_3 is finite of order at most $\mu(c)$. Since $|G : H_3|$ is finite we let R be the core of K_3 in G. Then $|G : R|$ is also finite and we let $K_2 = H_2 \cap R$. It is easy to see that the desired result now holds. □

As usual, if G is a group, we let $\Pi(G)$ be the set of primes occurring as divisors of the orders of the periodic elements of G.

Lemma 2.6. *Let F be a field, G be a locally finite group and A be an FG–module such that codim$_F\, \zeta_{FG}(A)$ or dim$_F A(\omega FG)$ is finite. If char$(F) \notin \Pi(G)$. Then $A = \zeta_{FG}(A) \bigoplus A(\omega FG)$.*

Proof. Suppose first that $\operatorname{codim}_F \zeta_{FG}(A) = c$ is finite. Let $\mathcal{L}$ be the local family consisting of all finite subgroups of G. If $K \in \mathcal{L}$, then $A = \zeta_{FK}(A) \oplus A(\omega FK)$, by [7, Corollary 5.16]. Clearly $\zeta_{FG}(A) \leq \zeta_{FK}(A)$, which implies that $\dim_F A(\omega FK) \leq c$. Let V be a finite subgroup of G such that $\dim_F A(\omega FV)$ is maximal. If $S \in \mathcal{L}$ then there exists a finite subgroup $W \in \mathcal{L}$ such that $\langle S, V\rangle \leq W$, so $A(\omega FS) \leq A(\omega FW)$ and $A(\omega FV) \leq A(\omega FW)$. The choice of V implies that $\dim_F A(\omega FV) = \dim_F A(\omega FW)$ and since $V \leq W$ we have $A(\omega FV) = A(\omega FW)$. Thus $A(\omega FS) \leq A(\omega FV)$ and since S is an arbitrary finite subgroup of G, we have $A(\omega FV) = A(\omega FG)$. Then

$$A = \zeta_{FV}(A) \oplus A(\omega FV) = \zeta_{FV}(A) \oplus A(\omega FG) = \zeta_{FG}(A) \oplus A(\omega FG).$$

This latter equality follows since $\operatorname{codim}_F \zeta_{FV}(A) = c = \operatorname{codim}_F \zeta_{FG}(A)$.

Finally we note that a similar proof works when $\dim_F A(\omega FG) = c$ and this is omitted. □

We need some information concerning the relationship between the section p-rank and the special rank of a group. Clearly, if G has finite special rank r, then G has finite section p-rank for all primes p, and $r_p(G) \leq r$ for all primes p. It is easy to exhibit examples of groups G for which $r_p(G)$ is finite for all primes p, but for which $r(G)$ is infinite. Of course even for finite groups the two invariants do not coincide in general. However for some classes of groups the section p-rank and the rank do coincide and we now exhibit two such classes. We let $\mathrm{Frat}(G)$ denote the Frattini subgroup of the group G.

Lemma 2.7. *Let p be a prime and G be a finite p–group. Then $r_p(G) = r(G)$.*

Proof. We noted above that $r_p(G) \leq r(G)$. Let $r_p(G) = s$ and let K be an arbitrary subgroup of G. Then $K/\mathrm{Frat}(K)$ is elementary abelian, so that $|K/\mathrm{Frat}(K)| \leq p^s$. However the number of generators of K coincides with the number of generators of $K/\mathrm{Frat}(K)$ and hence K has at most s generators. Therefore $r(G) \leq s = r_p(G)$, which proves that $r_p(G) = r(G)$. □

Corollary 2.8. *Let p be a prime and let G be a locally finite p–group. Then $r_p(G)$ is finite if and only if $r(G)$ is finite, and in this case $r_p(G) = r(G)$.*

Proof. We remarked earlier that if $r(G)$ is finite, then $r_p(G)$ is finite and $r_p(G) \leq r(G)$. Suppose that $r_p(G) = s$ is finite and let K be an arbitrary finite subgroup of G. Then $r_p(K) \leq s$. By Lemma 2.7 $r(K) = r_p(K) \leq s$ and since this is true for all finite subgroups we have $r(G) \leq r_p(G)$. □

Of course this result is not true for p-groups in general.

Lemma 2.9. *Let A have dimension n over F and let G be a periodic abelian subgroup of $GL_n(F)$. If $char(F) \notin \Pi(G)$, then $r(G) \leq n$.*

Proof. By [12, Corollary 1.6], there is an integer $t \leq n$ such that

$$A = \bigoplus_{1\leq j\leq t} A_j,$$

where each summand A_j is a simple FG-submodule. Also $G/C_G(A_j)$ is a locally cyclic group, by [6, Corollary 2.4]. Since $1 = C_G(A) = \bigcap_{1\leq j\leq t} C_G(A_j)$ Remak's theorem implies that

$$G \hookrightarrow G/C_G(A_1) \times \cdots \times G/C_G(A_t),$$

which shows that G has special rank at most t. □

3. Proof of Theorem A

In this section we prove the two separate results below, Theorems 3.1 and 3.2, which are dependent upon the characteristic of the field, but which combined yield Theorem A. We note that when $B = A(\omega FG)$ or $B = A/\zeta_{FG}(A)$ then $C_G(B)$ is an elementary abelian p-group when F has characteristic p and is torsion-free abelian when F has characteristic 0. We shall apply Lemmas 2.4 and 2.5 in these special cases.

Theorem 3.1. *Let F be a field of prime characteristic p and let G be a subgroup of $GL(F, A)$. Suppose that $codim_F\, \zeta_{FG}(A) = c$ is finite. If $r_p(G) = r$ is finite, then $A(\omega FG)$ has finite dimension and there exists a function κ such that $dim_F A(\omega FG) \leq \kappa(c, r)$.*

Proof. Suppose that G has a non-cyclic free subgroup. Then G must have a non-cyclic free subgroup S of free rank 2 and $S_1 = [S, S]$ is a free group of countable free rank (see [9, §36]). Then $U = S_1/[S_1, S_1]$ is a free abelian group of countably infinite $\mathbb{Z}$–rank. It follows that U/U^p is an infinite elementary abelian p–group, and we obtain a contradiction, which shows that G contains no non-cyclic free subgroups.

By Lemma 2.4, with $B = A/\zeta_{FG}(A)$, G has normal subgroups $P \leq K$ such that P is a bounded p–subgroup, K/P is abelian and G/K is locally finite. There is no loss of generality if we suppose that the torsion subgroup $\mathrm{Tor}(K/P)$ is a p'–group. Let $C = \zeta_{FG}(A)$. Since G is soluble-by-locally finite, the periodic subgroups of G are locally finite. Using Corollary 2.8 we deduce that P has finite special rank at most r. Then Corollary 2.3 shows that $\dim_F A(\omega FP) \leq rc$. Let $A_1 = A(\omega FP)$, and $T/P = \mathrm{Tor}(K/P)$. Then $P \leq C_G(A/A_1)$ and from Lemma 2.6 we deduce that

$$A/A_1 = \zeta_{FT}(A/A_1) \oplus (A/A_1)(\omega FT).$$

Next let $A_2/A_1 = (A/A_1)(\omega FT)$. Then $T \leq C_G(A/A_2)$ and since $(\zeta_{FG}(A) + A_1)/A_1 \leq \zeta_{FT}(A/A_1)$, we have $\dim_F(A_2/A_1) \leq c$, so that

$$\dim_F A_2 \leq rc + c = (r+1)c.$$

The factor group K/T is torsion-free abelian and we let

$$V/T = \mathrm{Dr}_{\lambda\in\Lambda}\langle v_\lambda\rangle$$

be a free abelian subgroup of K/T such that K/V is periodic. Then $r_0(K/T) = r_0(V/T)$ and $(V/T)^p = \mathrm{Dr}_{\lambda\in\Lambda}\langle v_\lambda^p\rangle$. Since $r_p(G) = r$ we have $|\Lambda| \leq r$ so that $r_0(K/T) \leq r$. Since K/T is torsion-free, the 0-rank of K/T coincides with its special rank and hence $r(K/T) \leq r$. Then Corollary 2.3 shows that

$$\dim_F (A/A_2)(\omega FK) \leq rc.$$

Put $A_3/A_2 = (A/A_2)(\omega FK)$, so that $K \leq C_G(A/A_3)$. We note that

$$\dim_F(A_3) \leq rc + (r+1)c = (r+2)c.$$

Let W/K be the kernel of the action of the locally finite group G/K on A/A_3 and let Q/W be an arbitrary q–subgroup of G/W, where q is a prime. If $q = p$, then by Corollary 2.8, $r(Q/W) = r_p(Q/W)$ and hence $r(Q/W) \leq r$. Suppose that $q \neq p$. Since

$$(\zeta_{FG}(A) + A_3)/A_3) \leq \zeta_{FG}(A/A_3)$$

we have $\mathrm{codim}_F\, \zeta_{FG}(A/A_3) \leq c$. Since $\mathrm{char}(F) = p$, it is easy to see that Q/W is isomorphic to a subgroup of $GL(c, F)$. Then Lemma 2.9 shows that every abelian subgroup of Q/W has special rank at most c. It follows that

$$r(Q/W) \leq \frac{c(5c+1)}{2},$$

by [10]. Then an application of the main result of the paper [3] proves that G/W has special rank at most

$$\frac{c(5c+1)}{2} + r + 1.$$

Finally, using Corollary 2.3, we deduce that $\dim_F (A/A_3)(\omega FG) \le (c(5c+1)/2 + r + 1)c$. Since $\dim_F A_3 \le c(r+2)$, it follows that

$$\dim_F A(\omega FG) \le \left(\frac{c(5c+1)}{2} + r + 1\right) c + c(r+2) = \frac{5c^3 + c^2 + 4rc + 6c}{2}.$$

Hence here we define

$$\kappa(c, r) = \frac{5c^3 + c^2 + 4rc + 6c}{2}.$$

□

Theorem 3.2. *Let F be a field of characteristic 0, and let G be a subgroup of $GL(F, A)$. Suppose that $codim_F\, \zeta_{FG}(A) = c$ is finite. If $sr_0(G) = r$ is finite, then $A(\omega FG)$ has finite dimension and there exists a function κ such that $dim_F\, A(\omega FG) \le \kappa(c, r)$.*

Proof. As in the proof of Theorem 3.1, G contains no non-cyclic free subgroups. Lemma 2.5 applies, setting $B = A/\zeta_{FG}(A)$, and we let $K = K_1 \le K_2 \le R$ be the subgroups of that Lemma. Let S/K_2 be a free abelian subgroup of R/K_2 such that R/S is periodic (and therefore locally finite). In a torsion-free abelian group the $\mathbb{Z}$-rank coincides with the special rank. From this it is easy to see that S has rank at most $r(c+1)$. Hence Corollary 2.3 implies that $\dim_F A(\omega FS) \le rc(c+1)$. Let $B = A(\omega FS)$. Since $S \le C_G(A/B)$ and R/S is locally finite, Lemma 2.6 shows that

$$A/B = \zeta_{FR}(A/B) \bigoplus (A/B)(\omega FR)$$

Put $C/B = (A/B)(\omega FR)$ so that $R \le C_G(A/C)$. Since $(\zeta_{FG}(A) + B)/B \le \zeta_{FR}(A/B)$, $\dim_F C/B \le c$ and so

$$\dim_F C \le rc(c+1) + c.$$

Since R is normal in G, $D = A(\omega FR)$ is an FG–module and clearly $D \le C$. Since G/R is finite, another application of Lemma 2.6 shows that

$$A/D = \zeta_{FG}(A/D) \oplus (A/D)(\omega FG).$$

Therefore

$$\dim_F A(\omega FG) \le c + \dim_F D \le c + rc(c+1) + c.$$

Hence here

$$\kappa(c, r) = c^2 r + cr + 2c.$$

□

4. Some Related Results

If R is a ring, G is a group and A is an RG-module we can construct the upper RG-central series of A. We let $\zeta^0_{RG}(A) = 0, \zeta^1_{RG}(A) = \zeta_{RG}(A)$ and for all ordinals α we set $\zeta^{\alpha+1}_{RG}(A)/\zeta^{\alpha}_{RG}(A) = \zeta_{RG}(A/\zeta^{\alpha}_{RG}(A))$, where, as usual, we write $\zeta^{\lambda}_{RG}(A) = \bigcup_{\mu<\lambda} \zeta^{\mu}_{RG}(A)$, for all limit ordinals λ. Thus we obtain the series

$$0 = \zeta^0_{RG}(A) \le \zeta^1_{RG}(A) \le \zeta^2_{RG}(A) \le \cdots \le \zeta^{\alpha}_{RG}(A) \le \zeta^{\alpha+1}_{RG} \le \cdots \le \zeta^{\gamma}_{RG}(A)$$

The last term $\zeta_{RG}^{\gamma}(A)$ of this series is called the upper RG-hypercentre of A. We observe that $\zeta_{RG}^{\alpha+1}(A)(\omega RG) \leq \zeta_{RG}^{\alpha}(A)$ for all $\alpha < \gamma$. Clearly $\zeta_{RG}^{\alpha}(A)$ is analogous to the corresponding term $\zeta_{\alpha}(G)$ of the upper central series of an arbitrary group G.

If the upper RG-hypercentre of A coincides with A, then A is called RG-hypercentral, unless γ is finite in which case we say that A is RG-nilpotent.

The lower RG-central series of A can also be defined. We let $A = \gamma_{RG}^{1}(A)$ and $\gamma_{RG}^{2}(A) = A(\omega RG)$. Then we let $\gamma_{RG}^{\alpha+1}(A) = \gamma_{RG}^{\alpha}(A)(\omega RG)$ for all ordinals α and, as usual, $\gamma_{RG}^{\lambda}(A) = \bigcap_{\mu<\lambda} \gamma_{RG}^{\mu}(A)$, for limit ordinals λ. This gives the series

$$A = \gamma_{RG}^{1}(A) \geq \gamma_{RG}^{2}(A) \geq \cdots \geq \gamma_{RG}^{\alpha}(A) \leq \gamma_{RG}^{\alpha+1}(A) \geq \cdots$$

Clearly $\gamma_{RG}^{\alpha}(A)$ is analogous to $\gamma_{\alpha}(G)$, the corresponding term of the lower central series of an arbitary group G.

R. Baer [2] generalized Schur's theorem, for arbitrary groups G, by proving that if $G/\zeta_k(G)$ is finite then $\gamma_{k+1}(G)$ is finite. The linear analogue of Baer's theorem is our Theorem B which we now prove.

`Proof of Theorem B`. We prove the result by induction on k. The case $k = 1$ is Theorem A and we define $\lambda(c, r, 1) = \kappa(c, r)$. Assume that $k > 1$ and suppose the natural inductive hypothesis holds. Let $C_j = \zeta_{FG}^{j}(A)$, for each $j \in \mathbb{N}$, so that $\mathrm{codim}_F\, C_k = c$. Since $C_1 \leq C_k$ it is clear that $\mathrm{codim}_{FG}\, C_k/C_1 = c$ and hence by the induction hypothesis $D/C_1 = \gamma_{FG}^{k}(A/C_1)$ has finite dimension at most $\lambda(c, r, k-1)$. Now $C_1 \leq \zeta_{FG}(D)$ and hence $\dim_F D/\zeta_{FG}(D) \leq \lambda(c, r, k-1)$. Hence, by Theorem A, $D(\omega FG)$ has dimension at most $\kappa(\lambda(c, r, k-1), r)$. However $\gamma_{FG}^{k}(A) \leq D$ and hence

$$\gamma_{FG}^{k+1}(A) = \gamma_{FG}^{k}(A)(\omega FG) \leq D(\omega FG).$$

This implies that $\gamma_{FG}^{k+1}(A)$ has dimension at most $\kappa(\lambda(c, r, k-1), r)$ which proves the result. We note that the function λ is defined recursively by $\lambda(c, r, 1) = \kappa(c, r)$ and $\lambda(c, r, k) = \kappa(\lambda(c, r, k-1), r)$. $\square$

Finally, we remark that P.Hall [4] proved a result dual to Baer's theorem. Hall's theorem states that if $\gamma_{k+1}(G)$ is finite for some natural number k then $G/\zeta_{2k}(G)$ is finite. We can also prove a linear analogue of this result using very similar arguments to those given above. In place of Theorem A we have:

Theorem 4.1. *Let G be a subgroup of $GL(F, A)$. Suppose that $\dim_F A(\omega FG) = c$ is finite.*

(A) *If $char(F) = 0$ and $sr_0(G) = r$ is finite, then $\zeta_{FG}(A)$ has finite codimension; and*

(B) *If $char(F) = p > 0$ and $r_p(G) = r$ is finite, then $\zeta_{FG}(A)$ has finite codimension.*

Moreover there exists a function α such that $\mathrm{codim}_F\, \zeta_{FG}(A) \leq \alpha(c, r)$.

The proof of this result is very similar to those used in the proofs of Theorems 3.1 and 3.2. When Lemmas 2.4 and 2.5 are invoked we set $B = A(\omega FG)$. We note also that with the stated hypotheses A has an FG-submodule of finite (c, r)-bounded codimension over F which intersects $A(\omega FG)$ trivially. Theorem 4.1 allows us to obtain the linear analogue of Hall's theorem, using arguments similar to those given in the proof of Theorem B. This is the content of Theorem C

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