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THE PRIME GRAPH CONJECTURE FOR INTEGRAL GROUP RINGS OF SOME ALTERNATING GROUPS

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ABSTRACT. We investigate the classical H. Zassenhaus conjecture for integral group rings of alternating groups A_9 and A_{10} of degree 9 and 10, respectively. As a consequence of our previous results we confirm the Prime Graph Conjecture for integral group rings of A_n for all $n \leq 10$.

1. **Introduction and main results**

Let G be a finite group and let $V(\mathbb{Z}G)$ denote the group of all normalized units of the integral group ring $\mathbb{Z}G$ of G. In [30], H. Zassenhaus proposed the following conjecture

(**ZC**): *Every torsion unit u in $V(\mathbb{Z}G)$ conjugates to some element g in G within the rational group algebra $\mathbb{Q}G$.*

Let $\pi(H)$ denote the Gruenberg-Kegel (prime) graph of a group H (not necessarily finite); i.e., the graph whose vertices are labeled by primes p for which there exists an element of order p in H and with an edge from p to a distinct prime q if and only if H has an element of order pq. In [25] (see also [23]), the following weaker version of (**ZC**) was proposed. We may call it the Prime Graph Conjecture:

$$(\mathbf{PGC}): \quad \pi(V(\mathbb{Z}G)) = \pi(G) \quad \textit{for any finite group } G.$$

The question about (**ZC**) remains open as no counterexample is known up to date. For nilpotent groups, (**ZC**) has been proved independently by K.W. Roggenkamp and L.L. Scott in [26] and by A. Weiss in [29]. But their method can not be applied to simple groups. However, using a new method based on the partial augmentation of a torsion unit, I.S. Luthar and I.B.S. Passi in [24] confirmed (**ZC**) for the alternating group A_5 of degree 5. Also, in [27, 28, 21] a positive answer for

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(PGC) was given for several new classes of groups, in particular for the alternating groups A_6, A_7 and A_8.

Recently, the (**PGC**) has been investigated in several papers. A positive answer has been given for solvable groups, Frobenius groups and almost for several simple groups in [23], [3] and [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19], respectively. Also, for non simple groups, see [3, 4, 16].

Here we continue our study of (**ZC**) for alternating groups. Our main results are given in the following two theorems.

Theorem 1.1. *Let G denote the alternating group A_9. For a torsion unit u in $V(\mathbb{Z}G)$ of order $|u|$, denote the partial augmentation of u by*

$$P(u) = (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b}, \nu_{7a}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{12a}, \nu_{15a}, \nu_{15b}) \in \mathbb{Z}^{17}.$$

The following hold:

(i) *There are no units of order 14, 21 and 35 in $V(\mathbb{Z}G)$.*

(ii) *If $|u| \in \{5, 7\}$, then u is rationally conjugate to some $g \in G$.*

(iii) *If $|u| = 2$, then the tuple of the partial augmentations of u belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{17} \mid (\nu_{2a}, \nu_{2b}) \in \{\ (0,1), (2,-1), (1,0), (-1,2)\ \},\ \nu_{kx} = 0,\ kx \notin \{2a, 2b\}\ \}.$$

(iv) *If $|u| = 3$, then the tuple of the partial augmentations of u belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{17} \mid (\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{\ (0,-1,2), (1,-1,1), (1,0,0), (-1,0,2), (0,0,1), (0,2,-1), (0,-2,3), (0,1,0), (-1,1,1)\ \},\ \nu_{kx} = 0,\ kx \notin \{3a, 3b, 3c\}\ \}.$$

(v) *If $|u| = 10$, then the tuple of the partial augmentations of u belongs to the set*

$$\{\ P(u) \in \mathbb{Z}^{17} \mid (\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}) \in \{\ (0,0,0,1), (1,1,0,-1)\ \},\ \nu_{kx} = 0,\ kx \notin \{2a, 2b, 5a, 10a\}\ \}.$$

Theorem 1.2. *Let G denote the alternating group A_{10}. For a torsion unit u in $V(\mathbb{Z}G)$ of order $|u|$, denote the partial augmentation of the element u by*

$$P(u) = (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{4c}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{7a}, \nu_{8a}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{12a}, \nu_{12b}, \nu_{15a}, \nu_{21a}, \nu_{21b}) \in \mathbb{Z}^{23}.$$

The following hold:

(i) *There are no units of order 14 and 35 in $V(\mathbb{Z}G)$.*

(ii) *If $|u| \in \{5, 7\}$, then u is rationally conjugate to some $g \in G$.*

(iii) *If* $|u| = 2$, *then the tuple of the partial augmentations of* u *belongs to the set*

$$\{ P(u) \in \mathbb{Z}^{23} \mid (\nu_{2a}, \nu_{2b}) \in \{ (0,1), (-2,3), (2,-1), (1,0), (-1,2) \},\ \nu_{kx} = 0,\ kx \notin \{2a, 2b\} \}.$$

(iv) *If* $|u| = 3$, *then the tuple of the partial augmentations of* u *belongs to the set*

$$\{ P(u) \in \mathbb{Z}^{23} \mid (\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{ (0,-1,2), (0,3,-2), (1,0,0), (0,0,1), (0,2,-1), (-1,2,0), (1,1,-1), (0,1,0), (-1,1,1) \},\ \nu_{kx} = 0,\ kx \notin \{3a, 3b, 3c\} \}.$$

As an immediate consequence of the first parts of Theorems 1.1, 1.2 and [21, 27, 28] we obtain the solution of the Prime Graph Conjecture for A_n:

Corollary 1.3. *For all* $n \leq 10$, *if* $G = A_n$, *then* $\pi(G) = \pi(V(\mathbb{Z}G))$.

2. Preliminaries

Let G be a finite group and let $\mathcal{C} = \{C_1, C_{kx} \mid x \in \{a, b, \ldots\},\ k \geq 2\}$ be the collection of all conjugacy classes of G, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Supposing that the torsion unit $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has order k, denote the partial augmentation of u with respect to the conjugacy class C_{nt} by $\nu_{nt} = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$. Denote the tuple of partial augmentations of the unit u by

$$P(u) = (\nu_{kx} \mid x \in \{a, b, \ldots\}, k \geq 2) \in \mathbb{Z}^l,$$

where $l + 1$ is the number of conjugacy classes of G.

From Higman-Berman's Theorem [1] one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1.$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}. Throughout the paper the p-Brauer character table of the group G will be denoted by $\mathfrak{BCT}(p)$, which can be found using the computational algebra system GAP [18]. Clearly, if $G \in \{A_9, A_{10}\}$, then the prime number p has value $p \in \{2, 3, 5, 7\}$.

Through the proofs of the main results we use the following propositions from [17, 20, 22, 24].

Proposition 2.1. *(see [24, 22]) Let either* $p = 0$ *or* p *be a prime divisor of* $|G|$ *and let* F *be the associated prime field. Suppose that* $u \in V(\mathbb{Z}G)$ *has finite order* k *and assume* k *and* p *are coprime in case* $p \neq 0$. *If* z *is a primitive* k*-th root of unity and* χ *is either a classical character or a* p*-Brauer character of* G *then for every integer* l *the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d \mid k} Tr_{F(z^d)/F} \{\chi(u^d) z^{-dl}\} \qquad (2.1)$$

is a non-negative integer.

For $p = 0$ we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Proposition 2.2. *(see [17]) The order of any unit $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

Proposition 2.3. *(see [24]) Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G. If $a \in C$ and p is a prime dividing the order of a but not the order of u then $\varepsilon_C(u) = 0$.*

M. Hertweck ([20], Proposition 3.1; [22], Lemma 5.6) obtained the next results. These already yield that several partial augmentations of the torsion units are zero.

Proposition 2.4. *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Proposition 2.5. *(see [24]) Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

3. Proof of the Theorems

Proof of Theorem 1.1. Let $G = A_9$. It is well known that $|G| = 181440 = 2^6 \cdot 3^4 \cdot 5 \cdot 7$ and $exp(G) = 1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$. The p-Brauer character tables are available for primes $p \in \{2, 3, 5, 7\}$. The group G possesses elements of orders 2, 3, 4, 5, 6, 7, 9, 10, 12 and 15. First we investigate units of orders $2, 3, 5, 7$ and 10. Secondly, according to Proposition 2.2, the order of each torsion unit divides the exponent of G, so the possible orders for units are: 14, 18, 20, 24, 30, 35, 45 and 63. We prove that units of orders 14, 21 and 35 do not appear in $V(\mathbb{Z}G)$.

• Let u be an involution. Then we have $\nu_{2a} + \nu_{2b} = 1$ by Propositions 2.3 and 2.4. According to (2.1) we get the following system of three inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(4\nu_{2a} + 8) \geq 0; \qquad \mu_1(u, \chi_2, *) = \tfrac{1}{2}(-4\nu_{2a} + 8) \geq 0;$$
$$\mu_0(u, \chi_2, 3) = \tfrac{1}{2}(3\nu_{2a} - \nu_{2b} + 7) \geq 0,$$

which has the four integral solutions listed in part (iii) of the Theorem.

• Let u be a unit of order 3. Then $\nu_{3a} + \nu_{3b} + \nu_{3c} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{3a} - \nu_{3b} + 2\nu_{3c}$ and $t_2 = 4\nu_{3a} + \nu_{3b} - 2\nu_{3c}$. Then by (2.1) we have that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{3}(2t_1 + 8) \geq 0; \qquad \mu_1(u, \chi_2, *) = \tfrac{1}{3}(-t_1 + 8) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \tfrac{1}{3}(18\nu_{3a} + 27) \geq 0; \quad \mu_1(u, \chi_5, *) = \tfrac{1}{3}(-9\nu_{3a} + 27) \geq 0;$$
$$\mu_0(u, \chi_3, 2) = \tfrac{1}{3}(-2t_2 + 8) \geq 0; \qquad \mu_1(u, \chi_3, 2) = \tfrac{1}{3}(t_2 + 8) \geq 0.$$

From the first two inequalities we get that $t_1 \in \{-4, -1, 2, 5, 8\}$ and from the next two we get that $\nu_{3a} \in \{-1, 0, 1, 2, 3\}$. Considering the last two inequalities, we obtain the 6 non-trivial and 3 trivial integral solutions listed in part (iv) of the Theorem.

• Let u be a unit of order either 5 or 7. Then by Propositions 2.3 and 2.4, the only nonzero partial augmentation is $\nu_{5a} = 1$ or $\nu_{7a} = 1$, respectively. According to Proposition 2.5, such unit u is rationally conjugate to an element $g \in G$.

• Let u be a unit of order 10. Then $\nu_{2a} + \nu_{2b} + \nu_{5a} + \nu_{10a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 4\nu_{2a} + 3\nu_{5a} - \nu_{10a}$, $t_2 = \nu_{2a} - 3\nu_{2b} + \nu_{5a} + \nu_{10a}$ and $t_3 = 7\nu_{2a} + 3\nu_{2b} + 2\nu_{5a} + 2\nu_{10a}$. Since u^5 is an involution, we need to consider the following four cases:

$$\chi(u^5) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b)\}.$$

Consider each case separately:

Case 1. Let $\chi(u^5) = \chi(2a)$ and $\chi(u^2) = \chi(5a)$. By (2.1) we obtain the following system of inequalities:

$$\begin{aligned}
&\mu_1(u, \chi_2, *) = \tfrac{1}{10}(t_1 + 1) \geq 0; && \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 16) \geq 0;\\
&\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 26) \geq 0; && \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 24) \geq 0;\\
&\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 19) \geq 0; && \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 18) \geq 0;\\
&\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 42) \geq 0; && \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 28) \geq 0.
\end{aligned}$$

It is easy to check that $t_1 = -1$, $t_2 = 1$ and $t_3 \in \{-8, 2\}$, which has the following integral solution: $(0, 0, 0, 1)$.

Case 2. Let $\chi(u^5) = \chi(2b)$ and $\chi(u^2) = \chi(5a)$. By (2.1) we have

$$\begin{aligned}
&\mu_0(u, \chi_2, *) = \tfrac{1}{10}(4t_1 + 20) \geq 0; && \mu_2(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 5) \geq 0;\\
&\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 22) \geq 0; && \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 28) \geq 0;\\
&\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 23) \geq 0; && \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 22) \geq 0;\\
&\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 38) \geq 0; && \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 32) \geq 0;
\end{aligned}$$
$$\mu_0(u, \chi_2, 3) = \tfrac{1}{10}(12\nu_{2a} - 4\nu_{2b} + 8\nu_{5a} - 8\nu_{10a} + 14) \geq 0.$$

It follows that $t_1 \in \{-5, 5\}$, $t_2 \in \{-3, 7\}$ and $t_3 \in \{-2, 8\}$, which has the following integral solution: $(1, 1, 0, -1)$.

Case 3. Let $\chi(u^5) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(5a)$. By (2.1) we have

$$\begin{aligned}
&\mu_1(u, \chi_2, *) = \tfrac{1}{10}(t_1 - 3) \geq 0; && \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 12) \geq 0;\\
&\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4t_2 + 30) \geq 0; && \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4t_2 + 20) \geq 0;\\
&\mu_1(u, \chi_3, *) = \tfrac{1}{10}(t_2 + 15) \geq 0; && \mu_1(u, \chi_5, *) = \tfrac{1}{10}(t_3 + 14) \geq 0;\\
&\mu_0(u, \chi_5, *) = \tfrac{1}{10}(4t_3 + 46) \geq 0; && \mu_5(u, \chi_5, *) = \tfrac{1}{10}(-4t_3 + 24) \geq 0;
\end{aligned}$$
$$\mu_0(u, \chi_7, *) = \tfrac{1}{10}(-20\nu_{2a} + 12\nu_{2b} + 22) \geq 0.$$

It follows that $t_1 = 3$, $t_2 \in \{-5, 5\}$ and $t_3 \in \{-4, 6\}$, which has no integral solution.

Case 4. Let $\chi(u^5) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Using (2.1) we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{10}(4t_1+16) \geq 0; \quad \mu_2(u,\chi_2,*) = \tfrac{1}{10}(-t_1+1) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{10}(4t_2+18) \geq 0; \quad \mu_5(u,\chi_3,*) = \tfrac{1}{10}(-4t_2+32) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{10}(t_2+27) \geq 0; \quad \mu_1(u,\chi_5,*) = \tfrac{1}{10}(t_3+26) \geq 0;$$
$$\mu_0(u,\chi_5,*) = \tfrac{1}{10}(4t_3+34) \geq 0; \quad \mu_5(u,\chi_5,*) = \tfrac{1}{10}(-4t_3+36) \geq 0.$$

It follows that $t_1 = 1$, $t_2 = 3$ and $t_3 \in \{-6, 4\}$, which has no integral solution.

• Let u be a unit of order 14. Then we have $\nu_{2a} + \nu_{2b} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 4\nu_{2a} + \nu_{7a}$ and $t_2 = \nu_{2a} - 3\nu_{2b}$. Since $\chi(u^7)$ has order 2, according to the previous cases we need to consider the following four cases:

$$\chi(u^7) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b)\}.$$

Consider each case separately:

Case 1. Let $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(6t_1+18) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{14}(-6t_1+10) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{14}(6t_2+22) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{14}(-6t_2+20) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{14}(t_2+20) \geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(6t_1+14) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{14}(-6t_1+14) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \tfrac{1}{14}(t_1+7) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{14}(6t_2+26) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{14}(-6t_2+16) \geq 0,$$

which has no integral solutions.

Case 3. Let $\chi(u^7) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we obtain the following system of inequalities

$$\mu_1(u,\chi_2,*) = \tfrac{1}{14}(t_1-1) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{14}(-6t_1+6) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{14}(6t_2+26) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{14}(-6t_2+16) \geq 0.$$

It follows that $t_1 = 1$ and $t_2 = -2$ which has no integral solutions.

Case 4. Let $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Then by (2.1) we have that

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(6t_1+10) \geq 0; \quad \mu_2(u,\chi_2,*) = \tfrac{1}{14}(-t_1+3) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{14}(6t_2+14) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{14}(-6tr_2+28) \geq 0.$$

It follows that $t_1 = 3$ and $t_2 = 0$ which has no integral solutions.

• Let u be a unit of order 21. Then $\nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{7a}$ and $t_2 = \nu_{3a} - \nu_{3b}$. Since $\chi(u^7)$ has order 3, according to previous cases we need to consider the following nine cases:

$$\chi(u^7) \in \{\chi(3a), \quad \chi(3b), \quad \chi(3c), \quad -2\chi(3b) + 3\chi(3c), \\ \chi(3a) - \chi(3b) + \chi(3c), \quad -\chi(3a) + \chi(3b) + \chi(3c) \\ -\chi(3a) + 2\chi(3c), \quad 2\chi(3b) - \chi(3c), \quad -\chi(3b) + 2\chi(3c)\}.$$

Consider each case separately:

Case 1. Let $\chi(u^7) = \chi(3a)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have that:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 24) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 9) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(18t_2 + 24) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 15) \geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(3b)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the following system of two inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 12) \geq 0; \quad \mu_3(u, \chi_2, *) = \tfrac{1}{21}(-2t_1 + 5) \geq 0.$$

Clearly, it has no integral solution.

Case 3. Let $\chi(u^7) = \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have that:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 18) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 12) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{21}(t_1 + 5) \geq 0.$$

It has no integral solution.

Case 4. Let $\chi(u^7) = -2\chi(3b) + 3\chi(3c)$ and $\chi(u^3) = \chi(7a)$. According to (2.1) we are able to construct the following system of inequalities:

$$\mu_1(u, \chi_2, *) = \tfrac{1}{21}(t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 6) \geq 0;$$
$$\mu_3(u, \chi_3, *) = \tfrac{1}{21}(6t_2 + 9) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 9) \geq 0,$$

which has no integral solutions.

Case 5. Let $\chi(u^7) = \chi(3a) - \chi(3b) + \chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the system:

$$\mu_1(u, \chi_2, *) = \tfrac{1}{21}(t_1 - 1) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 6) \geq 0;$$
$$\mu_3(u, \chi_3, *) = \tfrac{1}{21}(6t_2 + 9) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 9) \geq 0,$$

which has no integral solutions.

Case 6. Let $\chi(u^7) = -\chi(3b) + 2\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we have the system of four inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{21}(12t_1 + 24) \geq 0; \quad \mu_7(u, \chi_2, *) = \tfrac{1}{21}(-6t_1 + 9) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(18t_2 + 24) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{21}(-36t_2 + 15) \geq 0,$$

which has no integral solutions.

Case 7. Let $\chi(u^7) = -\chi(3a)+2\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we get the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(5t_1+12) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{21}(-12t_1+5) \geq 0,$$

which has no integral solutions.

Case 8. Let $\chi(u^7) = -\chi(3a)+\chi(3b)+\chi(3c)$ and $\chi(u^3) = \chi(7a)$. According to (2.1) we are able to construct the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(12t_1+6) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{21}(-2t_1-1) \geq 0,$$

which has no integral solutions.

Case 9. Let $\chi(u^7) = 2\chi(3b)-\chi(3c)$ and $\chi(u^3) = \chi(7a)$. Then by (2.1) we obtain the following unsolvable system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(t_1+6) \geq 0; \quad \mu_3(u,\chi_2,*) = \tfrac{1}{21}(-t_1-1) \geq 0.$$

• Let u be a unit of order 35. Then $\nu_{5a}+\nu_{7a} = 1$ by Propositions 2.3 and 2.4. Clearly $\chi(u^7) = \chi(5a)$ and $\chi(u^5) = \chi(7a)$. Put $t_1 = 3\nu_{5a}+\nu_{7a}$. Then by (2.1) we have the system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{35}(24t_1+26) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{35}(-6t_1+11) \geq 0,$$

which has no integral solution. □

Proof of Theorem 1.2. Let $G = A_{10}$. It is well known that $|G| = 1814400 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and $exp(G) = 2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. The group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15 and 21. First we investigate units of orders $2,3,5$ and 7. Secondly, according to Proposition 2.2, the order of each torsion unit divides the exponent of G, so the possible orders for units are: 14, 18, 20, 24, 30, 35, 45 and 63. We prove that units of orders 14 and 35 do not appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

• Let u be a unit of order 2. Then $\nu_{2a}+\nu_{2b} = 1$ by Propositions 2.3 and 2.4. Put $t = 5\nu_{2a}+\nu_{2b}$. By (2.1) we obtain that

$$\mu_0(u,\chi_2,*) = \tfrac{1}{2}(t+9) \geq 0; \qquad \mu_1(u,\chi_2,*) = \tfrac{1}{2}(-t+9) \geq 0,$$

which has the 5 integral solutions listed in part (iii) of the Theorem.

• Let u be a unit of order 3. Then we have $\nu_{3a}+\nu_{3b}+\nu_{3c} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 2\nu_{3a}+\nu_{3b}$, $t_2 = 14\nu_{3a}+2\nu_{3b}-\nu_{3c}$ and $t_3 = 8\nu_{3a}-4\nu_{3b}+2\nu_{3c}$. According to (2.1) we obtain the system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{3}(6t_1+9) \geq 0; \qquad \mu_1(u,\chi_2,*) = \tfrac{1}{3}(-3t_1+9) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{3}(2t_2+35) \geq 0; \quad \mu_1(u,\chi_3,*) = \tfrac{1}{3}(-t_2+35) \geq 0;$$
$$\mu_0(u,\chi_3,2) = \tfrac{1}{3}(-2t_3+16) \geq 0; \quad \mu_1(u,\chi_3,2) = \tfrac{1}{3}(t_3+16) \geq 0.$$

It is easy to see that this system has 6 non-trivial and 3 trivial integral solutions, which are listed in Theorem 1.2(iv).

• Let u be a unit of order 5. Then $\nu_{5a} + \nu_{5b} = 1$ by Propositions 2.3 and 2.4. The system of two inequalities constructed by (2.1)

$$\mu_0(u, \chi_2, *) = \tfrac{1}{5}(16\nu_{5a} - 4\nu_{5b} + 9) \geq 0;$$
$$\mu_0(u, \chi_3, 2) = \tfrac{1}{5}(-16\nu_{5a} + 4\nu_{5b} + 16) \geq 0,$$

has only two trivial integral solutions: $(\nu_{5a}, \nu_{5b}) \in \{(0,1),(1,0)\}$.

• Let u be a unit of order 7. Then by Propositions 2.3 and 2.4, $\nu_{7a} = 1$ and $\nu_{kx} = 0$ for $kx \neq 7a$. According to Propositions 2.5, the unit u is rationally conjugate to an element $g \in G$.

• Let u be a unit of order 14. Then $\nu_{2a} + \nu_{2b} + \nu_{7a} = 1$ by Propositions 2.3 and 2.4. Put $t_1 = 5\nu_{2a} + \nu_{2b} + 2\nu_{7a}$ and $t_2 = 11\nu_{2a} + 3\nu_{2b}$. Since

$$\chi(u^7) \in \{\chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b), \quad -2\chi(2a) + 3\chi(2b)\}$$

we need to consider the following five cases:

Case 1. Let $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7a)$. By (2.1) we obtain that

$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 + 2) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 16) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 46) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 24) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 24) \geq 0.$$

It follows that $t_1 = 5$ and $t_2 = 4$, which has no integral solution.

Case 2. Let $\chi(u^7) = \chi(2b)$ and $\chi(u^2) = \chi(7a)$. According to (2.1)

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 22) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 20) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 + 6) \geq 0.$$

From the first two equations we get $t_1 = 1$, which contradicts the third one. So this system has no integral solutions.

Case 3. Let $\chi(u^7) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(7a)$. By (2.1) we have

$$\mu_1(u, \chi_2, *) = \tfrac{1}{14}(t_1 - 2) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6t_1 + 12) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 54) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 16) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 16) \geq 0.$$

It follows that $t_1 = 2$ and $t_2 = 2$, which has no integral solution.

Case 4. Let $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = \chi(7a)$. By (2.1)

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 18) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{14}(-t_1 + 4) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 30) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 40) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 40) \geq 0.$$

Clearly, $t_1 = 4$ and $t_2 = 2$. It is easy to check that such system of equations has no integral solution.

Case 5. Let $\chi(u^7) = -2\chi(2a) + 3\chi(2b)$ and $\chi(u^2) = \chi(7a)$. By (2.1) we have the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6t_1 + 14) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{14}(-t_1) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{14}(6t_2 + 22) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(t_2 + 48) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6t_2 + 48) \geq 0.$$

It follows that $t_1 = 0$ and $t_2 = 8$, which has no integral solutions.

• Let u be of order 35. Then by Propositions 2.3 and 2.4 we get that

$$\nu_{5a} + \nu_{5b} + \nu_{7a} = 1.$$

Put $t_1 = 4\nu_{5a} - \nu_{5b} + 2\nu_{7a}$. We consider the following two cases:

Case 1. Let $\chi(u^7) = \chi(5a)$ and $\chi(u^5) = \chi(7a)$. Using (2.1) we obtain that

$$\mu_0(u, \chi_2, *) = \tfrac{1}{35}(24t_1 + 37) \geq 0; \qquad \mu_7(u, \chi_2, *) = \tfrac{1}{35}(-6t_1 + 17) \geq 0,$$

which has no integral solutions.

Case 2. Let $\chi(u^7) = \chi(5b)$ and $\chi(u^5) = \chi(7a)$. By (2.1) we construct the system of two inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{35}(24t_1 + 17) \geq 0; \qquad \mu_5(u, \chi_2, *) = \tfrac{1}{35}(-4t_1 + 3) \geq 0,$$

which has no integral solutions. □

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