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SUPERSOLUBLE CONDITIONS AND TRANSFER CONTROL

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Abstract. In this paper we give a new condition for a Sylow p-subgroup of a finite group to control transfer. Then it is deduced a characterization of supersoluble groups that can be seen as a generalization of the well known result concerning the supersolubility of finite groups with cyclic Sylow subgroups. Moreover a condition for a normal embedding of a strongly closed p-subgroup is given. These results make use of the properties of G-chains and Φ-chains.

1. Introduction

Let G be a finite group, P be a Sylow p-subgroup of G and $V \trianglelefteq P$. Then we say that V controls (strong) fusion in P with respect to G if for any $g \in G$ and any subset A of P such that $A^g \subseteq P$ then $\exists t \in N_G(V)$ such that $A^g = A^t$ (if it controls strong fusion then $g = ct$ with $c \in C_G(A)$ and $t \in N_G(V)$). Moreover we say that V controls transfer in P with respect to G if $P \cap G' = P \cap N_G(V)'$. It follows by the definitions and by the Focal Subgroup Theorem (see [16, p. 142]) that if V controls fusion, then it controls transfer.

These concepts play a fundamental role in the study of local properties of a finite group. We observe that some authors prefer to say that $N_G(V)$, instead of V, controls (strong) fusion or controls transfer.

In the papers [9] and [10] two types of chains have been introduced, namely the G-chain and the Φ-chain of a strongly closed subgroup whose definitions will be recalled below. These chains play a role, respectively, for the control of strong fusion and of transfer by a strongly closed subgroup. In particular, conditions for the existence of a normal Sylow p-complement can be deduced from their properties. In this paper we improve some criterions of [10] for the control of transfer from which we deduce

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new conditions for the existence of normal Sylow p-complements and the following characterization of supersoluble groups which can be seen as a generalization of the well known result concerning the supersolubility of finite groups with cyclic Sylow subgroups:

A finite group G is supersoluble if and only if there is a normal subgroup N such that G/N has cyclic Sylow p-subgroups and for every Sylow subgroup P of G we have that $P \cap N$ has a Φ-chain.

Using properties of G-chains and following some ideas from [10] we prove the supersolubility of a finite group G in which all cyclic subgroups of order p (for all primes p which divide $|G|$) or 4 are strongly closed. P. Csörgö and M. Herzog proved the same result, with different proof and by using the concept of $\mathcal{H}$-subgroup (see [3, Th. 8]). We observe that for a p-subgroup, the two concepts of strongly closed subgroup and of $\mathcal{H}$-subgroup are equivalent (see Lemma 2.2).

If G is p-soluble, then a p-subgroup V which is strongly closed can be characterized as the Sylow p-subgroup of its normal closure. A subgroup possessing this property is called normally embedded (see for instance [4, p. 250]). The above condition is not longer true in general. However, using properties of G-chains we can prove the following result:

Let G be a finite group, $p \in \pi(G)$ and $P \in Syl_p(G)$. Let V be strongly closed in P and suppose that there is a chain $1 = V_o \lhd V_1 \ldots \lhd V_n = V$ with V_i, $i = 1, \ldots, n$ weakly closed and $|V_i : V_{i-1}| = p$. Moreover, suppose that $(|N_G(V) : C_G(V)|, p-1) = 1$, then $V \in Syl_p(V^G)$.

2. Some definitions and preliminary results

All the groups considered in the paper are finite and the notation is usually standard. In particular $\pi(G)$ indicates the set of different primes which divide the order of a finite group G. If $p \in \pi(G)$ we set $p' = \{\pi(G) \setminus p\}$. If G is the semidirect product of the subgroups H and K with $H \trianglelefteq G$, we write $[H]K$. If $p \in \pi(G)$, then $Syl_p(G)$ is the set of all the Sylow p-subgroups of G. A finite group G is said to possess a Sylow tower if there is a normal series $1 = G_0 \lhd G_1 \lhd G_2 \ldots \lhd G_n = G$ in G such that G_i/G_{i-1}, $1 \leq i \leq n$ is isomorphic to a Sylow subgroup of G. $O_p(G)$ is the maximal normal p-subgroup of G and $O_{p'}(G)$ is the maximal normal subgroup whose order is not divisible by p. $O^p(G)$ indicates the minimal normal subgroup of G such that $G/O^p(G)$ is a p-group. If $O^p(G) = O_{p'}(G)$, then this is a normal Sylow p-complement of G and G is said to be p-nilpotent. If A is a subgroup of a finite group G, then $N_G(A)$ and $C_G(A)$ are respectively the normalizer and the centralizer of A in G. Moreover A^G is the normal closure of A i.e. the minimal normal subgroup of G containing A. Finally, if G is a p-group, then $\Omega_i(G)$ is the subgroup of G generated by the elements of G of order less or equal to p^i.

For the sake of the reader we remember some results and definitions.

Definition 2.1. *Let T be a subgroup of a finite group G and let A be a subset of T.*

a) *A is said to be weakly closed in T (with respect to G) if whenever $A^g \subseteq T$ with $g \in G$, we have $A^g = A$.*
b) *A is said to be strongly closed in T (with respect to G) if for any element $a \in A$ and for any $g \in G$, $a^g \in T$ implies $a^g \in A$ (that is $A^g \cap T \subseteq A$).*

c) *If A is a subgroup and $T = N_G(A)$, then A is said to be an $\mathcal{H}$-subgroup of G if $A^g \cap T \leq A$ for all $g \in G$ (see* [3]*).*

We often will omit the "*with respect to G*" if it is not necessary to specify in which group we consider the property.

We have the following properties of strongly closed subgroups (see [16, pg. 584] and [10]).

Lemma 2.2. *Let G be a finite group. Then*

(1) *If A is a strongly closed subset in a subgroup T of G with respect to G, then A is weakly closed in T with respect to G.*

(2) *Let V be a p-subgroup of G and let $P \in Syl_p(G)$ such that $V \leq P$. Then the following conditions on V are equivalent:*
- *The subgroup V is an $\mathcal{H}$-subgroup of G.*
- *If S is any p-subgroup of G such that $V \leq S$, then V is strongly closed in S with respect to G.*
- *The subgroup V is strongly closed in P with respect to G.*

(3) *Let V be a p-subgroup of G and let $P \in Syl_p(G)$ such that $V \leq P$. Suppose that V is a strongly closed subgroup in P with respect to G. Then the following properties hold:*

i) *For any normal subgroup N of G, $V \cap N$ is a strongly closed subgroup in P with respect to G.*

ii) *For any normal subgroup N of G, let $\overline{G} = G/N$. Then $\overline{V} = VN/N$ is a strongly closed subgroup in $\overline{P} = PN/N$ with respect to $\overline{G}$ and we have*

$$N_{\overline{G}}(\overline{V}) = N_G(V)N/N.$$

(4) *Let G be a finite group and $P \in Syl_p(G)$. Suppose that $V \trianglelefteq P$. Then*

a) *If $N \trianglelefteq G$ and V is weakly closed in P with respect to G then VN/N is weakly closed in PN/N with respect to G/N.*

b) *V is normal in G if and only if V is weakly closed in P and subnormal in G.*

Now we summarize and generalize some definitions given in previous papers (see [8] and [10]).

Definitions 2.3. a) *Let G be a finite group and $P \in Syl_p(G)$. Suppose that V_a and V_b are strongly closed subgroups of P with $V_a < V_b$. Let*

$$V_a = V_o \trianglelefteq V_1 \trianglelefteq \ldots \trianglelefteq V_n = V_b \tag{2.1}$$

a chain where $V_i, i = 1, \ldots n-1$ is weakly closed in P with respect to G and $V_i/V_{i-1} \leq Z(V_b/V_{i-1})$ for $i = 1, \ldots, n$. Then (2.1) is said to be a $p-G-$chain connecting V_a and V_b (or, if there is no ambiguity, a G-chain connecting V_a and V_b). If $V_a = 1$ we simply say that (2.1) is a G-chain of V_b.

b) *If in the above definition $|V_i : V_{i-1}| = p$ $(i = 1, \ldots, n)$, then (2.1) is said to be a strict $p-G-$chain connecting V_a and V_b.*

c) *Suppose that V is a strongly closed subgroup of $P \in Syl_p(G)$ and that $\Phi(V)$ is also strongly closed in P. Then a strict G-chain connecting $\Phi(V)$ and V is said to be a Φ-chain of V, that is*

$$\Phi(V) = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_n = V$$

with V_i weakly closed and $|V_i/V_{i-1}| = p$ $(i = 1, \ldots, n)$.

3. Supersoluble conditions

Let V be a normal subgroup of a Sylow p-subgroup P of a finite group G. Then V controls transfer, i.e. $P \cap G' = P \cap N_G(V)'$ if and only if $G/O^p(G)G' \cong N_G(V)/(O^p(N_G(V))N_G(V)')$ ([14, Lemma 6.15 d), p. 49]). On the other hand $G/O^p(G)G' \cong N_G(V)/(O^p(N_G(V))N_G(V)')$ if and only if $G/O^p(G) \cong N_G(V)/O^p(N_G(V))$ (see for instance [14, Th. 6.18, p. 51]). So, in the following, when we say that V controls transfer we mean

$$G/O^p(G) \cong N_G(V)/O^p(N_G(V)).$$

Lemma 3.1. *[[10], Lemma 21] Let G be a finite group and $P \in Syl_p(G)$. Suppose that V is a strongly closed subgroup of P. Moreover suppose that $\Phi(V)$ is strongly closed in P with respect to G. Then V controls transfer.*

Proposition 3.2. *Let G be a finite group and P a Sylow p-subgroup of G. Suppose that V is a strongly closed subgroup of P which possesses a Φ-chain and that P/V is cyclic. Then P controls transfer.*

Proof. Suppose that G is a minimal counterexample and let $N = N_G(P)$ and $N_1 = N_G(V)$. Since V possesses a Φ-chain, by Lemma 3.1 we have $G/O^p(G) \cong N_1/O^p(N_1)$. If $N_1 < G$, the hypotheses go to N_1 and so P controls transfer in N_1, that is

$$N_{N_1}(P)/O^p(N_{N_1}(P)) \cong N_1/O^p(N_1).$$

Since $N_G(P)/O^p(N_G(P)) = N_{N_1}(P)/O^p(N_{N_1}(P))$ we have

$$G/O^p(G) \cong N_G(P)/O^p(N_G(P)).$$

Therefore we can assume that $N_1 = G$, that is $V \trianglelefteq G$. Then we have $\Phi(V) \trianglelefteq G$ and $\Phi(V) \leq \Phi(P)$. Suppose that $\Phi(V) \neq 1$ and consider $\widehat{G} = G/\Phi(V)$. Then $N/\Phi(V) = N_{\widehat{G}}(P/\Phi(V))$. By minimality of G we have $G/\Phi(V)/O^p(G/\Phi(V)) \cong N/\Phi(V)/O^p(N/\Phi(V))$ that is

$$G/O^p(G)\Phi(V)) \cong N/O^p(N)\Phi(V).$$

Then $P \cap O^p(G)\Phi(V) = P \cap O^p(N)\Phi(V)$ that is $\Phi(V)(P \cap O^p(G)) = \Phi(V)(P \cap O^p(N))$. Since $\Phi(V) \leq \Phi(P)$ it follows that $\Phi(P)(P \cap O^p(G)) = \Phi(P)(P \cap O^p(N))$. We have that $G = PO^p(G)$ and $N = PO^p(N)$. So it follows that $\Phi(P)O^p(G) \trianglelefteq G$ and $\Phi(P)O^p(N) \trianglelefteq N$. On the other hand $P/(P \cap O^p(G)\Phi(P))$ is elementary abelian as well $P/(P \cap O^p(N)\Phi(P))$. Therefore $O^p(G)\Phi(P) \geq G'$ and then $O^p(G)\Phi(P) = O^p(G)G'\Phi(P)$. Similarly $O^p(N)\Phi(P) \geq N'$ and so $O^p(N)\Phi(P) = O^p(N)N'\Phi(P)$. In particular $O^p(G)\Phi(P)$ and $O^p(N)\Phi(P)$ are the smallest normal subgroups of G and N respectively, with elementary abelian quotient. Therefore, since $P \cap O^p(G)\Phi(P) = P \cap O^p(N)\Phi(P)$, we have, by Tate's result [5, p. 102], that $P \cap O^p(G) = P \cap O^p(N)$, that is

$$G/O^p(G) \cong N/O^p(N).$$

So we can suppose $\Phi(V) = 1$. Consider the chain

$$1 = V_o \lhd V_1 \lhd \ldots \lhd V_s = V \unlhd V_{s+1} \unlhd \ldots \unlhd V_n = P \tag{3.1}$$

where $1 = V_o \lhd V_1 \lhd \ldots \lhd V_s = V$ is the Φ-chain of V and $V = V_s \unlhd V_{s+1} \unlhd \ldots \unlhd V_n = P$ is a chain connecting V and P such that $|V_j/V_{j-1}| = p$, $j = s+1, \ldots, n$. Since P/V is cyclic, such a chain is uniquely determined. Moreover V_j is weakly closed since V is normal in G. So the chain (3.1) is a strict $p-G-$chain of P. Then P controls strong fusion by [9] and in particular it controls transfer, that is $G/O^p(G) \cong N/O^p(N)$.

q.e.d.

Corollary 3.3. *Let G be a finite group and p be the smallest prime in $\pi(G)$. Suppose that V is a strongly closed subgroup of $P \in Syl_p(G)$ which possesses a Φ-chain and that P/V is cyclic . Then G is p-nilpotent.*

Proof. By Proposition 3.2 it is enough to show that $N_G(P)$ is p-nilpotent. Let K be a Hall p'-subgroup of $N_G(P)$ which exists by Schur-Zassenhaus Theorem [12, Th. 2.1, p. 221]. K induces automorphisms of V that stabilize the Φ-chain of V since p is the smallest prime in $\pi(G)$. It follows by [12, p. 178] that K induces the identity on $V/\Phi(V)$ and by [12, p. 180] K fixes V. Also K fixes P/V since P/V is cyclic and p is the smallest prime. So K stabilizes the normal chain $1 \unlhd V \unlhd P$ and therefore, again by [12, p. 178], K induces the identity on P. It means that $N_G(P) = P \times K$. q.e.d.

Corollary 3.4. *A finite group G is supersoluble if and only if for all $p \in \pi(G)$, if $P \in Syl_p(G)$, then P possesses a strongly closed subgroup V with a Φ-chain and P/V cyclic.*

Proof. Let G be a minimal counterexample and let r be the smallest prime in $\pi(G)$. Then G has a normal Sylow r-complement K by Corollary 3.3. By minimality of G we have that K is supersoluble and so G has a Sylow tower with respect to the reverse natural order in $\pi(G)$. Let q be the biggest prime in $\pi(G)$ and let $Q \in Syl_q(G)$. Then $Q \unlhd G$. By the hypotheses Q has a strongly closed subgroup V which has Φ-chain and Q/V is cyclic. Since $\Phi(V)$ is strongly closed and subnormal, we have, by Lemma 2.2 ($4\,b$) that $\Phi(V) \unlhd G$. Let $\widehat{G} = G/\Phi(V)$ and suppose that $\Phi(V) \neq 1$. Since the hypotheses go to $\widehat{G}$ we have that $\widehat{G}$ is supersoluble and since $\Phi(V) \leq \Phi(G)$ we have that G is supersoluble. Then we can assume $\Phi(V) = 1$. Since Q/V is cyclic with $V \lhd G$ and V has a Φ-chain it follows, repeating an argument as in Proposition 3.2, that Q has a strict G-chain $1 = S_o \lhd S_1 \lhd S_2 \lhd \ldots \lhd S_n = Q$, where $S_i,\ i = 1, \ldots, n$, being weakly closed and subnormal, is normal in G by Lemma 2.2 ($4\,b$). Since G/Q is supersoluble by minimality of G we have that G is supersoluble. Vice versa the result is true by Corollary 2 of [8]. q.e.d.

As easy consequence of Corollary 3.4, considering property $3\,i$) of Lemma 2.2, we have the following:

Corollary 3.5. *A finite group G is supersoluble if and only if there is a normal subgroup N such that G/N has cyclic Sylow subgroups and for every Sylow subgroup P of G we have that $P \cap N$ has a Φ-chain.*

Remarks 3.6.

(1) Now we give two examples which show that if we exclude either that P/V is cyclic or that V has a Φ-chain, then the conclusion in Proposition 3.2 does not hold.

- Consider $G \cong Sym_4$ and let P be a Sylow 2-subgroup of G. Let V be the Klein subgroup which is normal in G. Then P/V is cyclic. On the other hand V does not possess a Φ-chain since an eventually weakly closed subgroup, contained properly in V, would be normal in G by Lemma 2.2 $(4\,b)$ and this is not the case. On the other hand $N_G(P) = P$. It follows that P does not control transfer, otherwise G would have a normal Sylow 2-complement and this is not true.
- Now consider $G \cong SL(2,7)$ and let $P \cong Q_{16}$ (the generalized quaternion group) be a Sylow 2-subgroup of G. Take $V = Z(P)$, then V is strongly closed in P and and $1 = \Phi(V) \lhd V$ is a Φ-chain of V, but P/V is not cyclic. On the other hand $N_G(P) = P$ and so, P does not control transfer, otherwise, as in the previous example, G would have a normal Sylow 2-complement in G and this is not the case.

(2) Now we show that if we suppose that $\Phi(V)$ is weakly closed, instead of strongly closed in the definition of Φ-chain of the strongly closed subgroup V, then the conclusion of Proposition 3.2 is not true.

For, consider $G \cong PSL(2,17)$ and let $V = P \in Syl_2(G)$. Then P is dihedral of order 16. We have that $\Phi(P)$ is the unique cyclic subgroup of order 4. So it is weakly closed in P but it is not strongly closed. Moreover we observe that there is a unique cyclic subgroup H of order 8 between $\Phi(P)$ and P. Therefore $\Phi(P) \lhd H \lhd P$ is a chain in which the members are weakly closed and the quotients are of order 2. We have that $N_G(P) = P$ but, as in previous examples, V does not control transfer since, otherwise G would have a normal Sylow 2-complement.

The next lemma extends [10, Th. 19] (see also [15]).

Lemma 3.7. *Let G be a finite group, p the smallest prime in $\pi(G)$ and $P \in Syl_p(G)$. Suppose that any subgroup of P of order p, when p is odd, or any cyclic subgroup of P of order less than or equal to 4, when $p = 2$, is strongly closed in P with respect to G. Then G is p-nilpotent.*

Proof. Let $\Omega(P) = \begin{cases} \Omega_1(P) \text{ for } p \text{ odd} \\ \Omega_2(P) \text{ for } p = 2 \end{cases}$

We have $\Omega_1(P) \leq Z(P)$. Moreover $\Omega_2(P)$, (for $p = 2$), is a product of strongly closed subgroups, then it is strongly closed by Theorem 2 of [1]. If $|\langle x\rangle| = 4$ then $\langle x\rangle\Omega_1(P) \leq Z(P/\Omega_1(P))$ and so $\Omega_2(P) \leq Z_2(P)$. Therefore $\Omega(P)$ has a G-chain, that is $1 \lhd \Omega_1(P) = \Omega(P)$ for p odd or $1 \lhd \Omega_1(P) \unlhd \Omega_2(P)$ for $p = 2$. We have that $\Omega(P)$ controls strong fusion by [9], therefore $\Omega(P)$ controls transfer, i.e.

$$N_G(\Omega(P))/O^p(N_G(\Omega(P)) \cong G/O^p(G) \tag{3.2}$$

Since any cyclic subgroup of order p, when p is odd, or any cyclic subgroup of order less than or equal to 4, when $p = 2$, is strongly closed in P with respect to $N_G(\Omega(P))$, we have, by induction, that $N_G(\Omega(P))$ is p-nilpotent if $N_G(\Omega(P)) < G$. Then G is p-nilpotent by (3.2).

So we can assume that $\Omega(P) \lhd G$. Since all the subgroups of order p, when p is odd, or the cyclic subgroups of order less than or equal to 4, when $p = 2$, are strongly closed and subnormal, then they are normal in G by Lemma 2.2 ($4\,b$).

Suppose that $|\langle x\rangle| = p$ with p odd. We have that $|N_G(\langle x\rangle)/C_G(\langle x\rangle)|$ divides $p-1$. Since p is the smallest prime in $\pi(G)$ we have $C_G(\langle x\rangle) = N_G(\langle x\rangle) = G$. It means $\langle x\rangle \leq Z(G)$. By [13, Satz 5.5, p. 435] we have that G is p-nilpotent.

Now assume that $p = 2$. Clearly if $|\langle x\rangle| = 2$, then $C_G(\langle x\rangle) = G$. So we suppose that $|\langle x\rangle| = 4$. We have $|N_G(\langle x\rangle) : C_G(\langle x\rangle)| \leq 2$. By $N_G(\langle x\rangle) = G$ it follows that $C_G(\langle x\rangle) \trianglelefteq G$. Since all the elements of order less than or equal to 4 of $P \cap C_G(\langle x\rangle)$ are strongly closed in $P \cap C_G(\langle x\rangle)$ with respect $C_G(\langle x\rangle)$, we have, by induction, that $C_G(\langle x\rangle)$ is 2-nilpotent. Suppose that $[G : C_G(\langle x\rangle)] = 2$. Since $O_{2'}(C_G\langle x\rangle)$ is characteristic in $C_G(\langle x\rangle)$, we have that $O_{2'}(C_G\langle x\rangle)$ is a normal Sylow 2-complement of G. So G is 2-nilpotent. Therefore we can suppose that all the cyclic subgroups of order less than or equal to 4 are in $Z(G)$. Then, again by [13, Satz 5.5, p. 435] we have that G is 2-nilpotent. q.e.d.

Applying Lemma 3.7 we get a new proof of the following condition for supersolubility. As we have observed in the introduction, this result is a restatement of Theorem 8 in [3].

Corollary 3.8. *Let G be a finite group and suppose that any cyclic subgroup of prime order p (for any $p \in \pi(G)$) or 4 is strongly closed in $P \in Syl_p(G)$ with respect to G. Then G is supersoluble.*

Proof. By Lemma 3.7 G has a Sylow tower with respect to the reverse natural order. Let $P \in Syl_p(G)$ where p is the biggest prime in $\pi(G)$. Then $P \trianglelefteq G$. By induction we have that G/P is supersoluble. Therefore, to prove the result, it is enough to show that each principal factor H/K of G under P has order p. For, let $P_o = \Omega_1(P) = \langle x_i : |x_i| = p\rangle$. Since $\langle x_i\rangle$ is strongly closed and subnormal we have, by Lemma 2.2 ($4\,b$) that it it is normal in G. Now, with the same proof as [2, Th. 6.7, p. 25] we can prove that $|H/K| = p$. q.e.d.

4. Strict p-G-chains and some applications

A p-subgroup V of a finite group G is said to be normally embedded in G if V is a Sylow p-subgroup of its normal closure (see [4, p. 250]). It is known that if G is a finite p-soluble group and V is a subgroup of $P \in Syl_p(G)$, then V is strongly closed in P with respect to G if and if it is normally embedded in G (see for instance [11] where this property is stated without a proof for its simplicity). However, for the convenience of the reader, we give an easy proof of it.

Lemma 4.1. *Let G be a finite p-soluble group with $P \in Syl_p(G)$. Then a subgroup V of P is strongly closed in P with respect to G if and only if it is normally embedded in G.*

Proof. First we suppose that V is normally embedded. Then there is a normal subgroup N of G such that $V = P \cap N$ for some $P \in Syl_p(G)$. So $V^g \cap P = (P \cap N)^g \cap P = P^g \cap N \cap P \leq V$. Therefore V is strongly closed in P with respect to G.

Vice versa, suppose that V is strongly closed in P with respect to G. Then we will show that V is normally embedded in G by induction on $|G|$. If $O_{p'}(G) \neq 1$, then $VO_{p'}(G)/O_{p'}(G)$ is strongly closed in $G/O_{p'}(G)$ by property 3 ii) of Lemma 2.2. So there is $N \trianglelefteq G$ with $N \geq O_{p'}(G)$ such that $VO_{p'}(G)/O_{p'}(G) = (PO_{p'}(G)/O_{p'}(G)) \cap (N/O_{p'}(G)) = (P \cap N)O_{p'}(G)/O_{p'}(G)$. Since $V \leq P \cap N$, it follows that $V = P \cap N$. Therefore we may argue that $O_{p'}(G) = 1$. Since G is p-soluble, it follows that $O_p(G) \neq 1$. By property 3 i) of Lemma 2.2 we have that $V \cap O_p(G)$ is strongly closed in P with respect to G. Moreover, by properties 1) and 4 b) of Lemma 2.2, we have that $V \cap O_p(G) \trianglelefteq G$. Since G is p-soluble and $O_{p'}(G) = 1$, we have, by [12, Th. 3.3, p. 228], that $C_G(O_p(G)) \leq O_p(G)$. So $V \cap O_p(G) \neq 1$. Then, by induction $V/V \cap O_p(G)$ is normally embedded in $G/V \cap O_p(G)$. Therefore V is normally embedded in G. q.e.d.

However the property of Lemma 4.1 is not true in general. Now we give a sufficient condition for a strongly closed subgroup to be normally embedded.

Proposition 4.2. *Let G be a finite group, $p \in \pi(G)$ and $P \in Syl_p(G)$. Suppose that V is a strongly closed subgroup of P which possesses a strict $p - G-$ chain and $(|N_G(V) : C_G(V)|, p-1) = 1$. Then $V \in Syl_p(V^G)$.*

Proof. Let $1 = V_o \lhd V_1 \lhd \ldots \lhd V_n = V$ a strict $p - G$-chain of V. First observe that the condition $(|N_G(V) : C_G(V)|, p-1) = 1$ implies $N_G(V) \leq C_G(V_1)$. In fact, suppose that x is a p'-element in $N_G(V)$. Since $V_i, i = 1, \ldots, n$, is weakly closed in P, we have that V_i is invariant by x and so it induces an automorphism of $V_i/V_{i-1}, i = 1, \ldots, n$. Since $|Aut(V_i/V_{i-1})| = p - 1$ the hypotheses $(|N_G(V) : C_G(V)|, p-1) = 1$ implies that x centralizes V_i/V_{i-1} for all $i = \{1, \ldots, n\}$. Then by [12, p. 178] x centralizes V, in particular $x \in C_G(V_1)$. If x is a p-element, then $x \in Q \in Syl_p(N_G(V))$. We observe that $Q \in Syl_p(G)$ and that $V_1 \leq Z(Q)$, so $x \in C_G(V_1)$. If x is any element of $N_G(V)$, then x is a product of a p-element and a p'-element and so we are done.

Now we prove that $V \in Syl_p(V^G)$ and we suppose that G is a minimal counterexample. We consider $N = N_G(V_1)$ and we distinguish two cases.

a) $N = G$. If $V_1 = V$ we are done. So we can suppose $V_1 < V$. Then we consider $\overline{G} = G/V_1$ and $\overline{V} = V/V_1$. Since $N_{\overline{G}}\overline{V} = N_G(V)/V_1$ by Lemma 2.2 and considering that $C_{\overline{G}}(\overline{V}) \geq C_G(V)/V_1$ we have that $\overline{G}$ satisfies the same hypotheses as G. Therefore $\overline{V}$ is a Sylow p-subgroup of $(\overline{V})^{\overline{G}}$. But $(\overline{V})^{\overline{G}} = V^G/V_1$ and then V is a Sylow p-subgroup of V^G.

b) $N < G$. Obviously $V \leq P \leq N$, so the hypotheses go to N. It follows that $V \in Syl_p(V^N)$. Now let $g \in N \setminus C_G(V_1)$. By Theorem A of [7] we have that V^N has a normal Sylow p-complement K. Since K is characteristic in V^N we have that K is normal in N. So $K \leq C_G(V_1)$. Since $V^g \in Syl_p(V^N)$ we have $V^g = V^k$ where $k \in K$. It follows $gk^{-1} \in N_G(V)$. Then $g = nk$ with $n \in N_G(V) \leq C_G(V_1)$. So $g \in C_G(V_1)$ against the assumption. It follows $N_G(V_1) = C_G(V_1)$.

This condition implies that if $\langle x\rangle = V_1$ then x is not conjugate to its powers. Furthermore, since V_1 is weakly closed in P with respect to G, we have that x does not commute with its conjugates. So either by [6, Proposition 2.2] for the case p odd or by the Z^*-Theorem of Glauberman if $p = 2$ (see [16, p. 315]) we have that $V_1O_{p'}(G)/O_{p'}(G) \leq Z(G/O_{p'}(G))$. Observe that, since V_1 is not normal in G, we can suppose $O_{p'}(G) \neq 1$. So, by minimality of G we get that $VO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $(VO_{p'}(G)/O_{p'}(G))^{G/O_{p'}(G)}$. Let $M/O_{p'}(G) = (VO_{p'}(G)/O_{p'}(G))^{G/O_{p'}(G)}$. Since $M \geq V^G$ and $(|V|, |O_{p'}(G)|) = 1$, we have that $V \in Syl_p(V^G)$ and so we are done.

q.e.d.

Corollary 4.3. *Let G be a finite group and $P \in Syl_p(G)$. Suppose that $(|G|, p-1) = 1$ and that V is strongly closed in P with a strict $p-G-$chain. Then $VO_{p'}(G)$ is normal in G.*

Proof. By Proposition 4.2 we have that V is a Sylow p-subgroup of V^G. We have that $N_G(V)/C_G(V)$ is a p-group. In fact suppose that x is a p'-element of $N_G(V)$. If $1 = V_o \trianglelefteq \ldots \trianglelefteq V_n = V$ is a $p-G-$chain of V, we have that V_i is invariant by x since V_i is weakly closed in P. So x induces an automorphism of V_{i+1}/V_i, $(i = 0, \ldots, n-1)$, of order dividing $p-1$. Then, by assumption x induces the identity. It follows, by [12, p. 178] that x induces the identity on V, that is $x \in C_G(V)$. Then by [7] we have that V^G has a normal Sylow p-complement, say K. Of course $K \leq O_{p'}(G)$. If $O_{p'}(G) = 1$, then $V = V^G$ and we are done. So we may assume that $O_{p'}(G) \neq 1$. But $G/O_{p'}(G)$ satisfies the same hypotheses as G. So by induction $(VO_{p'}(G)/O_{p'}(G))O_{p'}(G/O_{p'}(G)) = VO_{p'}(G)/O_{p'}(G)$ is normal in $G/O_{p'}(G)$, that is $VO_{p'}(G) \trianglelefteq G$. q.e.d.

Remarks 4.4. *We prove that if we exclude one of the two hypotheses in Proposition 4.2, then the conclusion is not true.*

a) Consider the simple group $PSL(2, 17)$. Then the Sylow p-subgroups for p odd are cyclic. In particular a Sylow 3-subgroup P has order 9. Let V be a subgroup of order 3. Then V is strongly closed and obviously has a $p-G-$chain. However V is not a Sylow 3-subgroup of its normal closure. The reason depends on the fact that $|N_G(V) : C_G(V)|, 3-1) \neq 1$.
b) Let $G \cong Sz(2^{2n+1})$ be a Suzuki group and $P \in Syl_2(G)$. Then $Z(P) = \Omega_1(P)$ and so $V = Z(P)$ is strongly closed in P with respect to G. Here, obviously we have $(|N_G(V) : C_G(V)|, 2-1) = 1$, but V is not a Sylow 2-subgroup of V^G. The reason depends on the fact that V has not strict $p-G-$chain although it has obviously a $p-G-$chain.

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