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ON SOME SUBGROUPS ASSOCIATED WITH THE TENSOR SQUARE OF A GROUP

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Abstract. In this paper we present some results about subgroup which is generalization of the subgroup $R_2^{\otimes}(G) = \{a \in G | [a, g] \otimes g = 1_{\otimes}, \forall g \in G\}$ of right $2_{\otimes}$-Engel elements of a given group G. If p is an odd prime, then with the help of these results, we obtain some results about tensor squares of p-groups satisfying the law $[x, g, y] \otimes g = 1_{\otimes}$, for all $x, g, y \in G$. In particular p-groups satisfying the law $[x, g, y] \otimes g = 1_{\otimes}$ have abelian tensor squares. Moreover, we can determine tensor squares of two-generator p-groups of class three satisfying the law $[x, g, y] \otimes g = 1_{\otimes}$.

1. Introduction

Throughout this paper p is an odd prime and a *p-group* is a group in which every element has order a power of p.

For any group G, *the non-abelian tensor square* $G \otimes G$ is a group generated by the symbols $g \otimes h$, subject to the relations,

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h),$$
$$g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'}),$$

for all $g, g', h, h' \in G$, where $g^h = h^{-1}gh$ is conjugation on the right. The non-abelian tensor square is a special case of the non-abelian tensor product which has its origins in homotopy theory. It was introduced by R. Brown and J. L. Loday in [3] and [4], extending ideas of J. H. C. Whitehead in [13]. In [2], R. Brown, D. L. Johnson, and E. F. Robertson start the investigation of non-abelian tensor squares as group theoretical objects.

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The set of all elements a in G such that $a \otimes g = 1_{\otimes}$, for all g in G, is called *the tensor centre* of G and denoted by $Z^{\otimes}(G)$. This concept was introduced by G. J. Ellis [5].
Following [9] *The set of right $n_{\otimes}$-Engel elements* of a group G is defined as

$$R_n^{\otimes}(G) = \{a \in G : [a,_{n-1} x] \otimes x = 1_{\otimes} \quad \forall x \in G\}.$$

Here commutators are denoted by $[x, y] = x^{-1}y^{-1}xy$ and $[x_1, \ldots, x_n, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$. Also for $x \in G$, the set of all x^g for $g \in G$ is called, *the conjugacy class* of x in G and is denoted by x^G.
The set of right $n_{\otimes}$-Engel elements in groups have been studied by several authors (see for example [1], [9], [10]). One of the results of [1] shows that $R_2^{\otimes}(G)$ is always a characteristic subgroup of G. Moravec in [9] investigated the properties of right $2_{\otimes}$-Engel elements of a group G. The subgroup $B^{\otimes}(G)$ studied in this paper is a generalization of $R_2^{\otimes}(G)$.

Definition 1.1. *Let G be a group. We define*
$C^{\otimes}(G) = \{a \in G | [ay, g, x] \otimes g = [y, g, x] \otimes g \ \ \forall g, y, x \in G\}$,
$B^{\otimes}(G) = \{a \in G | [a, g, x] \otimes g = 1_{\otimes} \ \ \forall g, x \in G\}$.

One can easily check that $C^{\otimes}(G)$ is a characteristic subgroup of G. It will be shown that $B^{\otimes}(G) = C^{\otimes}(G)$. Thus $B^{\otimes}(G)$ is a characteristic subgroup of G.
At first, we determine some information about $B^{\otimes}(G)$. Next we show that every p-group satisfying the law $[x, g, y] \otimes g = 1_{\otimes}$ has an abelian tensor square and $\gamma_3(G) \leq Z^{\otimes}(G)$. With the help of this we can compute tensor squares of two-generator p-groups of class 3 satisfying the law $[x, g, y] \otimes g = 1_{\otimes}$.

2. Preliminary Results

In this section we summarize some basic results which will be used in the proof of our main results.

Lemma 2.1. *([12, 5.1.5]) Let G be a group and $x, y, z \in G$. Then*
$[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$. (2.1.1)
$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$. (2.1.2)
$[g^{-1}, h]^g = [g, h]^{-1} = [g, h^{-1}]^h$. (2.1.3)
$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$. (2.1.4)

Lemma 2.2. *([2]) Let $g, g', h, h' \in G$. The following relations hold in $G \otimes G$:*
$(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$. (2.2.1)
$(g' \otimes h')^{g \otimes h} = (g' \otimes h')^{[g,h]}$. (2.2.2)
$[g, h] \otimes g' = (g \otimes h)^{-1}(g \otimes h)^{g'}$. (2.2.3)
$g' \otimes [g, h] = (g \otimes h)^{-g'}(g \otimes h)$. (2.2.4)
$[g, h] \otimes [g', h'] = [g \otimes h, g' \otimes h']$. (2.2.5)

Proposition 2.3. *([2]) For a given group G, there exists a homomorphism*
$\lambda : G \otimes G \to G'$ such that $\lambda(g \otimes h) = [g, h]$, for all $g, h \in G$.

Lemma 2.4. *([1,9]) Let G be a group, $a \in R_2^{\otimes}(G)$ and $x, y, z \in G$. Then*
$[a,x] \otimes y = ([a,y] \otimes x)^{-1}$. (2.4.1)
$[a,x]^z \otimes x = 1_{\otimes}$. (2.4.2)
$[a,x] \in R_2^{\otimes}(G)$. (2.4.3)
$[a,x] \otimes [y,z] = 1_{\otimes}$. (2.4.4)
$[x,y] \otimes a = ([x,a] \otimes y)^2$ *and* $a \otimes [x,y] = ([a,x] \otimes y)^2$. (2.4.5)
$([a,x,y] \otimes z)^2 = 1_{\otimes}$. (2.4.6)

Theorem 2.5. *([7]) The variety of groups which are nilpotent of class n or less (where n is a fixed integer greater than 2) may be defined by the law*

$$[x, y_1, \ldots, y_{n-1}, x] = 1.$$

3. Results for $B^{\otimes}(G)$

The goal of this section is to prove the following results for $B^{\otimes}(G)$, which are tensor analogues to the work of Kappe in [6].

Lemma 3.1. *Let G be a group, $g, x, d, d', d'' \in G$, $i, j \in \mathbb{Z}$ and $a \in B^{\otimes}(G)$. Then*
$[a,g,x]^d \otimes g^{d'} = 1_{\otimes}$. (3.1.1)
$[a,g^i,x]^d \otimes g^{d'} = 1_{\otimes}$. (3.1.2)
$[[a,g^i]^d,x]^{d'} \otimes (g^j)^{d''} = 1_{\otimes}$. (3.1.3)
$[a,g,x]^d \otimes a^{d'} = 1_{\otimes}$. (3.1.4)

Proof. Commutator expansion gives,

$$\begin{aligned} 1_{\otimes} &= [a,g,xd] \otimes g \\ &= [a,g,d][a,g,x]^d \otimes g \\ &= ([a,g,d] \otimes g)^{[a,g,d]^d}([a,g,x]^d \otimes g) \\ &= [a,g,x]^d \otimes g, \end{aligned}$$

lead to (3.1.1).
To prove (3.1.2) for $i > 0$, we use induction on i. (3.1.1) is as the basis of an inductive proof.
Now assume that (3.1.2) follows when i is a fixed positive k, we prove that (3.1.2) is true for $i = k+1$.

$$\begin{aligned} [a,g^{k+1},x]^d \otimes g^{d'} &= [a,g^k.g,x]^d \otimes g^{d'} \\ &= [[a,g][a,g^k]^g,x]^d \otimes g^{d'} \\ &= ([a,g,x]^w[[a,g^k]^g,x])^d \otimes g^{d'} \\ &= ([a,g,x]^{wd} \otimes g^{d'})^{w'}([[a,g^k]^g,x]^d \otimes g^{d'}) \\ &= [a,g^k,x^{g^{-1}}]^{gd} \otimes g^{d'} \\ &= 1_{\otimes}, \end{aligned}$$

where $w = [a, g^k]^g$ and $w' = [[a, g^k]^g, x]^d$.
If $i < 0$, then put $t = -i$. Hence,

$$\begin{aligned}
[a, g^i, x]^d \otimes g^{d'} &= [a, g^{-t}, x]^d \otimes g^{d'} \\
&= [[a, g^t]^{-g^{-t}}, x]^d \otimes g^{d'} \quad by (2.1.3) \\
&= [[a, g^t]^{-1}, x^{g^t}]^{g^{-t}d} \otimes g^{d'} \\
&= [a, g^t, x^{g^t}]^{-wd} \otimes g^{d'} \quad by (2.1.3) \\
&= ([a, g^t, x^{g^t}]^{wd} \otimes g^{d'})^{-w'} \quad by (2.2.1) \\
&= 1_\otimes,
\end{aligned}$$

where $w = [a, g^t]^{-1} g^{-t}$ and $w' = [a, g^t, x^{g^t}]^{-wd}$.
These imply (3.1.2).
(3.1.3) is proved in a way similar to (3.1.2).
For the proof of (3.1.4) note that,

$$\begin{aligned}
1_\otimes &= [a, ag, x]^d \otimes (ag)^{d'} \\
&= [[a, ag], x]^d \otimes a^{d'} g^{d'} \\
&= ([a, g, x]^d \otimes g^{d'})([a, g, x]^d \otimes a^{d'})^{g^{d'}} \\
&= ([a, g, x]^d \otimes a^{d'})^{g^{d'}}, \quad by (3.1.1)
\end{aligned}$$

proving (3.1.4). □

Theorem 3.2. *Let G be a group. Then we have*
$B^\otimes(G) = C^\otimes(G)$. (3.2.1)
$R_2^\otimes(G) \subseteq B^\otimes(G)$. (3.2.2)
$[a, g, x, h] \otimes g = 1_\otimes$, *for* $a \in B^\otimes(G)$ *and* $g, x, h \in G$. (3.2.3)
$[a, g, x, h] \otimes h = 1_\otimes$, *for* $a \in B^\otimes(G)$ *and* $g, x, h \in G$. (3.2.4)
$([a, g, x, b, c] \otimes h)^2 = 1_\otimes$, *for* $a \in B^\otimes(G)$ *and* $g, x, b, c, h \in G$. (3.2.5)
$[a, g, x] \otimes h = ([a, h, x] \otimes g)^{-1}$, *for* $a \in B^\otimes(G)$ *and* $g, x, h \in G$. (3.2.6)

Proof. To prove (3.2.1) let $a \in B^\otimes(G)$. Then $[[a, g]^y, x]^{[y,g]} \otimes g = 1_\otimes$ by (3.1.3). Hence,

$$\begin{aligned}
[ay, g, x] \otimes g &= ([[a, g]^y, x]^{[y,g]} \otimes g)^{[y,g,x]}([y, g, x] \otimes g) \\
&= [y, g, x] \otimes g
\end{aligned}$$

i.e., $B^\otimes(G) \subseteq C^\otimes(G)$ and hence (3.2.1) follows since clearly $C^\otimes(G) \subseteq B^\otimes(G)$.
To prove (3.2.2), let $a \in R_2^\otimes(G)$. Then by proposition 2.3, $[a, g, g] = 1$. Hence, by (2.4.3) and (2.4.1)

we have

$$\begin{aligned}[a,g,x]\otimes g &= ([[a,g],g]\otimes x)^{-1}\\ &= (1\otimes x)^{-1}\\ &= 1_{\otimes},\end{aligned}$$

proving $R_2^{\otimes}(G)\subseteq B^{\otimes}(G)$.
To prove (3.2.3), we have

$$\begin{aligned}[a,g,x,h]\otimes g &= [[a,g]^{-1}[a,g]^{x},h]\otimes g\\ &= [a,g,h]^{-w}[[a,g]^{x},h]\otimes g\\ &= ([a,g,h]^{w}\otimes g)^{-w'}([[a,g]^{x},h]\otimes g)\\ &= 1_{\otimes},\quad by (3.1.3)\end{aligned}$$

where $w=[a,g]^{-1}[a,g]^{x}$ and $w'=[a,g,h]^{-w}[[a,g]^{x},h]$.
To prove (3.2.4), since $a\in B^{\otimes}(G)$, then by proposition 2.3 we have $1=[a,gh,x,gh]$. Hence, commutator expansion gives

$$\begin{aligned}1 &= [[a,h][a,g]^{h},x,gh]\\ &= [[a,h,x]^{[a,g]^{h}}[[a,g]^{h},x],gh]\\ &= [y_1y_2,gh],\end{aligned}$$

where $y_1=[a,h,x]^{[a,g]^{h}}$ and $y_2=[[a,g]^{h},x]$. Now by (3.1.3), $y_1\otimes h=1_{\otimes}$ and $y_2\otimes g=1_{\otimes}$. Hence, by proposition 2.3 we have $[y_1,h]=1$ and $[y_2,g]=1$. Then the commutator expansion of $1=[y_1y_2,gh]$ gives

$$1=[y_1,gh]^{y_2}[y_2,gh]=[y_1,h]^{y_2}[y_1,g]^{hy_2}[y_2,h][y_2,g]^{h}.$$

Hence, $1=[y_1,g]^{hy_2}[y_2,h]$ and so

$$\begin{aligned}1_{\otimes} &= [y_1,g]^{hy_2}[y_2,h]\otimes h\\ &= ([y_1,g]^{hy_2}\otimes h)^{[y_2,h]}([y_2,h]\otimes h).\end{aligned}$$

Now we prove $[y_1,g]^{hy_2}\otimes h=1_{\otimes}$. We have

$$\begin{aligned}1_{\otimes} &= y_1^{ghy_2}\otimes h\quad by (3.1.3)\\ &= (y_1[y_1,g])^{hy_2}\otimes h\\ &= (y_1^{hy_2}\otimes h)^{[y_1,g]^{hy_2}}([y_1,g]^{hy_2}\otimes h)\\ &= [y_1,g]^{hy_2}\otimes h.\quad by (3.1.3)\end{aligned}$$

Hence, we obtain

$$1_{\otimes}=[y_2,h]\otimes h=[[a,g]^{h},x,h]\otimes h.$$

The substitution of x^h for x finally gives

$$1_\otimes = [[a,g]^h, x^h, h] \otimes h = ([a,g,x,h] \otimes h)^h,$$

proving (3.2.4).
To prove (3.2.5), $[a,g,x] \in R_2^\otimes(G)$ by (3.2.4). Hence, (3.2.5) follows by (2.4.6).
Finally for (3.2.6), expansion of $1_\otimes = [a, gh, x] \otimes gh$, as in the proof (3.2.4), leads to

$$1_\otimes = (y_1 \otimes g)^{hy_2}(y_2 \otimes h),$$

where $y_1 = [a,h,x]^{[a,g]^h}$ and $y_2 = [[a,g]^h, x]$. Now by (3.1.4) we have $[a,h,x] \otimes [a,g]^h = 1_\otimes$, since $[a,g]^h \in \langle a^G \rangle$. Hence, $[[a,h,x],[a,g]^h] = 1$ by proposition 2.3. Therefore $y_1 = [a,h,x]$. Now we have

$$\begin{aligned}(y_1 \otimes g)^{hy_2} &= ([a,h,x] \otimes g)([[a,h,x],g] \otimes hy_2) \qquad \text{by (2.2.3)}\\ &= ([a,h,x] \otimes g)([[a,h,x],g] \otimes y_2)([[a,h,x],g] \otimes h)^{y_2}\\ &= ([a,h,x] \otimes g)([[a,h,x],g] \otimes y_2) \qquad \text{by (3.2.3)}\\ &= [a,h,x] \otimes g \qquad \text{by (3.1.4), since } y_2 \in \langle a^G \rangle.\end{aligned}$$

To simplify $y_2 \otimes h$, write $[a,g]^h = [a,g][a,g,h]$ and expand

$$\begin{aligned}y_2 \otimes h &= [a,g,x]^{w_0}[a,g,h,x] \otimes h\\ &= ([a,g,x]^{w_0} \otimes h)^{w_1}([a,g,h,x] \otimes h)\\ &= ([a,g,x]^{w_0} \otimes h)^{w_1} \qquad \text{since } [a,g] \in B^\otimes(G)\\ &= ([a,g,x]^{w_0} \otimes h)([[a,g,x]^{w_0}, h] \otimes w_1), \qquad \text{by (2.2.3)}\end{aligned}$$

where $w_0 = [a,g,h]$ and $w_1 = [w_0, x]$. Now $[a,g,x] \otimes w_0 = 1_\otimes$ by (3.1.4), since $w_0 \in \langle a^G \rangle$. Hence, $[[a,g,x], w_0] = 1$ by proposition 2.3. Therefore $[a,g,x]^{w_0} = [a,g,x]$. Also $([a,g,x]^{w_0}, h] \otimes w_1) = 1_\otimes$ by (3.1.4), since $w_1 \in \langle a^G \rangle$. Hence,

$$y_2 \otimes h = [a,g,x] \otimes h.$$

Now, altogether we have

$$1_\otimes = (y_1 \otimes g)^{hy_2}(y_2 \otimes h) = ([a,h,x] \otimes g)([a,g,x] \otimes h),$$

proving (3.2.6). □

4. **The variety with the tensor square**

A variety is a class of groups defined by laws, or identical relations.

Lemma 4.1. *Every group G of the variety with the law*

$$[x,g,y] \otimes g = 1_\otimes, \qquad (4.1.1)$$

for all $x, g, y \in G$ is nilpotent of class at most 3.

Proof. Expansion of $1_\otimes = [x, xg, y] \otimes xg$ implies that

$$1_\otimes = [x, g, y] \otimes x.$$

Thus G is nilpotent of class at most 3 by proposition 2.3 and theorem 2.5. □

Proposition 4.2. *Let G be a group satisfying the law* (4.1.1) *and $a, b, x, y \in G$. Then*

$G' \subseteq R_2^\otimes(G)$. (4.2.1)

$[a, b] \otimes [x, y] = ([a, b, x] \otimes y)^2$ *and* $[a, b] \otimes [x, y] = ([a, [x, y]] \otimes b)^2$. (4.2.2)

$([a, b] \otimes [x, y])^2 = 1_\otimes$. (4.2.3)

Proof. To prove (4.2.1), by (2.1.4) and lemma 4.1 we have

$$[x^{-1}, y^{-1}, z][y, z^{-1}, x^{-1}][z, x, y] = 1,$$

for all $x, y, z \in G$. Hence, $1_\otimes = [x^{-1}, y^{-1}, z][y, z^{-1}, x^{-1}][z, x, y] \otimes y^{-1}$. The equation yields

$$1_\otimes = [z, x, y] \otimes y,$$

proving (4.2.1).

Next, (4.2.2) follows from (4.2.1) and (2.4.5).

Finally for (4.2.3), we have

$$\begin{aligned}
[a, b] \otimes [x, y] &= ([x, y, a]^{-1} \otimes b)^2 && \text{by } (4.2.2)\\
&= ([x, y, a] \otimes b)^{-2} && \text{by } (2.2.1), (2.2.3) \text{ and } 4.1\\
&= ([x, y, b] \otimes a)^2 && \text{by } (4.2.1) \text{ and } (2.4.1)\\
&= ([x, a, b] \otimes y)^{-2} && \text{by } (3.2.6)\\
&= ([a, x, b] \otimes y)^2 && \text{by } (2.2.1), (2.2.3) \text{ and } 4.1\\
&= ([a, x, y] \otimes b)^{-2} && \text{by } (4.2.1) \text{ and } (2.4.1)\\
&= ([a, b, y] \otimes x)^2 && \text{by } (3.2.6)\\
&= ([a, b, x] \otimes y)^{-2} && \text{by } (4.2.1) \text{ and } (2.4.1)\\
&= ([a, b] \otimes [x, y])^{-1}, && \text{by } (4.2.2)
\end{aligned}$$

proving, (4.2.3) □

Theorem 4.3. *A group G has a finite covering by subgroups satisfying the law* (4.1.1) *if and only if* $|G : B^\otimes(G)| < \infty$.

Proof. Suppose that $G = \bigcup_{i=1}^n H_i$ where H_i are subgroups of G satisfying the law (4.1.1). The result of B. H. Neumann [11], shows that we may assume that $|G : H_i| < \infty$ for every i. Hence, the subgroup $D = \bigcap_{i=1}^n H_i$ has a finite index in G. It is clear that $D \leq B^\otimes(G)$ and hence, $|G : B^\otimes(G)| < \infty$.

Conversely, let $\{g_1, \ldots, g_n\}$ be a transversal of $B^\otimes(G)$ in G and let $H_i = \langle g_i \rangle B^\otimes(G)$. We have $G = \bigcup_{i=1}^n H_i$, hence it suffices to prove that each H_i satisfying the law (4.1.1).

Let $x = g^i a, y = g^j b$ and z be arbitrary elements of $\langle g \rangle B^\otimes(G)$, where $i, j \in \mathbb{Z}$ and $a, b \in B^\otimes(G)$. Since $a, b \in B^\otimes(G)$, using (3.1.4), we obtain,

$$\begin{aligned}
[x, y, z] \otimes y &= [g^i a, g^j b, z] \otimes g^j b\\
&= [[g^i, g^j b]^a [a, g^j b], z] \otimes g^j b
\end{aligned}$$

$$
\begin{aligned}
&= [[g^i, g^j b]^a, z]^w [a, g^j b, z] \otimes g^j b \\
&= ([[g^i, g^j b]^a, z]^w \otimes g^j b)^{w'} ([a, g^j b, z] \otimes g^j b) \\
&= ([[g^i, b]^a, z]^w \otimes b)^{w'} ([[g^i, b]^a, z]^w \otimes g^j)^{bw'} \quad a \in B^{\otimes}(G) \\
&= ([[g^i, b]^a, z]^w \otimes g^j)^{bw'} \quad b \in B^{\otimes}(G) \\
&= ([[b, g^i]^a, z]^v \otimes g^j)^{-v'} \\
&= 1_{\otimes}, \quad b \in B^{\otimes}(G)
\end{aligned}
$$

where $w = [a, g^j b], w' = [a, g^j b, z], v = [b, g^i]^{-a} w, v' = [[b, g^i]^a, z]^{-v} bw'$, as required. □

Theorem 4.4. *Let G be a p-group satisfying the law* (4.1.1)*. Then*
$G \otimes G$ *is abelian.* (4.4.1)
$\gamma_3(G) \leq Z^{\otimes}(G)$. (4.4.2)

Proof. If G is an abelian p-group, then the result is obvious. Otherwise, $G \otimes G$ has no element of order 2, since G is a p-group and p is an odd prime. Hence, $[a, b] \otimes [x, y] = 1_{\otimes}$ by (4.2.3). So (2.2.5) implies (4.4.1). Next we have $([a, b, x] \otimes y)^2 = 1_{\otimes}$ by (4.2.2) since $[a, b] \otimes [x, y] = 1_{\otimes}$. Thus $1_{\otimes} = [a, b, x] \otimes y$, proving (4.4.2). □

Corollary 4.5. *Let G be a two-generator p-group of class 3 satisfying the law* (4.1.1)*. Then we can determine tensor square of G.*

Proof. If G is a p-group satisfying the law (4.1.1), then by (4.4.2), $\gamma_3(G) \leq Z^{\otimes}(G)$. Now using the result of G. J. Ellis [5], we see that $G \otimes G \cong \frac{G}{\gamma_3(G)} \otimes \frac{G}{\gamma_3(G)}$, hence the calculations of tensor squares reduce to the calculations of tensor squares of class 2 groups. From [8], we obtain the complete classification of tensor square of two-generator p-groups of class 2. Hence by Theorem 36 from [8]

$$
G \otimes G \cong \begin{cases} C_{p^\alpha} \times C_{p^\beta} \times C_{p^\beta} \times C_{p^\beta} \times C_{p^\rho} \times C_{p^\rho} & if \rho \leq \sigma \\ C_{p^\alpha} \times C_{p^\beta} \times C_{p^\beta} \times C_{p^{\beta+\tau}} \times C_{p^\sigma} \times C_{p^\sigma} & if \rho > \sigma \end{cases}
$$

where $\tau = \min(\alpha - \beta, \rho - \sigma)$ and $\alpha, \beta, \gamma, \rho$ and σ are defined integers as in Theorem 1 from [8](see [8] for more information). □

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