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CHARACTERIZATION OF THE SYMMETRIC GROUP BY ITS NON-COMMUTING GRAPH

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Abstract. The non-commuting graph $\nabla(G)$ of a non-abelian group G is defined as follows: its vertex set is $G - Z(G)$ and two distinct vertices x and y are joined by an edge if and only if the commutator of x and y is not the identity. In this paper we prove that if G is a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, then $G \cong \mathbb{S}_n$, where $\mathbb{S}_n$ is the symmetric group of degree n, where n is a natural number.

1. Introduction

Let G be a group. The non-commuting graph $\nabla(G)$ of G is defined as follows: the set of vertices of $\nabla(G)$ is $G - Z(G)$, where $Z(G)$ is the center of G and two vertices are connected whenever they do not commute. Also we define the prime graph $\Gamma(G)$ of G as follows: the vertices of $\Gamma(G)$ are the prime divisors of the order of G and two distinct vertices p and q are joined by an edge and we write $p \sim q$, if there is an element in G of order pq. We denote by $\pi_e(G)$ the set of orders of elements of G. The connected components of $\Gamma(G)$ are denoted by π_i, $i = 1, 2, \ldots, t(G)$, where $t(G)$ is the number of components. We can express the order of G as a product of some positive integer m_i, $i = 1, 2, \ldots, t(G)$ with $\pi(m_i) = \pi_i$. The integers m_is are called the order components of G. In 2006, A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture in [1] as follows.

AAM's Conjecture: If M is a finite non abelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$.

Ron Solomon and Andrew Woldar proved the above conjecture in [6]. In this paper we will prove that if G is a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, then $G \cong \mathbb{S}_n$, where $\mathbb{S}_n$ is the symmetric group of degree

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n. Therefore we extend the AAM's conjecture to the case, where M is not necessarily a finite simple group.

2. Preliminaries

The following result was proved in part(1) of Theorem 3.16 of [1].

Lemma 2.1. *Let G be a finite group such that $\nabla(G) \cong \nabla(\mathbb{S}_n)$, $n \geq 3$. Then $|G| = |\mathbb{S}_n|$.*

Lemma 2.2. *Let G and H be two non-abelian groups. If $\nabla(G) \cong \nabla(H)$, then*

$$\nabla(C_G(A)) \cong \nabla(C_H(\varphi(A)))$$

for all $\varnothing \neq A \subseteq G - Z(G)$, where φ is the isomorphism from $\nabla(G)$ to $\nabla(H)$ and $C_G(A)$ is non-abelian.

Proof. It is sufficient to show that $\varphi\mid_{V(C_G(A))}: V(C_G(A)) \longrightarrow V(C_H(\varphi(A)))$ is onto, where $\varphi\mid_{V(C_G(A))}$ is the restriction of φ to $V(C_G(A))$ and

$$V(C_G(A)) = C_G(A) - Z(C_G(A)),$$
$$V(C_H(\varphi(A))) = C_H(\varphi(A)) - Z(C_H(\varphi(A))).$$

Assume d is an element of $V(C_H(\varphi(A)))$, then $d \in H - Z(H)$ and so there exists an element c of $G - Z(G)$ such that $\varphi(c) = d$. From

$$d = \varphi(c) \in C_H(\varphi(A)),$$

it follows that $[\varphi(c), \varphi(g)] = 1$ for all $g \in A$ and since φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$, $[c, g] = 1$ for all $g \in A$. Therefore $c \in C_G(A)$. But $d \notin Z(C_H(\varphi(A)))$, so for an element $x \in C_H(\varphi(A))$ we have $[x, d] \neq 1$. Hence x is an element of H that does not commute with $d \in H$. This implies that $x \in H - Z(H)$. Thus there exists $x' \in G - Z(G)$, such that $\varphi(x') = x$. It is easy to see that $[x', c] \neq 1$ and therefore $c \notin Z(C_G(A))$. Hence

$$c \in C_G(A) - Z(C_G(A)) = V(C_G(A))$$

and $\varphi(c) = d$. □

The following result was proved by E. Artin in [2] and [3] and together with the classification of finite simple groups can be stated as follows:

Lemma 2.3. *Let G and M be finite simple groups, $|G| = |M|$, then the following holds:*

1. *If $|G| = |\mathbb{A}_8| = |L_3(4)|$, then $G \cong \mathbb{A}_8$ or $G \cong L_3(4)$;*
2. *If $|G| = |B_n(q)| = |C_n(q)|$, where $n \geq 3$, and q is odd, then $G \cong B_n(q)$ or $G \cong C_n(q)$;*
3. *If M is not in the above cases, then $G \cong M$.*

As an immediate consequence of Lemma 2.3, we get the following corollary.

Corollary 2.4. *Let G be a finite simple group with $|G| = |\mathbb{A}_n|$, where n is a natural number, $n \geq 5$, $n \neq 8$, then $G \cong \mathbb{A}_n$.*

Lemma 2.5. *Let G and H be two finite groups with $\nabla(G) \cong \nabla(H)$ and $|G| = |H|$. Then $p_1p_2\cdots p_t \in \pi_e(G)$ if and only if $p_1p_2\cdots p_t \in \pi_e(H)$, where p_i's are distinct prime numbers for $i = 1, 2, \ldots, t$. In particular, $\Gamma(G) = \Gamma(H)$.*

Proof. If φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$ and $|G| = |H|$, then we can easily see that

$$|Z(C_G(x))| = |Z(C_H(\varphi(x)))|$$

for all $x \in G$. If $p_1p_2\cdots p_t \in \pi_e(G)$, then there exists an element $z \in G$ such that $o(z) = p_1p_2\cdots p_t$. Thus

$$p_1p_2\cdots p_t = |\langle z\rangle|\big||Z(C_G(z))|$$

and so

$$p_1p_2\cdots p_t\big||Z(C_H(\varphi(z)))|.$$

Hence H has an abelian subgroup of order $p_1p_2\cdots p_t$, which is a cyclic group. Therefore $p_1p_2\cdots p_t \in \pi_e(H)$. By a similar argument we see that if $p_1p_2\cdots p_t \in \pi_e(H)$, then $p_1p_2\cdots p_t \in \pi_e(G)$. □

Lemma 2.6. *Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, where $3 \le n \le 8$ or $11 \le n \le 14$, then $G \cong \mathbb{S}_n$.*

Proof. Since $\nabla(G) \cong \nabla(\mathbb{S}_n)$, by Lemma 2.1, $|G| = |\mathbb{S}_n|$. Also by Lemma 2.5 $\Gamma(G) = \Gamma(\mathbb{S}_n)$, where Γ denotes the prime graph. Thus the order components of G and $\mathbb{S}_n$ are the same. In [7] it is proved that $\mathbb{S}_p$ and $\mathbb{S}_{p+1}$ are characterizable by their order components, where $p \ge 3$ is a prime number. Hence $\mathbb{S}_n$, where $3 \le n \le 8$ or $11 \le n \le 14$ is characterizable by their order components and so $G \cong \mathbb{S}_n$, where $3 \le n \le 8$ or $11 \le n \le 14$ □

Lemma 2.7. *Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, $n = 9, 10, 15, 16$, then $G \cong \mathbb{S}_n$*

Proof. We give the proof in the case $n = 9$, the proof in other cases is similar. Set

$$T = \{\alpha \in \mathbb{S}_9 | (i)\alpha = i, i = 4, 5, \ldots, 9\}.$$

Obviously

$$T \le \mathbb{S}_9,$$
$$T \cong \mathbb{S}_3$$

and $C_{\mathbb{S}_9}(T - \{1\}) \cong \mathbb{S}_6$. By Lemma 2.2 we have

$$\nabla(C_{\mathbb{S}_9}(T - \{1\})) \cong \nabla(C_G(\varphi(T - \{1\}))),$$

where φ is an isomorphism from $\nabla(\mathbb{S}_9)$ to $\nabla(G)$. Thus by Lemma 2.6 $C_G(\varphi(T - \{1\})) \cong \mathbb{S}_6$.
Let N be a minimal normal subgroup of G. If

$$N \cap C_G(\varphi(T - \{1\})) = 1,$$

then since

$$|NC_G(\varphi(T-\{1\}))|\big||G| = 9!$$

and

$$|C_G(\varphi(T-\{1\}))| = 6!,$$

we have $|N|\big|9{\cdot}8{\cdot}7$. We know that N is a union of conjugacy classes of G and the size of conjugacy class of G containing x is equal to the size of conjugacy class of $\mathbb{S}_9$ containing $\varphi^{-1}(x)$ for all $x \in G-\{1\}$. We can see that all conjugacy class sizes in $\mathbb{S}_9$ less than $9\cdot8\cdot7$ are 1, $\frac{9\cdot8}{2}$, $\frac{9\cdot8\cdot7}{3}$ and $\frac{9\cdot8\cdot7\cdot6}{8}$.
Let y be an arbitrary element in $N-\{1\}$. Thus the size of conjugacy class of y in G and so the size of conjugacy class of $\varphi^{-1}(y)$ in $\mathbb{S}_9$ is equal to $\frac{9\cdot8}{2}$, $\frac{9\cdot8\cdot7}{3}$ or $\frac{9\cdot8\cdot7\cdot6}{8}$.
Therefore we have one of the possibilities: $\varphi^{-1}(y)$ is a 2-cycle, $\varphi^{-1}(y)$ is a 3-cycle or $\varphi^{-1}(y)$ is a permutation of type 2^2.
In any case there exists a subgroup of $\mathbb{S}_9$, say K isomorphic to $\mathbb{S}_3$ such that

$$\varphi^{-1}(y) \in C_{\mathbb{S}_9}(K-\{1\})$$

and

$$C_{\mathbb{S}_9}(K-\{1\}) \cong \mathbb{S}_6.$$

Hence

$$y \in N \cap C_G(\varphi(K-\{1\})).$$

By Lemma 2.6 $C_G(\varphi(K-\{1\})) \cong \mathbb{S}_6$ and since

$$N \cap C_G(\varphi(K-\{1\})) \neq 1,$$

$\mathbb{A}_6$ is embedded in N.
If

$$N \cap C_G(\varphi(T-\{1\})) \neq 1,$$

then since $C_G(\varphi(T-\{1\}) \cong \mathbb{S}_6$ and

$$N \cap C_G(\varphi(T-\{1\})) \trianglelefteq C_G(\varphi(T-\{1\})),$$

we conclude that $\mathbb{A}_6$ is embedded in N in this case too.
Thus $2^3\cdot3^2\cdot5\big||N|$. We know that N is a direct product of isomorphic simple groups. But $5\big||N|$ and $5^2 \nmid |N|$, hence N is a simple group.
Moreover $5 \nsim 7$ in $\Gamma(\mathbb{S}_9)$ and since $\Gamma(G)=\Gamma(\mathbb{S}_9)$ by Lemma 2.5 ,$5 \nsim 7$ in $\Gamma(G)$ too. By Frattini's argument $N_G(N_5)N = G$, where N_5 is a Sylow 5-subgroup of N and since $7\big||G|$, $7\big||N_G(N_5)|$ or $7\big||N|$. If $7\big||N_G(N_5)|$, then there exists an element z of order 7 in $N_G(N_5)$ and so $\langle z\rangle N_5$ is a subgroup of $N_G(N_5)$ of order 5.7. Hence $\langle z\rangle N_5$ is a cyclic group. It means that $5 \sim 7$ in $\Gamma(G)$, which is a contradiction. Thus $7\big||N|$.
Now we assert that $C_G(N)=1$. Otherwise there is a minimal normal subgroup T of G such that

$T \leq C_G(N)$. By the same argument as above we see that $2^3 \cdot 3^2 \cdot 5 \cdot 7 \big| |T|$. Therefore $2^3 \cdot 3^2 \cdot 5 \cdot 7 \big| |C_G(N)|$. Hence $5 \big| |C_G(N)|$ and so there is an element $a \in C_G(N)$ such that $o(a) = 5$ and since $7 \big| |N|$, there is an element of order 7, say b in N. $o(ab) = 5 \cdot 7$, because $ab = ba$. But $5 \nsim 7$ in $\Gamma(G)$ and this is a contradiction. Thus $C_G(N) = 1$.

It implies that

$$G \cong \frac{G}{1} = \frac{G}{C_G(N)} \hookrightarrow Aut(N).$$

Therefore

$$9! = |G| \big| |Aut(N)|.$$

So we proved that N is a simple group with

$$2^3 \cdot 3^2 \cdot 5 \cdot 7 \big| |N|,$$
$$9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \big| |Aut(N)|$$

and $|N| \big| 2^7 \cdot 3^4 \cdot 5 \cdot 7$. By table 1 of [5] ,we conclude that $N \cong \mathbb{A}_9$. But

$$G \hookrightarrow Aut(N),$$
$$|G| = |\mathbb{S}_9|$$

and

$$Aut(N) \cong Aut(\mathbb{A}_9) \cong \mathbb{S}_9.$$

Hence $G \cong \mathbb{S}_9$. □

Lemma 2.8. *Let T be a finite group and $T \cong S_1 \times S_2 \times \cdots \times S_t$, where S_is are isomorphic simple groups, $1 \leq i \leq t$. Let T contain a copy of the alternating group $\mathbb{A}_{n-3}$, $n \geq 16$ and $|T| \big| n!$. Then T is a simple group.*

Proof. Without loss of generality we may assume that

$$T = S_1 \times S_2 \times \cdots \times S_t.$$

Suppose that $\pi_1 : S_1 \times S_2 \times \cdots \times S_t \to S_1 \times 1 \times \cdots \times 1$ is defined by

$$\pi_1(s_1, s_2, \ldots, s_t) = (s_1, 1, \ldots, 1)$$

and K is a subgroup of T isomorphic to $\mathbb{A}_{n-3}$. Set

$$\overline{S_1} = S_1 \times 1 \times \cdots \times 1$$

and

$$\overline{S_2 \times \cdots \times S_t} = 1 \times S_2 \times \cdots \times S_t.$$

Now we consider the following three cases.

Case 1) $K \cap \overline{S_1} = K \cap \overline{S_2 \times \cdots \times S_t} = 1$.

In this case $\phi : K \to \pi_1(K)$ defined by $\phi(k) = \pi_1(k)$ for all $k \in K$ is an isomorphism from K onto $\pi_1(K)$. This means that $K \cong \pi_1(K)$. Thus we have

$$\mathbb{A}_{n-3} \cong K \cong \pi_1(K) \leq \overline{S_1} \cong S_1$$

and so

$$\frac{(n-3)!}{2} = |\mathbb{A}_{n-3}|\big||S_1|.$$

But S_is are isomorphic simple groups, $1 \leq i \leq t$ and thus

$$\frac{(n-3)!}{2} = |\mathbb{A}_{n-3}|\big||S_i|, \quad 1 \leq i \leq t.$$

Therefore

$$\left[\frac{(n-3)!}{2}\right]^t \big||T|$$

and since $|T|\big|n!$, we obtain $\left[\frac{(n-3)!}{2}\right]^t\big|n!$. But

$$\left[\frac{(n-3)!}{2}\right]^2 \nmid n!$$

for $n \geq 16$ and so $t = 1$ and T is a simple group.

Case 2) $K \cap \overline{S_1} \neq 1$
Since $\overline{S_1} \trianglelefteq T$, we have

$$1 \neq K \cap \overline{S_1} \trianglelefteq K \cong \mathbb{A}_{n-3},$$

which implies that $K \cap \overline{S_1} = K$ and so

$$\mathbb{A}_{n-3} \cong K \leq \overline{S_1} \cong S_1.$$

Now similar argument as in Case (1) shows that T is a simple group .

Case 3) $K \cap \overline{S_2 \times \cdots \times S_t} \neq 1$
Since

$$\overline{S_2 \times \cdots \times S_t} \trianglelefteq T,$$

we have

$$1 \neq K \cap \overline{S_2 \times \cdots \times S_t} \trianglelefteq K \cong \mathbb{A}_{n-3},$$

which implies that

$$K \cap \overline{S_2 \times \cdots \times S_t} = K$$

and so

$$\mathbb{A}_{n-3} \cong K \leq \overline{S_2 \times \cdots \times S_t} \cong S_2 \times \cdots \times S_t.$$

Thus $\mathbb{A}_{n-3}$ is embedded in $S_2 \times \cdots \times S_t$.
By repeating above argument for

$$T_i = S_i \times \cdots \times S_t, \quad 2 \leq i \leq t,$$

we conclude that T is a simple group. □

Lemma 2.9. *Let a, b be two natural numbers. Then:*

1) $a^b.b! \leq (ab)!$ *and* $a^0.0! = (a.0)!$
2) *If* $a \geq 4$, *then* $a^{b-1}.b! \leq (a(b-1))!$
3) $3^{b-1}b! \leq (3b-3)!$
4) *If* $b \geq 3$, *then* $2^{b-1}b! \leq (2b-2)!$
5) *If* $b \geq 5$, *then* $2^{b-2}b! \leq 2(2b-4)!$
6) *If* $b \geq 4$, *then* $3^{b-2}b! \leq 2(3b-6)!$

Proof. 1) We prove Lemma 2.9 part 1 by induction on b. If $b = 1$, then clearly (1) holds. Suppose that $a^k k! \leq (ak)!$. We prove that $a^{k+1}(k+1)! \leq (ak+a)!$.
By induction hypothesis

$$a^{k+1}(k+1)! \leq (ak)!a(k+1).$$

But clearly

$$(ak)!a(k+1) \leq (ak+a)!$$

and so

$$a^{k+1}(k+1)! \leq (ak+a)!$$

and this completes the proof of (1).
2) We prove part 2 by induction on b. If $b = 1$, then clearly (2) holds. Suppose that

$$a^{k-1}k! \leq (a(k-1))!$$

for $k \geq 1$ and $a \geq 4$. We prove that

$$a^k(k+1)! \leq (ak)!.$$

By induction hypothesis,

$$a^k(k+1)! \leq (a(k-1))!a(k+1).$$

But since $ak \geq 4$, we have $ak - 1 \geq 3$ and so

$$(ak)(ak-1)\cdots(ak-a+1) \geq ak+a.$$

Thus $(ak)! \geq (ak-a)!a(k+1)$. Hence

$$a^k(k+1)! \leq (ak)!$$

and this completes the proof of (2).

3) We prove this part by induction on b too. If $b = 1$, then (3) clearly holds. Suppose that

$$3^{k-1}k! \leq (3k-3)!.$$

We prove that

$$3^k(k+1)! \leq (3k)!.$$

By induction hypothesis we obtain

$$3^k(k+1)! \leq (3k-3)!3(k+1).$$

It is easy to know that

$$k+1 \leq k(3k-1)(3k-2)$$

for $k \geq 1$. Thus

$$(3k-3)!3(k+1) \leq (3k)!$$

and so

$$3^k(k+1)! \leq (3k)!$$

and this completes the proof of (3).

4) We prove part (4) by induction on b. If $b = 3$, then clearly (4) holds. Suppose that

$$2^{k-1}k! \leq (2k-2)!$$

for $k \geq 3$. We prove that

$$2^k(k+1)! \leq (2k)!.$$

By induction hypothesis we obtain $2^k(k+1)! \leq (2k-2)!2(k+1)$. It is easy to see that $k+1 \leq k(2k-1)$ for $k \geq 3$. Thus

$$(2k-2)!2(k+1) \leq (2k)!$$

and so

$$2^k(k+1)! \leq (2k)!$$

and this completes the proof of (4).

5) We prove this part by induction on b. If $b = 5$, then (5) clearly holds. Suppose that

$$2^{k-2}k! \leq 2(2k-4)!$$

for $k \geq 5$. We prove that

$$2^{k-1}(k+1)! \leq 2(2k-2)!.$$

By induction hypothesis

$$2^{k-1}(k+1)! \leq 2(2k-4)!2(k+1).$$

But since

$$k^2 - 3k + 1 \geq 0$$

for $k \geq 5$, we have

$$k+1 \leq (k-1)(2k-3)$$

and so

$$2(2k-4)!2(k+1) \leq 2(2k-2)!.$$

Hence

$$2^{k-1}(k+1)! \leq 2(2k-2)!$$

and this completes the proof of (5).

6) We prove (6) by induction on b too. If $b = 4$, then (6) clearly holds. Suppose that

$$3^{k-2}k! \leq 2(3k-6)!$$

for $k \geq 4$. We prove that

$$3^{k-1}(k+1)! \leq 2(3k-3)!.$$

By induction hypothesis

$$3^{k-1}(k+1)! \leq 2(3k-6)!3(k+1).$$

It is easy to see that

$$3(k+1) \leq (3k-3)(3k-4)(3k-5)$$

for $k \geq 4$ and so

$$2(3k-6)!3(k+1) \leq 2(3k-3)!$$

for $k \geq 4$. Hence

$$3^{k-1}(k+1)! \leq 2(3k-3)!$$

and this completes the proof of (6). □

Lemma 2.10. *Let $a \geq 0$, $b \geq 0$ be two integers. Then $a!b! \leq (a+b)!$.*

Proof. If $a \geq 1$, $b \geq 1$, then since

$$a+b > b,$$
$$a+b-1 > b-1, \ldots,$$
$$a+1 > 1,$$

we have

$$(a+b)(a+b-1)\cdots(a+1) > b!$$

and so

$$(a+b)!$$
$$= (a+b)(a+b-1)\cdots(a+1)a! > b!a!.$$

If $a = 0$ or $b = 0$, then clearly $a!b! = (a+b)!$. □

Lemma 2.11. *Let $a_1, a_2, \ldots, a_m$ be integers with $a_i \geq 0$, $1 \leq i \leq m$. Then $a_1!a_2!\cdots a_m! \leq (a_1+\cdots+a_m)!$.*

Proof. We prove Lemma by induction on m. If $m = 1$, then clearly Lemma holds. Assume that

$$a_1!a_2!\cdots a_k!$$
$$\leq (a_1+a_2+\cdots+a_k)!.$$

We prove that

$$a_1!a_2!\cdots a_k!a_{k+1}!$$
$$\leq (a_1+a_2+\cdots+a_k+a_{k+1})!.$$

By induction hypothesis

$$a_1!a_2!\cdots a_k!a_{k+1}!$$
$$\leq (a_1+a_2+\cdots+a_k)!a_{k+1}!.$$

But by Lemma 2.10 we have

$$(a_1+\cdots+a_k)!a_{k+1}!$$
$$\leq (a_1+a_2+\cdots+a_k+a_{k+1})!.$$

Thus

$$a_1!a_2!\cdots a_{k+1}!$$
$$\leq (a_1+a_2+\cdots+a_{k+1})!.$$

□

Lemma 2.12. *Let l, m, n be three natural numbers with $n \geq 13$. Then the following holds.*

1) *If there exists a m-cycle,* $m \geq 4$ *in a cycle type of* $x \in \mathbb{S}_n$*, then* $|C_{\mathbb{S}_n}(x)| \leq m(n-m)!$
2) *If there exists two l-cycles in a cycle type of* $x \in \mathbb{S}_n$*, where* $l = 2$ *or* $l = 3$*, then* $|C_{\mathbb{S}_n}(x)| \leq l^2 2!(n-2l)!$
3) *If there exist a 2-cycle and a 3-cycle in a cycle type of* $x \in \mathbb{S}_n$*, then* $|C_{\mathbb{S}_n}(x)| \leq 2.3.(n-5)!$.

Proof. 1) Assume that $x \in \mathbb{S}_n$ is a permutation of type

$$1^{\alpha_1} \cdot 2^{\alpha_2} \cdots m^{\alpha_m} \cdots n^{\alpha_n},$$

where $\alpha_i \geq 0$, $1 \leq i \leq n$. By assumption $\alpha_m \geq 1$. Thus

$$|C_{\mathbb{S}_n}(x)| = 1^{\alpha_1}\alpha_1! \cdots m^{\alpha_m}\alpha_m! \cdots n^{\alpha_n}\alpha_n!,$$

where $\alpha_m \geq 1$. By Lemma 2.9 part 1 and 2 we conclude that

$$|C_{\mathbb{S}_n}(x)| \leq \alpha_1!(2\alpha_2)! \cdots m(m(\alpha_m - 1))! \cdots (n\alpha_n)!$$

and so by Lemma 2.11, we have

$$\begin{aligned} |C_{\mathbb{S}_n}(x)| &\leq m(\alpha_1 + 2\alpha_2 + \cdots + m(\alpha_m - 1) + \cdots + n\alpha_n)! \\ &= m(n-m)! \end{aligned}$$

and this completes the proof of (1).
2) Assume that $x \in \mathbb{S}_n$ is a permutation of type

$$1^{\alpha_1} \cdots 2^{\alpha_2} \cdots n^{\alpha_n}$$

where $\alpha_i \geq 0$, $1 \leq i \leq n$. By assumption $\alpha_l \geq 2$, where $l = 2$ or $l = 3$. First suppose that $l = 2$. We have

$$|C_{\mathbb{S}_n}(x)| = 1^{\alpha_1}\alpha_1! 2^{\alpha_2}\alpha_2! \cdots n^{\alpha_n}\alpha_n!.$$

If $\alpha_2 \geq 5$, then by Lemma 2.9 part 5 and 1 we conclude that

$$|C_{\mathbb{S}_n}(x)| \leq \alpha_1! 2^3.(2\alpha_2 - 4)! \cdots (n\alpha_n)!$$

and so by Lemma 2.11 we have

$$\begin{aligned} |C_{\mathbb{S}_n}(x)| &\leq 2^3(\alpha_1 + 2\alpha_2 - 4 + \cdots + n\alpha_n)! \\ &= 2^3(n-4)! = 2^2 2!(n-4)!. \end{aligned}$$

If $\alpha_2 = 2$, then

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&= 1^{\alpha_1}\alpha_1!2^2 \cdot 2! \cdots n^{\alpha_n}\alpha_n!.\end{aligned}$$

By part 1 of Lemma 2.9 and Lemma 2.11 we conclude that

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\le 2^2.2!\alpha_1!(3\alpha_3)! \cdots (n\alpha_n)!\\&\le 2^2 2!(n-4)!.\end{aligned}$$

If $\alpha_2 = 3$ or $\alpha_2 = 4$, then similar argument as case $\alpha_2 = 2$ shows us that

$$|C_{\mathbb{S}_n}(x)| \le 2^3 3!(n-6)!$$

or

$$|C_{\mathbb{S}_n}(x)| \le 2^4 4!(n-8)!$$

respectively and since

$$\begin{aligned}&2^3 3!(n-6)!\\&\le 2^2.2!(n-4)!\end{aligned}$$

and

$$\begin{aligned}&2^4 4!(n-8)!\\&\le 2^2 2!(n-4)!\end{aligned}$$

for $n \ge 13$, we have

$$|C_{\mathbb{S}_n}(x)| \le 2^2 2!(n-4)!$$

in this case too.

Now suppose that $l = 3$. If $\alpha_3 \ge 4$, then by Lemma 2.9 part 6 and 1 we have

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\le \alpha_1!(2\alpha_2)!3^2 2(3\alpha_3 - 6)! \cdots (n\alpha_n)!\end{aligned}$$

and so by Lemma 2.11 we have

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\le 3^2.2!(\alpha_1 + 2\alpha_2 + 3\alpha_3 - 6 + \cdots + n\alpha_n)!\\&= 3^2 2!(n-6)!.\end{aligned}$$

If $\alpha_3 = 2$, then

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&= 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^2 2!\cdots n^{\alpha_n}\alpha_n!.\end{aligned}$$

By Lemma 2.9 part 1 and Lemma 2.11 we conclude that

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\leq 3^2 2!\alpha_1!(2\alpha_2)!(4\alpha_4)!\cdots(n\alpha_n)!\\&\leq 3^2 2!(n-6)!.\end{aligned}$$

If $\alpha_3 = 3$, then similar argument as case $\alpha_3 = 2$ shows us that

$$|C_{\mathbb{S}_n}(x)| \leq 3^3.3!.(n-9)!$$

and since

$$3^3 3!(n-9)! \leq 3^2 2!(n-6)!$$

for $n \geq 13$, we have

$$|C_{\mathbb{S}_n}(x)| \leq 3^2 2!(n-6)!$$

in this case too and so the proof of (2) is complete.
3) Again assume that $x \in \mathbb{S}_n$ is a permutation of type

$$1^{\alpha_1} \cdot 2^{\alpha_2} \cdots n^{\alpha_n},$$

where $\alpha_i \geq 0$, $1 \leq i \leq n$. By assumption $\alpha_2 \geq 1$ and $\alpha_3 \geq 1$. We have

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&= 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^{\alpha_3}\alpha_3!\cdots n^{\alpha_n}\alpha_n!.\end{aligned}$$

If $\alpha_2 \geq 3$, then by Lemma 2.9 part 4,3 and 1 we have

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\leq \alpha_1!2(2\alpha_2-2)!3(3\alpha_3-3)!\cdots(n\alpha_n)!\end{aligned}$$

and so by Lemma 2.11

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\leq 2.3.(\alpha_1+2\alpha_2-2+3\alpha_3-3+\cdots+n\alpha_n)!\\&= 2.3.(n-5)!.\end{aligned}$$

If $\alpha_2 = 1$, then we have

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&= 1^{\alpha_1}\alpha_1!.2.3^{\alpha_3}\alpha_3!\cdots n^{\alpha_n}\alpha_n!.\end{aligned}$$

By Lemma 2.9 part 1 and 3 we have

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\leq \alpha_1!\cdot 2\cdot 3\cdot(3\alpha_3-3)!\cdots(n\alpha_n)!\end{aligned}$$

and so by Lemma 2.11

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\leq 2\cdot 3\cdot(\alpha_1+3\alpha_3-3+\cdots+n\alpha_n)!\\&=2\cdot 3\cdot(n-5)!.\end{aligned}$$

If $\alpha_2=2$, then similar argument as case $\alpha_2=1$ shows us that

$$\begin{aligned}&|C_{\mathbb{S}_n}(x)|\\&\leq \alpha_1!2^2\cdot 2!\cdot 3\cdot(3\alpha_3-3)!\cdots(n\alpha_n)!\\&\leq 2^2\cdot 2!\cdot 3(\alpha_1+3\alpha_3-3+\cdots+n\alpha_n)!\\&=2^2\cdot 2!\cdot 3(n-7)!\end{aligned}$$

and since

$$\begin{aligned}&2^2.2!.3(n-7)!\\&\leq 2.3(n-5)!\end{aligned}$$

for $n\geq 13$, we have

$$|C_{\mathbb{S}_n}(x)|\leq 2.3(n-5)!$$

in this case too and the proof of (3) is complete. □

Lemma 2.13. *Let l, k be two natural numbers with $l>1$ and $1<l+k<n-1$, where $n\geq 13$ is a natural number. Then $l(n-l)!>(l+k)(n-l-k)!$*

Proof. We prove Lemma 2.12 by induction on k. If $k=1$, then since $n-l>2$, $l>1$, we have $l(n-l)>l+1$ and so

$$l(n-l)!>(l+1)(n-l-1)!.$$

Thus the lemma holds whenever $k=1$. Suppose that if

$$\begin{aligned}&1<l+k<n-1,\\&l>1,\end{aligned}$$

then

$$l(n-l)!>(l+k)(n-l-k)!.$$

We prove the lemma for $k+1$.
Suppose that

$$1 < l+k+1 < n-1,$$
$$l > 1.$$

Since

$$(n-l-k) > 2,$$
$$l+k > 1,$$

we have

$$(l+k)(n-l-k)$$
$$> 2(l+k) > l+k+1$$

and so

$$(l+k)(n-l-k)!$$
$$> (l+k+1)(n-l-k-1)!.$$

Thus by induction hypothesis we conclude that

$$l(n-l)!$$
$$> (l+k+1)(n-l-k-1)!.$$

Hence the lemma is proved. □

Lemma 2.14. *Let l, m, n be three natural numbers with $l > 1$, $n \geq 13$, $m \neq n$ and $l \leq m$. Then $l(n-l)! \geq m(n-m)!$*

Proof. If $l = m$, then clearly Lemma holds. If $l < m$ and $1 < m < n-1$, then since $l > 1$, Lemma 2.14 concluded from Lemma 2.13. But if $l < m$ and $m = n-1$, then we have

$$m(n-m)!$$
$$= (n-1)1!$$
$$= n-1.$$

We have $(n-1) < (n-2)2$ for $n \geq 13$ and since $1 < n-2 < n-1$ by above argument for all $1 < l \leq n-2$ we have

$$l(n-l)! \geq (n-2)2!.$$

Hence $l(n-l)! > n-1$, also if $l = n-1$, clearly

$$l(n-l)! \geq n-1.$$

So the proof is complete. □

Lemma 2.15. *If $x \in \mathbb{S}_n$ and $|x^{\mathbb{S}_n}| \leq n(n-1)(n-2)$, where $x^{\mathbb{S}_n}$ is the conjugacy class of $\mathbb{S}_n$, $n \geq 13$ containing x. Then $x = 1$, x is a 2-cycle or x is a 3-cycle and $|x^{\mathbb{S}_n}| = 1$, $|x^{\mathbb{S}_n}| = \frac{n(n-1)}{2}$ or $|x^{\mathbb{S}_n}| = \frac{n(n-1)(n-2)}{3}$.*

Proof. Suppose that $|x^{\mathbb{S}_n}| \leq n(n-1)(n-2)$. Then

$$|C_{\mathbb{S}_n}(x)| \geq \frac{n!}{n(n-1)(n-2)} = (n-3)!.$$

If there exists a m-cycle, $m \geq 4$ in a cycle type of x, then by Lemma 2.12 part 1

$$|C_{\mathbb{S}_n}(x)| \leq m(n-m)!$$

and by Lemma 2.14 we conclude that if $m \neq n$, then

$$m(n-m)! \leq 4(n-4)!.$$

But if $m = n$, then $m(n-m)! = n$. It is easy to know that

$$n < 4(n-4)!$$

for $n \geq 13$. Therefore if there exists a m-cycle, $m \geq 4$ in a cycle type of x, then

$$|C_{\mathbb{S}_n}(x)| \leq 4(n-4)!.$$

But we have $|C_{\mathbb{S}_n}(x)| \geq (n-3)!$ and so

$$(n-3)! \leq 4(n-4)!,$$

which is a contradiction, because $n \geq 13$. Thus there is no m-cycle, $m \geq 4$ in a cycle type of x. If there exist two 2-cycles or two 3-cycles in a cycle type of x, then by Lemma 2.12 part 2 we conclude that

$$|C_{\mathbb{S}_n}(x)| \leq 2^2 2!(n-4)!$$

or

$$|C_{\mathbb{S}_n}(x)| \leq 3^2 2!(n-6)!$$

respectively and so

$$(n-3)! \leq 2^2 2!(n-4)!$$

or

$$(n-3)! \leq 3^2 2!(n-6)!,$$

which is a contradiction, because $n \geq 13$. Also if there exists a 3-cycle and a 2-cycle in a cycle type of x, then by Lemma 2.12 part 3 we conclude that

$$|C_{\mathbb{S}_n}(x)| \leq 2.3.(n-5)!$$

and so

$$(n-3)! \leq 2.3.(n-5)!,$$

which is a contradiction with $n \geq 13$. Thus $x = 1$ or x is a 2-cycle or x is a 3-cycle. Hence $|x^{\mathbb{S}_n}| = 1$ or $|x^{\mathbb{S}_n}| = \frac{n(n-1)}{2}$ or $x^{\mathbb{S}_n} = \frac{n(n-1)(n-2)}{3}$. □

Lemma 2.16. *Let x be an element of $\mathbb{S}_n$, $n \geq 13$. If $|C_{\mathbb{S}_n}(x)| = 3(n-3)!$, then x is a 3-cycle.*

Proof. If $|C_{\mathbb{S}_n}(x)| = 3(n-3)!$, then

$$|C_{\mathbb{S}_n}(x)| \geq (n-3)!$$

and so by Lemma 2.15 we conclude that $x = 1$ or x is a 2-cycle or x is a 3-cycle. But if $x = 1$ or x is a 2-cycle, then clearly

$$|C_{\mathbb{S}_n}(x)| \neq 3(n-3)!.$$

($n! \neq 3(n-3)!$ and $2(n-2)! \neq 3(n-3)!$) and so x is a 3-cycle. □

3. Main result

In this section we will prove our main result.

Theorem 3.1. *Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, where $\mathbb{S}_n$ is the symmetric group of degree n and $n \geq 3$, then $G \cong \mathbb{S}_n$.*

Proof. By Lemma 2.1, we have $|G| = |\mathbb{S}_n|$. Since $\nabla(G) \cong \nabla(\mathbb{S}_n)$,

$$|G - Z(G)| = |\mathbb{S}_n - Z(\mathbb{S}_n)| = |\mathbb{S}_n| - 1$$

and so $|Z(G)| = 1$.
By Lemmas 2.6 and 2.7 we may assume that $n \geq 16$. Without loss of generality we can assume that $\varphi : \mathbb{S}_n \to G$ and $\varphi(1) = 1$, where φ is an isomorphism from $\nabla(\mathbb{S}_n)$ to $\nabla(G)$.
Now we prove the theorem by induction on n, where $n \geq 16$. If $n = 16$, then theorem holds by Lemma 2.7. Suppose the theorem is true for all $m < n$ and assume that $n \geq 16$. We will prove that the result is valid for $\mathbb{S}_n$.
Set

$$A = \{\alpha \in \mathbb{S}_n | (i)\alpha = i, i = 4, 5, \ldots, n\}.$$

Clearly

$$A \leq \mathbb{S}_n,$$

$$A \cong \mathbb{S}_3.$$

By Lemma 2.2 we have

$$\nabla(C_{\mathbb{S}_n}(A)) \cong \nabla(C_G(\varphi(A)))$$

and since $C_{\mathbb{S}_n}(A) \cong \mathbb{S}_{n-3}$, we have

$$\nabla(\mathbb{S}_{n-3}) \cong \nabla(C_G(\varphi(A))).$$

Thus by induction hypothesis $C_G(\varphi(A)) \cong \mathbb{S}_{n-3}$. Therefore G has a subgroup isomorphic to $\mathbb{S}_{n-3}$ i.e. $C_G(\varphi(A))$.

Let $H = C_G(\varphi(A))$. Now we assume that N is an arbitrary minimal normal subgroup of G. We will prove that N is a simple group and that

$$\mathbb{A}_{n-3} \hookrightarrow N \cap P$$

for all subgroups P of G isomorphic to $\mathbb{S}_{n-3}$. In particular N contains all even permutations of P, for all

$$P \leq G,$$
$$P \cong \mathbb{S}_{n-3}.$$

Let P be an arbitrary subgroup of G isomorphic to $\mathbb{S}_{n-3}$. We have $N \cap P \trianglelefteq P$. We assert that $N \cap P \neq 1$. If $N \cap P = 1$, then we have

$$|NP| = |N||P| \big| |G| = n!.$$

Thus

$$|N|.(n-3)! \big| n!,$$

since $|P| = (n-3)!$. This implies that $|N| \big| n(n-1)(n-2)$. Moreover N is a union of conjugacy classes of G and the size of conjugacy class of G containing x is equal to the size of conjugacy class of $\mathbb{S}_n$ containing $\varphi^{-1}(x)$ for all $x \in G - \{1\}$.

By Lemma 2.15 we see that all conjugacy class sizes less than $n(n-1)(n-2)$ in $\mathbb{S}_n$, $n \geq 16$ are 1, $\frac{n(n-1)}{2}$ and $\frac{n(n-1)(n-2)}{3}$.

Let y be an arbitrary element of $N - \{1\}$. Thus the size of the conjugacy class of G containing y and so the size of conjugacy class of $\mathbb{S}_n$ containing $\varphi^{-1}(y)$ is equal to $\frac{n(n-1)}{2}$ or $\frac{n(n-1)(n-2)}{3}$. Also by Lemma 2.15 $\varphi^{-1}(y)$ is a 2-cycle or $\varphi^{-1}(y)$ is a 3-cycle.

In any case there exists a subgroup of $\mathbb{S}_n$, say E isomorphic to $\mathbb{S}_3$ such that $\varphi^{-1}(y) \in C_{\mathbb{S}_n}(E)$ and

$$C_{\mathbb{S}_n}(E) \cong \mathbb{S}_{n-3}.$$

So $y \in C_G(\varphi(E))$, also we know that $y \in N - \{1\}$. Therefore

$$y \in N \cap C_G(\varphi(E))$$

and

$$N \cap C_G(\varphi(E)) \neq 1.$$

By Lemma 2.2

$$\begin{aligned}\nabla(\mathbb{S}_{n-3}) &\cong \nabla(C_{\mathbb{S}_n}(E)) \\ &\cong \nabla(C_G(\varphi(E)))\end{aligned}$$

and so by induction hypothesis

$$C_G(\varphi(E)) \cong \mathbb{S}_{n-3}.$$

Since

$$\begin{aligned}1 &\neq N \cap C_G(\varphi(E)) \\ &\trianglelefteq C_G(\varphi(E)) \cong \mathbb{S}_{n-3},\end{aligned}$$

we conclude that

$$\mathbb{A}_{n-3} \hookrightarrow N \cap C_G(\varphi(E)).$$

Set $R = N \cap C_G(\varphi(E))$. Therefore

$$\frac{(n-3)!}{2} \,\Big|\, |R|.$$

Since $P \cap N = 1$,

$$\begin{aligned}&P \cap R \\ &\subseteq P \cap N = 1\end{aligned}$$

and so $P \cap R = 1$. Thus $|PR| = |P||R|$. On the other hand $|P| = (n-3)!$ and

$$\frac{(n-3)!}{2} \,\Big|\, |R|.$$

So

$$\frac{[(n-3)!]^2}{2} \,\Big|\, |P||R| = |PR|.$$

But since $PR \subseteq G$, we have

$$|PR| \leq |G| = n!.$$

So

$$\frac{[(n-3)!]^2}{2} \leq n!,$$

which is a contradiction, since we assumed that $n \geq 16$. Hence $P \cap N \neq 1$ for all subgroup P of G isomorphic to $\mathbb{S}_{n-3}$. In particular $N \cap H \neq 1$. Also

$$1 \neq N \cap P \trianglelefteq P \cong \mathbb{S}_{n-3}$$

implies that

$$\mathbb{A}_{n-3} \hookrightarrow N \cap P$$

for all $P \leq G$, $P \cong \mathbb{S}_{n-3}$.
Since N is a minimal normal subgroup of G, N is a direct product of isomorphic simple group, say

$$N \cong S_1 \times \cdots \times S_t,$$

where S_i's are isomorphic simple groups, $1 \leq i \leq t$. Also since

$$\mathbb{A}_{n-3} \hookrightarrow N \cap H,$$

$\mathbb{A}_{n-3} \hookrightarrow N$. Thus by Lemma 2.8 N is a simple group.

Next set

$$B = \{\beta \in \mathbb{S}_n | (i)\beta = i, i = 1, 2, \ldots, n-3\}.$$

Clearly

$$B \leq \mathbb{S}_n,$$
$$B \cong \mathbb{S}_3$$

and

$$C_{\mathbb{S}_n}(B) \cong \mathbb{S}_{n-3}.$$

It is easy to see that

$$C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B) \cong \mathbb{S}_{n-6}.$$

By Lemma 2.2 we have

$$\begin{aligned}
&\nabla(\mathbb{S}_{n-6}) \\
&\cong \nabla(C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B)) \\
&= \nabla(C_{\mathbb{S}_n}(A \cup B)) \\
&\cong \nabla(C_G(\varphi(A \cup B))) \\
&= \nabla(C_G(\varphi(A) \cup \varphi(B))) \\
&= \nabla(C_G(\varphi(A)) \cap C_G(\varphi(B)))
\end{aligned}$$

and so by induction hypothesis

$$C_G(\varphi(A)) \cap C_G(\varphi(B)) \cong \mathbb{S}_{n-6}.$$

Similarly $C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$.
By above argument

$$\mathbb{A}_{n-3} \hookrightarrow N \cap C_G(\varphi(A))$$

and

$$\mathbb{A}_{n-3} \hookrightarrow N \cap C_G(\varphi(B)).$$

Let

$$L \leq N \cap C_G(\varphi(A)),$$
$$K \leq N \cap C_G(\varphi(B))$$

and $L \cong K \cong \mathbb{A}_{n-3}$. We have

$$\begin{aligned} & L \cap K \\ & \leq N \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) \\ & \leq C_G(\varphi(A)) \cap C_G(\varphi(B)) \\ & \cong \mathbb{S}_{n-6}. \end{aligned}$$

Now we will prove the following claim.

Claim: $L \cap K \neq C_G(\varphi(A)) \cap C_G(\varphi(B))$

Suppose by way of contradiction, that

$$L \cap K = C_G(\varphi(A)) \cap C_G(\varphi(B)).$$

Assume that $a = (1\ 2\ 3) \in \mathbb{S}_n$. Clearly $a \in C_{\mathbb{S}_n}(B)$. Since

$$\begin{aligned} & |C_{\mathbb{S}_n}(B) \cap C_{\mathbb{S}_n}(a)| \\ & = |C_{C_{\mathbb{S}_n}(B)}(a)| = 3(n-6)!, \end{aligned}$$

we conclude that

$$\begin{aligned} & |C_G(\varphi(B)) \cap C_G(\varphi(a))| \\ & = |C_{C_G(\varphi(B))}(\varphi(a))| = 3(n-6)!. \end{aligned}$$

But

$$C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$$

and by Lemma 2.16 if

$$\begin{aligned} & y \in \mathbb{S}_{n-3}, \\ & |C_{\mathbb{S}_{n-3}}(y)| = 3(n-6)!, \\ & n \geq 16, \end{aligned}$$

then y is a 3-cycle. Thus $\varphi(a)$ is a 3-cycle in $C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$.
Therefore $\varphi(a)$ is an even permutation in $C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$ and so

$$\varphi(a) \in K \cong \mathbb{A}_{n-3}$$

(Note that $K \leq C_G(\varphi(B))$). Also we have

$$C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B) \subseteq C_{\mathbb{S}_n}(a),$$

so

$$L \cap K = C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_G(\varphi(a)),$$

which implies that

$$C_G(\varphi(A)) \cap C_G(\varphi(B))$$
$$\leq C_G(\varphi(a)) \cap K = C_K(\varphi(a).$$

On the other hand $\langle \varphi(a) \rangle \leq C_K(\varphi(a))$. Since

$$C_{\mathbb{S}_n}(C_{\mathbb{S}_n}(a)) \cap C_{\mathbb{S}_n}(A)$$
$$\cap C_{\mathbb{S}_n}(B) = 1,$$

we have

$$\langle \varphi(a) \rangle \cap C_G(\varphi(A)) \cap C_G(\varphi(B))$$
$$\subseteq C_G(C_G(\varphi(a))) \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) = 1$$

and so

$$\langle \varphi(a) \rangle \cap C_G(\varphi(A))$$
$$\cap C_G(\varphi(B)) = 1.$$

Therefore

$$|\langle \varphi(a) \rangle C_G(\varphi(A)) \cap C_G(\varphi(B))|$$
$$= |\langle \varphi(a) \rangle||C_G(\varphi(A)) \cap C_G(\varphi(B))| = 3(n-6)!.$$

Moreover since a commutes with all elements of $C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B)$, $\varphi(a)$ commutes with all elements of

$$C_G(\varphi(A)) \cap C_G(\varphi(B)).$$

So

$$\langle \varphi(a) \rangle C_G(\varphi(A))$$
$$\cap C_G(\varphi(B)) \leq G.$$

But we have

$$\langle\varphi(a)\rangle \leq C_K(\varphi(a)),$$
$$C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_K(\varphi(a))$$

and thus

$$\langle\varphi(a)\rangle C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_K(\varphi(a)).$$

Hence

$$3(n-6)! \big| |C_K(\varphi(a))|,$$

where $K \cong \mathbb{A}_{n-3}$. But This is impossible, because $\varphi(a)$ is a 3-cycle in K and so

$$|C_K(\varphi(a))| = |C_{\mathbb{A}_{n-3}}(\varphi(a))|$$
$$= 3.\frac{(n-6)!}{2}.$$

Hence

$$L \cap K \neq C_G(\varphi(A)) \cap C_G(\varphi(B))$$

and the claim is proved. For the order of N we will prove the followings:

1. $|N| > \frac{n!}{4}$

We know that $L, K \leq N$ and $|L| = |K| = \frac{(n-3)!}{2}$. Also

$$L \cap K \lneqq C_G(\varphi(A)) \cap C_G(\varphi(B))$$

and so

$$|L \cap K| \leq \frac{|C_G(\varphi(A)) \cap C_G(\varphi(B))|}{2}.$$

From $L, K \leq N$, we deduce that $LK \leq N$. Thus

$$|N| \geq |LK|$$
$$= \frac{|L||K|}{|L \cap k|}$$
$$\geq \frac{|L||K|}{\frac{|C_G(\varphi(A)) \cap C_G(\varphi(B))|}{2}}$$
$$= \frac{\frac{(n-3)!}{2}\frac{(n-3)!}{2}}{\frac{(n-6)!}{2}}.$$

On the other hand

$$\frac{\frac{(n-3)!}{2}\frac{(n-3)!}{2}}{\frac{(n-6)!}{2}} > \frac{n!}{4},$$

for $n \geq 16$. Thus $|N| > \frac{n!}{4}$.

2. $|N| \neq \frac{n!}{3}$

We know that

$$C_G(\varphi(A)) \cong C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$$

and

$$N \cap C_G(\varphi(A)) \neq 1$$

and

$$N \cap C_G(\varphi(B)) \neq 1.$$

If $C_G(\varphi(A)) \leq N$ and $C_G(\varphi(B)) \leq N$, then

$$C_G(\varphi(A))C_G(\varphi(B)) \subseteq N.$$

Thus

$$\begin{aligned} |N| &\geq |C_G(\varphi(A))C_G(\varphi(B))| \\ &= \frac{|C_G(\varphi(A))||C_G(\varphi(B))|}{|C_G(\varphi(A)) \cap C_G(\varphi(B))|} \\ &= \frac{(n-3)!(n-3)!}{(n-6)!}. \end{aligned}$$

But

$$\frac{(n-3)!(n-3)!}{(n-6)!} > \frac{n!}{2}$$

for $n \geq 16$, which implies that $|N| = |G|$ and since N is an arbitrary minimal normal subgroup of G, we conclude that G is a simple group. By assumption $\nabla(G) \cong \nabla(\mathbb{S}_n)$ and [6] we have $G \cong \mathbb{S}_n$, so $\mathbb{S}_n$ must be a simple group too, which is a contradiction.
Hence

$$N \cap C_G(\varphi(A)) \neq C_G(\varphi(A))$$

or

$$N \cap C_G(\varphi(B)) \neq C_G(\varphi(B)).$$

Suppose that

$$N \cap C_G(\varphi(A)) \neq C_G(\varphi(A)).$$

We know that

$$\begin{aligned} 1 &\neq N \cap C_G(\varphi(A)) \\ &\trianglelefteq C_G(\varphi(A)) \cong \mathbb{S}_{n-3}. \end{aligned}$$

Therefore

$$|N \cap C_G(\varphi(A))| = |\mathbb{A}_{n-3}| = \frac{(n-3)!}{2}$$

and so we have

$$\begin{aligned} &|NC_G(\varphi(A))| \\ &= \frac{|N||C_G(\varphi(A))|}{|N \cap C_G(\varphi(A))|} \\ &= \frac{|N|(n-3)!}{\frac{(n-3)!}{2}} = 2|N|. \end{aligned}$$

Moreover $N \trianglelefteq G$ implies that $NC_G(\varphi(A)) \leq G$. Thus

$$|NC_G(\varphi(A))| = 2|N| \big| |G| = n!.$$

Now if $|N| = \frac{n!}{3}$, then we have $\frac{2n!}{3} \big| n!$, a contradiction. This shows that $|N| \neq \frac{n!}{3}$.

3. $|N| = \frac{n!}{2}$

From $|N| > \frac{n!}{4}$ and $|N| \big| |G| = n!$, we conclude that $|N|$ is equal to one of $\frac{n!}{3}$, $\frac{n!}{2}$ or $n!$. By 2 $|N| \neq \frac{n!}{3}$. If $|N| = |G| = n!$, then G is a simple group, since N is an arbitrary minimal normal subgroup of G. By assumption $\nabla(G) \cong \nabla(\mathbb{S}_n)$. Now since G is a simple group, by [6] $G \cong \mathbb{S}_n$. So $\mathbb{S}_n$ must be a simple group too, a contradiction. Hence $|N| = \frac{n!}{2}$.

From $|N| = \frac{n!}{2}$, simplicity of N and by corollary 2.4, $N \cong \mathbb{A}_n$. We assert that $C_G(N) = 1$. Otherwise there is a minimal normal subgroup of G, say M such that $M \leq C_G(N)$. We proved that all minimal normal subgroups of G are isomorphic to $\mathbb{A}_n$. Thus $M \cong \mathbb{A}_n$ and since

$$\begin{aligned} &N \cap C_G(N) = Z(N) = 1, \\ &M \cap N = 1. \end{aligned}$$

On the other hand $MN \leq G$ and so

$$|MN| = |M||N| \big| |G|.$$

It follows that $(\frac{n!}{2})^2 \big| |G| = n!$, a contradiction. Hence $C_G(N) = 1$ and so

$$\begin{aligned} G &\cong \frac{G}{1} \\ &= \frac{G}{C_G(N)} \hookrightarrow Aut(N) \end{aligned}$$

and since for $n \geq 16$,

$$Aut(N) \cong Aut(\mathbb{A}_n) \cong \mathbb{S}_n,$$

we conclude that G is embedded into $\mathbb{S}_n$. But $|G| = |\mathbb{S}_n|$ and so $G \cong \mathbb{S}_n$.

□

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