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ON THE NUMBER OF THE IRREDUCIBLE CHARACTERS OF FACTOR GROUPS

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Dedicated to Professor Hossein Doostie on the occasion of his retirement

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ABSTRACT. Let G be a finite group and let N be a normal subgroup of G. Suppose that Irr(G|N) is the set of the irreducible characters of G that contain N in their kernels. In this paper, we classify solvable groups G in which the set $C(G) = {Irr(G|N)|1 \neq N \leq G}$ has at most three elements. We also compute the set C(G) for such groups.

1. Introduction

Let G be a finite group and let N be a normal subgroup of G. Suppose that Irr(G|N) is the set of the irreducible characters of G that contain N in their kernels. Our aim in this paper is to study the set $\mathcal{C}(G) = \{Irr(G|N) | 1 \neq N \leq G\}$. Indeed, we classify finite solvable groups G in which the set $\mathcal{C}(G)$ has at most three elements and compute the set $\mathcal{C}(G)$ for these groups. We are motivated by the article [4], where the author and S. Zandi considered a similar problem for conjugacy classes of G. They defined $\xi(N)$ to be the number of the conjugacy classes of G, contained in the normal subgroup N and classified finite solvable groups G in which the set $\mathcal{K}(G) = \{\xi(N)|N \leq G, N \neq G\}$ contains at most three element. It is easy to see that $|\mathcal{K}(G)| = 1$ if and only if $|\mathcal{C}(G)| = 1$. This is equivalent to the simplicity of the group G. It is also routine to check that for solvable groups G, $|\mathcal{C}(G)| = 2$ if and only if $|\mathcal{K}(G)| = 2$. However, we give examples of solvable groups G with $|\mathcal{K}(G)| \neq |\mathcal{C}(G)|$. In this paper we only consider finite solvable groups. An elementary abelian p-group of order p^n is denoted by \mathbb{C}_p^n . By a Frobenius group G of type $\mathbb{C}_p^n \rtimes \mathbb{C}_q^m$, we mean that G has an elementary abelian

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kernel of order p^n and a cyclic complement of order q^m , where p^n, q^m are prime powers. The set of the irreducible characters of G is denoted by Irr(G). Recall that for a normal subgroup N of G, there exists a one to one correspondence between Irr(G|N) and Irr(G/N). We write $G = K \rtimes_{Irr} H$ if K is an elementary abelian p-group and the action of H on K is non-trivial and irreducible. A monolith is a group with a unique minimal normal subgroup. Our notations are standard and mainly obtained from [3]. The main result of this paper is the following:

Theorem A. Let G be a solvable group. If $|\mathcal{C}(G)| = 3$, then one of the following holds $(p, q \text{ and } r \text{ are prime numbers and } p \neq q)$:

- (i) G is an abelian group of order pq and $\mathcal{C}(G) = \{1, p, q\}$.
- (ii) G is a group of order p^3 and $\mathcal{C}(G) = \{1, p, p^2\}.$
- (iii) G is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\operatorname{Irr}} \mathbb{C}_{q^2}$ and $\mathcal{C}(G) = \{1, q, q^2\}.$
- (iv) $G = K \rtimes_{\operatorname{Irr}} H$, where $K \cong \mathbb{C}_p^n$ and \mathbb{C}_{q^2} , where $p^n 1 = q^2(q-1)$ and the action is non-faithful. Also, $\mathcal{C}(G) = \{1, q, q^2\}$.
- (v) G is a Frobenius group of type $\mathbb{C}_p^{2n} \rtimes \mathbb{C}_q$ and contains two minimal normal subgroups of order p^n . Also, we have $\mathcal{C}(G) = \{1, q, q + (p^n 1)/q\}.$
- (vi) G has exactly two non-trivial proper normal subgroups, namely N and G', where N < G'. Also, $C(G) = \{1, q, q + (|G':N|-1)/q\}$, where q = |G:G'|.

Remark 1.1. By the results of [4], a group G satisfies the statement (vi) if and only if G is one of the following groups:

- $G = K \rtimes H$, K is a p-group which is either special or abelian, $H \cong \mathbb{C}_q$ and $\Phi(K)$, K are the only non-trivial proper normal subgroups of G.
- $G = L \rtimes_{\operatorname{Irr}} H$, where L = G'' and H is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\operatorname{Irr}} \mathbb{C}_q$.

Compairing Theorem A with the main theorem of [4], we get the following result.

Corollary B. Let G be a solvable group and let $C(G) \leq 3$. Then |C(G)| = |K(G)|, unless G is not a monolith and G' is a minimal normal subgroup of G.

We will give examples of solvable groups in which $|\mathcal{C}(G)| \neq |\mathcal{K}(G)|$. According to Corollary B, our examples are of type (iv).

2. Preliminaries

We start this section with some easy results.

Lemma 2.1. Let G be a group with $|\mathcal{C}(G)| = t$. Assume that $N_1, ..., N_t$ are non-trivial normal subgroups of G. If $N_1 \leq ... \leq N_t$, then $N_t = G$.

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Lemma 2.2. Let G be an abelian group of order n. Then $C(G) = D(n) - \{n\}$, where D(n) is the set of the positive integers dividing n.

Lemma 2.3. Let G be a group and let N be a subgroup of G. If $G' \leq N$, then $|\operatorname{Irr}(G|N)| = |G:N|$. In particular, if N_1, N_2 contain G', then $|\operatorname{Irr}(G|N_1)| = |\operatorname{Irr}(G|N_2)|$ if and only if $|N_1| = |N_2|$.

Lemma 2.4. [3, Theorem 5.6] Let G be a group with an abelian Sylow p-subgroup. Then $G' \cap Z(G)$ is a p'-group.

Lemma 2.5. [3, Lemma 12.11] Let G be a non-abelian group. Then $cd(G) = \{1, p\}$, where p is a prime if and only if one of the followings hold.

- (1) There exists an abelian $A \leq G$ with |G:A| = p.
- (2) $|G:Z(G)| = p^3$.

Lemma 2.6. [4, Lemma 2.6] Let G be a solvable group and assume that N is a proper normal subgroup of G. Then $G'N \lneq G$. In particular if G' is a maximal subgroup of G, then it contains all normal subgroups of G.

Lemma 2.7. [4, Lemma 2.8] Let G be a group and $G = A \rtimes H$, where A is an abelian normal subgroup of G and $H \cong \mathbb{C}_p$ for a prime p. If Z(G) = 1, then G is a Frobenius group with kernel A.

Lemma 2.8. [3, Lemma 12.3] Let G be a solvable group. If G' is the unique minimal normal subgroup of G, then one of the followings holds:

- (1) G is a p-group, |G'| = p and Z(G) is cyclic.
- (2) G is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\mathrm{Irr}} \mathbb{C}_{q^m}$.

Theorem 2.9. [1, 10.4](Gaschütz's Theorem) Let G be a group and assume that P is a Sylow psubgroup of G. If K is an abelian subgroup of P and $K \leq G$, then G splits over K if and only if P splits over K.

Theorem 2.10. [1, 18.1] (Schur-Zassenhaus Theorem) Let G be a group and assume that H is a Hall normal subgroup of G. Then G splits over H.

3. Main reults

In this section, we prove Theorem A. First, we prove the case when G is nilpotent.

Proposition 3.1. Let G be a nilpotent group with $|\mathcal{C}(G)| = 3$. Then one of the followings hold:

- (1) G is an abelian group of order pq, where p, q are distinct prime numbers. Also, $\mathcal{C}(G) = \{1, p, q\}$.
- (2) G is a group of order p^3 and $\mathcal{C}(G) = \{1, p, p^2\}.$

Proof. If G is not a p-group, then we may choose maximal subgroups M, L of G such that $|G: M| \neq |G: L|$. Then, $G' \leq L \cap M$ and we deduce by Lemma 2.3 that $|\operatorname{Irr}(G|L)| \neq |\operatorname{Irr}(G|M)|$. Therefore, G' = 1 and G is abelian. Consequently, Lemma 2.2 implies that |G| = pq and $\mathcal{C}(G) = \{1, p, q\}$. Next,

assume that G is a p-group. Thus by Lemma 2.1, G must be of order p^3 . If G is abelian, then it is clear that $\mathcal{C}(G) = \{1, p, p^2\}$. Now, suppose that G is not abelian. Then, |G'| = p and all non-trivial normal subgroups of G contain G'. Hence, we deduce by Lemma 2.3 that $\mathcal{C}(G) = \{1, p, p^2\}$. The proof is now completed.

We are now ready to prove the main result of this paper. During the proof, we will frequently use the following well-known equality:

(3.1)
$$|G| = |G:G'| + \sum_{\chi \in \operatorname{Irr}_1(G)} \chi(1)^2.$$

Here $Irr_1(G)$ is the set of nonlinear irreducible characters of G.

Proof of Theorem A. If G is nilpotent, then by Proposition 3.1, G is in case (i) or (ii). So, we may assume that G is not nilpotent. We finish the proof through the following steps:

Step 1. If G is a monolith, then G is either in case (iii) or (vi).

Let N be the unique minimal normal subgroup of G. If N < G', then G' is a maximal subgroup of G and by Lemma 2.6, it contains all proper normal subgroup of G. Therefore by Lemma 2.1, N and G' are the only non-trivial proper normal subgroups of G. Hence, G is of type (vi). Note that |G : G'| = q is a prime number. Also, G'/N is an abelian maximal subgroup of G/N. So by Lemma 2.5, $cd(G/N) = \{1, q\}$ and we may write:

(3.2)
$$G/N = |G:G'| + (|\operatorname{Irr}(G|N)| - |G:G'|)q^2.$$

This implies that $|\operatorname{Irr}(G|N)| = q + (|G':N|-1)/q$. Next, assume that G' is the unique minimal normal subgroup of G. Then, Lemma 2.8 implies that G is a Frobenius group of type $\mathbb{C}_{P}^{n} \rtimes_{\operatorname{Irr}} \mathbb{C}_{q^{m}}$. Note that $|G:G'| = q^{m}$ and G has normal subgroups of index q^{i} for i = 0, 1, ..., m. Since $|\mathcal{C}(G)| = 3$, we conclude by Lemma 2.1 that m = 2. That is, G is of type (*iii*). Now, choose a proper normal subgroup of N of G with G' < N. Then, N is of index q. On the other hand, since the action is irreducible and Z(G) = 1, we conclude that G' is the unique minimal normal subgroup of G. So, all non-trivial proper normal subgroups of G have index q. Hence, $\mathcal{C}(G) = \{1, q, q^2\}$.

Step 2. If G is not a monolith and G' is a minimal normal subgroup, then G is in case (iv).

Let N be a minimal normal subgroup of G and $N \neq G'$. Then, N is central. We claim that N = Z(G), otherwise, considering the subgroups G', N, Z(G) and G'Z(G), we may find at least four elements in $\mathcal{C}(G)$ which is a contradictin. So, N = Z(G) and G has no other minimal normal subgroups. Let |Z(G)| = q and $|G'| = p^n$. If r is a prime divisor of G apart from p, q, then G contains normal subgroups of order rp^n and qp^n , containing G'. Therefore, $|\mathcal{C}(G)| \ge 4$, a contradiction. So, p, q are the

only prime divisors of |G|. Now, Lemma 2.5 implies that, $cd(G) = \{1, q\}$ where q = |G : G'Z(G)|. Applying the equality (3.1) to G/Z(G) we have:

(3.3)

$$qp^{n} = q + q^{2} \left(|\operatorname{Irr} (G|N)| - q \right)$$

$$\Rightarrow qp^{n} = q + q^{2} \left(q^{2} - q \right)$$

$$\Rightarrow q^{2} \left(q - 1 \right) = p^{n} - 1$$

On the other hand, by Theorem 2.10, G splits over G' and we have $G \cong G' \rtimes G/G'$. Therefore, G is in case (*iv*). Now assume that G is of type (*iv*). We show that $\mathcal{C}(G) = \{1, q, q^2\}$. It is easy to see that K = Z(G) and G' = H' and G'Z(G) are the only non-trivial proper normal subgroups of G. The equality (3.3) garantees that $|\operatorname{Irr}(G|Z(G))| = |\operatorname{Irr}(G|G')|$. Therefore, $\mathcal{C}(G) = \{1, q, q^2\}$. This completes the proof of this step.

Step 3. If G is not a monolith and G' is not a minimal normal subgroup, then G is in case (v).

Let N, L be distinct minimal normal subgroups of G. Then NL = G' and G' is a maximal subgroup of index q. Since G'/L and G'/N are abelian, we conclude by Lemma 2.5 that cd(G/L) = cd(G/N) = $\{1,q\}$. Also note that |Irr(G|N)| = |Irr(G|L)|. Thus, applying the equality (3.1) to the groups G/Land G/N, we conclude that |N| = |L|. Hence, G' is a p-group and we deduce by Theorem 2.10 that $G \cong \mathbb{C}_p^{2n} \rtimes \mathbb{C}_q$. Now by Lemma 2.4 and Lemma 2.7, we get G is in case (v). Next, we show that $\mathcal{C}(G) = \{1, q, q + (p^n - 1)/q\}$. Let N be an arbitrary minimal normal subgroup G and $N \neq G'$. As Z(G) = 1, we must have N < G'. So, G' contains all minimal normal subgroups of G. This implies that $|G'| = p^{2n}$ and that G has exactly two minimal normal subgroups. Certainly, |Irr(G|G')| = q. Let N be a minimal normal subgroup. Since G'/N is an abelian maximal subgroup, then the equality (3.2) is valid. Therefore, |Irr(G|N)| = q + (|G':N| - 1)/q. As $|G':N| = p^n$, the result follows. \Box

Example 3.2. Consider the group G = SmallGroup(20,1), the first group of order 20 in the library of GAP [2]. Then, it is easy to see that G is case (iv) in Theorem A. Therefore, $C(G) = \{1, 2, 4\}$. However, we may check that $\mathcal{K}(G) = \{1, 2, 5, 6\}$. Now, assume that H = SmallGroup(63, 1). Then, $\mathcal{K}(G) = \{1, 3, 9\}$, while $C(G) = \{1, 3, 5, 9\}$. In both examples, as Corollary B implies, the groups contain more than one minimal normal subgroup, one of which is the derived subgroup.

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