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ON THE NUMBER OF THE IRREDUCIBLE CHARACTERS OF FACTOR GROUPS

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Dedicated to Professor Hossein Doostie on the occasion of his retirement

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Abstract. Let G be a finite group and let N be a normal subgroup of G. Suppose that $\mathrm{Irr}(G|N)$ is the set of the irreducible characters of G that contain N in their kernels. In this paper, we classify solvable groups G in which the set $\mathcal{C}(G) = \{\mathrm{Irr}(G|N) | 1 \neq N \trianglelefteq G\}$ has at most three elements. We also compute the set $\mathcal{C}(G)$ for such groups.

1. Introduction

Let G be a finite group and let N be a normal subgroup of G. Suppose that $\mathrm{Irr}(G|N)$ is the set of the irreducible characters of G that contain N in their kernels. Our aim in this paper is to study the set $\mathcal{C}(G) = \{\mathrm{Irr}(G|N) | 1 \neq N \trianglelefteq G\}$. Indeed, we classify finite solvable groups G in which the set $\mathcal{C}(G)$ has at most three elements and compute the set $\mathcal{C}(G)$ for these groups. We are motivated by the article [4], where the author and S. Zandi considered a similar problem for conjugacy classes of G. They defined $\xi(N)$ to be the number of the conjugacy classes of G, contained in the normal subgroup N and classified finite solvable groups G in which the set $\mathcal{K}(G) = \{\xi(N) | N \trianglelefteq G, N \neq G\}$ contains at most three element. It is easy to see that $|\mathcal{K}(G)| = 1$ if and only if $|\mathcal{C}(G)| = 1$. This is equivalent to the simplicity of the group G. It is also routine to check that for solvable groups G, $|\mathcal{C}(G)| = 2$ if and only if $|\mathcal{K}(G)| = 2$. However, we give examples of solvable groups G with $|\mathcal{K}(G)| \neq |\mathcal{C}(G)|$. In this paper we only consider finite solvable groups. An elementary abelian p-group of order p^n is denoted by $\mathbb{C}_p^n$. By a Frobenius group G of type $\mathbb{C}_p^n \rtimes \mathbb{C}_{q^m}$, we mean that G has an elementary abelian

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kernel of order p^n and a cyclic complement of order q^m, where p^n, q^m are prime powers. The set of the irreducible characters of G is denoted by $\mathrm{Irr}(G)$. Recall that for a normal subgroup N of G, there exists a one to one correspondence between $\mathrm{Irr}(G|N)$ and $\mathrm{Irr}(G/N)$. We write $G = K \rtimes_{\mathrm{Irr}} H$ if K is an elementary abelian p-group and the action of H on K is non-trivial and irreducible. A monolith is a group with a unique minimal normal subgroup. Our notations are standard and mainly obtained from [3]. The main result of this paper is the following:

Theorem A. *Let G be a solvable group. If $|\mathcal{C}(G)| = 3$, then one of the following holds (p, q and r are prime numbers and $p \neq q$):*

(i) G is an abelian group of order pq and $\mathcal{C}(G) = \{1, p, q\}$.
(ii) G is a group of order p^3 and $\mathcal{C}(G) = \{1, p, p^2\}$.
(iii) G is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\mathrm{Irr}} \mathbb{C}_{q^2}$ and $\mathcal{C}(G) = \{1, q, q^2\}$.
(iv) $G = K \rtimes_{\mathrm{Irr}} H$, where $K \cong \mathbb{C}_p^n$ and $\mathbb{C}_{q^2}$, where $p^n - 1 = q^2(q-1)$ and the action is non-faithful. Also, $\mathcal{C}(G) = \{1, q, q^2\}$.
(v) G is a Frobenius group of type $\mathbb{C}_p^{2n} \rtimes \mathbb{C}_q$ and contains two minimal normal subgroups of order p^n. Also, we have $\mathcal{C}(G) = \{1, q, q + (p^n - 1)/q\}$.
(vi) G has exactly two non-trivial proper normal subgroups, namely N and G', where $N < G'$. Also, $\mathcal{C}(G) = \{1, q, q + (|G' : N| - 1)/q\}$, where $q = |G : G'|$.

Remark 1.1. *By the results of* [4]*, a group G satisfies the statement* (vi) *if and only if G is one of the following groups:*

- *$G = K \rtimes H$, K is a p-group which is either special or abelian, $H \cong \mathbb{C}_q$ and $\Phi(K), K$ are the only non-trivial proper normal subgroups of G.*
- *$G = L \rtimes_{\mathrm{Irr}} H$, where $L = G''$ and H is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\mathrm{Irr}} \mathbb{C}_q$.*

Compairing Theorem A with the main theorem of [4], we get the following result.

Corollary B. *Let G be a solvable group and let $\mathcal{C}(G) \leq 3$. Then $|\mathcal{C}(G)| = |\mathcal{K}(G)|$, unless G is not a monolith and G' is a minimal normal subgroup of G.*

We will give examples of solvable groups in which $|\mathcal{C}(G)| \neq |\mathcal{K}(G)|$. According to Corollary B, our examples are of type (iv).

2. Preliminaries

We start this section with some easy results.

Lemma 2.1. *Let G be a group with $|\mathcal{C}(G)| = t$. Assume that $N_1, ..., N_t$ are non-trivial normal subgroups of G. If $N_1 \lneqq ... \lneqq N_t$, then $N_t = G$.*

Lemma 2.2. *Let G be an abelian group of order n. Then $\mathcal{C}(G) = \mathcal{D}(n) - \{n\}$, where $\mathcal{D}(n)$ is the set of the positive integers dividing n.*

Lemma 2.3. *Let G be a group and let N be a subgroup of G. If $G' \leq N$, then $|\mathrm{Irr}(G|N)| = |G : N|$. In particular, if N_1, N_2 contain G', then $|\mathrm{Irr}(G|N_1)| = |\mathrm{Irr}(G|N_2)|$ if and only if $|N_1| = |N_2|$.*

Lemma 2.4. [3, Theorem 5.6] *Let G be a group with an abelian Sylow p-subgroup. Then $G' \cap Z(G)$ is a p'-group.*

Lemma 2.5. [3, Lemma 12.11] *Let G be a non-abelian group. Then $\mathrm{cd}(G) = \{1, p\}$, where p is a prime if and only if one of the followings hold.*

(1) *There exists an abelian $A \trianglelefteq G$ with $|G : A| = p$.*
(2) *$|G : Z(G)| = p^3$.*

Lemma 2.6. [4, Lemma 2.6] *Let G be a solvable group and assume that N is a proper normal subgroup of G. Then $G'N \lneqq G$. In particular if G' is a maximal subgroup of G, then it contains all normal subgroups of G.*

Lemma 2.7. [4, Lemma 2.8]*Let G be a group and $G = A \rtimes H$, where A is an abelian normal subgroup of G and $H \cong \mathbb{C}_p$ for a prime p. If $Z(G) = 1$, then G is a Frobenius group with kernel A.*

Lemma 2.8. [3, Lemma 12.3] *Let G be a solvable group. If G' is the unique minimal normal subgroup of G, then one of the followings holds:*

(1) *G is a p-group, $|G'| = p$ and $Z(G)$ is cyclic.*
(2) *G is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\mathrm{Irr}} \mathbb{C}_{q^m}$.*

Theorem 2.9. [1, 10.4](Gaschütz's Theorem) *Let G be a group and assume that P is a Sylow p-subgroup of G. If K is an abelian subgroup of P and $K \trianglelefteq G$, then G splits over K if and only if P splits over K.*

Theorem 2.10. [1, 18.1](Schur-Zassenhaus Theorem) *Let G be a group and assume that H is a Hall normal subgroup of G. Then G splits over H.*

3. Main reults

In this section, we prove Theorem A. First, we prove the case when G is nilpotent.

Proposition 3.1. *Let G be a nilpotent group with $|\mathcal{C}(G)| = 3$. Then one of the followings hold:*

(1) *G is an abelian group of order pq, where p, q are distinct prime numbers. Also, $\mathcal{C}(G) = \{1, p, q\}$.*
(2) *G is a group of order p^3 and $\mathcal{C}(G) = \{1, p, p^2\}$.*

Proof. If G is not a p-group, then we may chooose maximal subgroups M, L of G such that $|G : M| \neq |G : L|$. Then, $G' \leq L \cap M$ and we deduce by Lemma 2.3 that $|\mathrm{Irr}(G|L)| \neq |\mathrm{Irr}(G|M)|$. Therefore, $G' = 1$ and G is abelian. Consequently, Lemma 2.2 implies that $|G| = pq$ and $\mathcal{C}(G) = \{1, p, q\}$. Next, assume that G is a p-group. Thus by Lemma 2.1, G must be of order p^3. If G is abelian, then it is clear that $\mathcal{C}(G) = \{1, p, p^2\}$. Now, suppose that G is not abelian. Then, $|G'| = p$ and all non-trivial normal subgroups of G contain G'. Hence, we deduce by Lemma 2.3 that $\mathcal{C}(G) = \{1, p, p^2\}$. The proof is now completed.

□

We are now ready to prove the main result of this paper. During the proof, we will frequently use the following well-known equality:

$$|G| = \left|G : G'\right| + \sum_{\chi \in \mathrm{Irr}_1(G)} \chi(1)^2. \tag{3.1}$$

Here $\mathrm{Irr}_1(G)$ is the set of nonlinear irreducible characters of G.

Proof of Theorem A. If G is nilpotent, then by Proposition 3.1, G is in case (i) or (ii). So, we may assume that G is not nilpotent. We finish the proof through the following steps:

Step 1. If G is a monolith, then G is either in case (iii) or (vi).

Let N be the unique minimal normal subgroup of G. If $N < G'$, then G' is a maximal subgroup of G and by Lemma 2.6, it contains all proper normal subgroup of G. Therefore by Lemma 2.1, N and G' are the only non-trivial proper normal subgroups of G. Hence, G is of type (vi). Note that $|G : G'| = q$ is a prime number. Also, G'/N is an abelian maximal subgroup of G/N. So by Lemma 2.5, $\mathrm{cd}(G/N) = \{1, q\}$ and we may write:

$$G/N = |G : G'| + (|\mathrm{Irr}(G|N)| - |G : G'|)q^2. \tag{3.2}$$

This implies that $|\mathrm{Irr}(G|N)| = q + (|G' : N| - 1)/q$. Next, assume that G' is the unique minimal normal subgroup of G. Then, Lemma 2.8 implies that G is a Frobenius group of type $\mathbb{C}_p^n \rtimes_{\mathrm{Irr}} \mathbb{C}_{q^m}$. Note that $|G : G'| = q^m$ and G has normal subgroups of index q^i for $i = 0, 1, ..., m$. Since $|\mathcal{C}(G)| = 3$, we conclude by Lemma 2.1 that $m = 2$. That is, G is of type (iii). Now, choose a proper normal subgroup of N of G with $G' < N$. Then, N is of index q. On the other hand, since the action is irreducible and $Z(G) = 1$, we conclude that G' is the unique minimal normal subgroup of G. So, all non-trivial proper normal subgroups of G have index q. Hence, $\mathcal{C}(G) = \{1, q, q^2\}$.

Step 2. If G is not a monolith and G' is a minimal normal subgroup, then G is in case (iv).

Let N be a minimal normal subgroup of G and $N \neq G'$. Then, N is central. We claim that $N = Z(G)$, otherwise, considering the subgroups G', N, $Z(G)$ and $G'Z(G)$, we may find at least four elements in $\mathcal{C}(G)$ which is a contradictin. So, $N = Z(G)$ and G has no other minimal normal subgroups. Let $|Z(G)| = q$ and $|G'| = p^n$. If r is a prime divisor of G apart from p, q, then G contains normal subgroups of order rp^n and qp^n, containing G'. Therefore, $|\mathcal{C}(G)| \geq 4$, a contradiction. So, p, q are the only prime divisors of $|G|$. Now, Lemma 2.5 implies that, $\mathrm{cd}(G)) = \{1, q\}$ where $q = |G : G'Z(G)|$. Applying the equality (3.1) to $G/Z(G)$ we have:

$$\begin{aligned} &qp^n = q + q^2 \left(|\mathrm{Irr}\,(G|N)| - q\right) \\ \Rightarrow\ &qp^n = q + q^2 \left(q^2 - q\right) \\ \Rightarrow\ &q^2 \left(q - 1\right) = p^n - 1 \end{aligned} \tag{3.3}$$

On the other hand, by Theorem 2.10, G splits over G' and we have $G \cong G' \rtimes G/G'$. Therefore, G is in case (iv). Now assume that G is of type (iv). We show that $\mathcal{C}(G) = \{1, q, q^2\}$. It is easy to see that $K = Z(G)$ and $G' = H'$ and $G'Z(G)$ are the only non-trivial proper normal subgroups of G. The equality (3.3) garantees that $|\mathrm{Irr}(G|Z(G))| = |\mathrm{Irr}(G|G')|$. Therefore, $\mathcal{C}(G) = \{1, q, q^2\}$. This completes the proof of this step.

Step 3. If G is not a monolith and G' is not a minimal normal subgroup, then G is in case (v).

Let N, L be distinct minimal normal subgroups of G. Then $NL = G'$ and G' is a maximal subgroup of index q. Since G'/L and G'/N are abelian, we conclude by Lemma 2.5 that $\mathrm{cd}(G/L) = \mathrm{cd}(G/N) = \{1, q\}$. Also note that $|\mathrm{Irr}(G|N)| = |\mathrm{Irr}(G|L)|$. Thus, applying the equality (3.1) to the groups G/L and G/N, we conclude that $|N| = |L|$. Hence, G' is a p-group and we deduce by Theorem 2.10 that $G \cong \mathbb{C}_p^{2n} \rtimes \mathbb{C}_q$. Now by Lemma 2.4 and Lemma 2.7, we get G is in case (v). Next, we show that $\mathcal{C}(G) = \{1, q, q + (p^n - 1)/q\}$. Let N be an arbitrary minimal normal subgroup G and $N \neq G'$. As $Z(G) = 1$, we must have $N < G'$. So, G' contains all minimal normal subgroups of G. This implies that $|G'| = p^{2n}$ and that G has exactly two minimal normal subgroups. Certainly, $|\mathrm{Irr}(G|G')| = q$. Let N be a minimal normal subgroup. Since G'/N is an abelian maximal subgroup, then the equality (3.2) is valid. Therefore, $|\mathrm{Irr}(G|N)| = q + (|G' : N| - 1)/q$. As $|G' : N| = p^n$, the result follows. □

Example 3.2. *Consider the group $G = SmallGroup(20, 1)$, the first group of order 20 in the library of GAP [2]. Then, it is easy to see that G is case (iv) in Theorem A. Therefore, $\mathcal{C}(G) = \{1, 2, 4\}$. However, we may check that $\mathcal{K}(G) = \{1, 2, 5, 6\}$. Now, assume that $H = SmallGroup(63, 1)$. Then, $\mathcal{K}(G) = \{1, 3, 9\}$, while $\mathcal{C}(G) = \{1, 3, 5, 9\}$. In both examples, as Corollary B implies, the groups contain more than one minimal normal subgroup, one of which is the derived subgroup.*

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