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CHARACTERIZATION OF THE SYMMETRIC GROUP BY ITS NON-COMMUTING GRAPH

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ABSTRACT. The non-commuting graph $\nabla(G)$ of a non-abelian group G is defined as follows: its vertex set is G - Z(G) and two distinct vertices x and y are joined by an edge if and only if the commutator of x and y is not the identity. In this paper we prove that if G is a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, then $G \cong \mathbb{S}_n$, where \mathbb{S}_n is the symmetric group of degree n, where n is a natural number.

1. Introduction

Let G be a group. The non-commuting graph $\nabla(G)$ of G is defined as follows: the set of vertices of $\nabla(G)$ is G - Z(G), where Z(G) is the center of G and two vertices are connected whenever they do not commute. Also we define the prime graph $\Gamma(G)$ of G as follows: the vertices of $\Gamma(G)$ are the prime divisors of the order of G and two distinct vertices p and q are joined by an edge and we write $p \sim q$, if there is an element in G of order pq. We denote by $\pi_e(G)$ the set of orders of elements of G. The connected components of $\Gamma(G)$ are denoted by π_i , $i = 1, 2, \ldots, t(G)$, where t(G) is the number of components. We can express the order of G as a product of some positive integer m_i , $i = 1, 2, \ldots, t(G)$ with $\pi(m_i) = \pi_i$. The integers m_i s are called the order components of G. In 2006, A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture in [1] as follows.

AAM's Conjecture: If M is a finite non abelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$.

Ron Solomon and Andrew Woldar proved the above conjecture in [6]. In this paper we will prove that if G is a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, then $G \cong \mathbb{S}_n$, where \mathbb{S}_n is the symmetric group of degree

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n. Therefore we extend the AAM's conjecture to the case, where M is not necessarily a finite simple group.

2. Preliminaries

The following result was proved in part(1) of Theorem 3.16 of [1].

Lemma 2.1. Let G be a finite group such that $\nabla(G) \cong \nabla(\mathbb{S}_n)$, $n \ge 3$. Then $|G| = |\mathbb{S}_n|$.

Lemma 2.2. Let G and H be two non-abelian groups. If $\nabla(G) \cong \nabla(H)$, then

$$\nabla(C_G(A)) \cong \nabla(C_H(\varphi(A)))$$

for all $\emptyset \neq A \subseteq G - Z(G)$, where φ is the isomorphism from $\nabla(G)$ to $\nabla(H)$ and $C_G(A)$ is non-abelian.

Proof. It is sufficient to show that $\varphi \mid_{V(C_G(A))} : V(C_G(A)) \longrightarrow V(C_H(\varphi(A)))$ is onto, where $\varphi \mid_{V(C_G(A))}$ is the restriction of φ to $V(C_G(A))$ and

$$V(C_G(A)) = C_G(A) - Z(C_G(A)),$$

$$V(C_H(\varphi(A))) = C_H(\varphi(A)) - Z(C_H(\varphi(A))).$$

Assume d is an element of $V(C_H(\varphi(A)))$, then $d \in H - Z(H)$ and so there exists an element c of G - Z(G) such that $\varphi(c) = d$. From

$$d = \varphi(c) \in C_H(\varphi(A)),$$

it follows that $[\varphi(c), \varphi(g)] = 1$ for all $g \in A$ and since φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$, [c,g] = 1 for all $g \in A$. Therefore $c \in C_G(A)$. But $d \notin Z(C_H(\varphi(A)))$, so for an element $x \in C_H(\varphi(A))$ we have $[x,d] \neq 1$. Hence x is an element of H that does not commute with $d \in H$. This implies that $x \in H - Z(H)$. Thus there exists $x' \in G - Z(G)$, such that $\varphi(x') = x$. It is easy to see that $[x', c] \neq 1$ and therefore $c \notin Z(C_G(A))$. Hence

$$c \in C_G(A) - Z(C_G(A)) = V(C_G(A))$$

and $\varphi(c) = d$.

The following result was proved by E. Artin in [2] and [3] and together with the classification of finite simple groups can be stated as follows:

Lemma 2.3. Let G and M be finite simple groups, |G| = |M|, then the following holds:

- (1) If $|G| = |A_8| = |L_3(4)|$, then $G \cong A_8$ or $G \cong L_3(4)$;
- (2) If $|G| = |B_n(q)| = |C_n(q)|$, where $n \ge 3$, and q is odd, then $G \cong B_n(q)$ or $G \cong C_n(q)$;
- (3) If M is not in the above cases, then $G \cong M$.

As an immediate consequence of Lemma 2.3, we get the following corollary.

Corollary 2.4. Let G be a finite simple group with $|G| = |A_n|$, where n is a natural number, $n \ge 5$, $n \ne 8$, then $G \cong A_n$.

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Lemma 2.5. Let G and H be two finite groups with $\nabla(G) \cong \nabla(H)$ and |G| = |H|. Then $p_1 p_2 \cdots p_t \in \pi_e(G)$ if and only if $p_1 p_2 \cdots p_t \in \pi_e(H)$, where p_i 's are distinct prime numbers for $i = 1, 2, \ldots, t$. In particular, $\Gamma(G) = \Gamma(H)$.

Proof. If φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$ and |G| = |H|, then we can easily see that

$$|Z(C_G(x))| = |Z(C_H(\varphi(x)))|$$

for all $x \in G$. If $p_1 p_2 \cdots p_t \in \pi_e(G)$, then there exists an element $z \in G$ such that $o(z) = p_1 p_2 \cdots p_t$. Thus

$$p_1 p_2 \cdots p_t = |\langle z \rangle| ||Z(C_G(z))|$$

and so

$$p_1 p_2 \cdots p_t ||Z(C_H(\varphi(z)))|$$

Hence H has an abelian subgroup of order $p_1p_2\cdots p_t$, which is a cyclic group. Therefore $p_1p_2\cdots p_t \in \pi_e(H)$. By a similar argument we see that if $p_1p_2\cdots p_t \in \pi_e(H)$, then $p_1p_2\cdots p_t \in \pi_e(G)$.

Lemma 2.6. Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, where $3 \le n \le 8$ or $11 \le n \le 14$, then $G \cong \mathbb{S}_n$.

Proof. Since $\nabla(G) \cong \nabla(\mathbb{S}_n)$, by Lemma 2.1, $|G| = |\mathbb{S}_n|$. Also by Lemma 2.5 $\Gamma(G) = \Gamma(\mathbb{S}_n)$, where Γ denotes the prime graph. Thus the order components of G and \mathbb{S}_n are the same. In [7] it is proved that \mathbb{S}_p and \mathbb{S}_{p+1} are characterizable by their order components, where $p \ge 3$ is a prime number. Hence \mathbb{S}_n , where $3 \le n \le 8$ or $11 \le n \le 14$ is characterizable by their order components and so $G \cong \mathbb{S}_n$, where $3 \le n \le 8$ or $11 \le n \le 14$

Lemma 2.7. Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, n = 9, 10, 15, 16, then $G \cong \mathbb{S}_n$

Proof. We give the proof in the case n = 9, the proof in other cases is similar. Set

$$T = \{ \alpha \in \mathbb{S}_9 | (i)\alpha = i, i = 4, 5, \dots, 9 \}.$$

Obviously

 $T \leq \mathbb{S}_9,$ $T \cong \mathbb{S}_3$

and $C_{\mathbb{S}_9}(T - \{1\}) \cong \mathbb{S}_6$. By Lemma 2.2 we have

$$\nabla(C_{\mathbb{S}_9}(T-\{1\})) \cong \nabla(C_G(\varphi(T-\{1\}))),$$

where φ is an isomorphism from $\nabla(\mathbb{S}_9)$ to $\nabla(G)$. Thus by Lemma 2.6 $C_G(\varphi(T - \{1\})) \cong \mathbb{S}_6$. Let N be a minimal normal subgroup of G. If

$$N \cap C_G(\varphi(T - \{1\})) = 1,$$

then since

$$|NC_G(\varphi(T - \{1\}))|||G| = 9!$$

and

$$|C_G(\varphi(T - \{1\}))| = 6!,$$

we have $|N||9\cdot 8\cdot 7$. We know that N is a union of conjugacy classes of G and the size of conjugacy class of G containing x is equal to the size of conjugacy class of \mathbb{S}_9 containing $\varphi^{-1}(x)$ for all $x \in G - \{1\}$. We can see that all conjugacy class sizes in \mathbb{S}_9 less than $9 \cdot 8 \cdot 7$ are $1, \frac{9\cdot 8}{2}, \frac{9\cdot 8\cdot 7}{3}$ and $\frac{9\cdot 8\cdot 7\cdot 6}{8}$.

Let y be an arbitrary element in $N - \{1\}$. Thus the size of conjugacy class of y in G and so the size of conjugacy class of $\varphi^{-1}(y)$ in \mathbb{S}_9 is equal to $\frac{9.8}{2}$, $\frac{9.8\cdot7}{3}$ or $\frac{9.8\cdot7\cdot6}{8}$. Therefore we have one of the perceptilities: $(\varphi^{-1}(y))$ is a 2 grade $(\varphi^{-1}(y))$ is a 2 grade or $(\varphi^{-1}(y))$ is a

Therefore we have one of the possibilities: $\varphi^{-1}(y)$ is a 2-cycle, $\varphi^{-1}(y)$ is a 3-cycle or $\varphi^{-1}(y)$ is a permutation of type 2².

In any case there exists a subgroup of S_9 , say K isomorphic to S_3 such that

$$\varphi^{-1}(y) \in C_{\mathbb{S}_9}(K - \{1\})$$

and

$$C_{\mathbb{S}_9}(K-\{1\})\cong \mathbb{S}_6.$$

Hence

$$y \in N \cap C_G(\varphi(K - \{1\})).$$

By Lemma 2.6 $C_G(\varphi(K - \{1\})) \cong \mathbb{S}_6$ and since

$$N \cap C_G(\varphi(K - \{1\})) \neq 1,$$

 \mathbb{A}_6 is embedded in N.

 If

$$N \cap C_G(\varphi(T - \{1\})) \neq 1,$$

then since $C_G(\varphi(T - \{1\}) \cong \mathbb{S}_6$ and

$$N \cap C_G(\varphi(T - \{1\})) \trianglelefteq C_G(\varphi(T - \{1\})),$$

we conclude that \mathbb{A}_6 is embedded in N in this case too.

Thus $2^3 \cdot 3^2 \cdot 5 ||N|$. We know that N is a direct product of isomorphic simple groups. But 5 ||N| and $5^2 \nmid |N|$, hence N is a simple group.

Moreover $5 \approx 7$ in $\Gamma(\mathbb{S}_9)$ and since $\Gamma(G) = \Gamma(\mathbb{S}_9)$ by Lemma 2.5, $5 \neq 7$ in $\Gamma(G)$ too. By Frattini's argument $N_G(N_5)N = G$, where N_5 is a Sylow 5-subgroup of N and since $7||G|, 7||N_G(N_5)|$ or 7||N|. If $7||N_G(N_5)|$, then there exists an element z of order 7 in $N_G(N_5)$ and so $\langle z \rangle N_5$ is a subgroup of $N_G(N_5)$ of order 5.7. Hence $\langle z \rangle N_5$ is a cyclic group. It means that $5 \sim 7$ in $\Gamma(G)$, which is a contradiction. Thus 7||N|.

Now we assert that $C_G(N) = 1$. Otherwise there is a minimal normal subgroup T of G such that

 $T \leq C_G(N)$. By the same argument as above we see that $2^3 \cdot 3^2 \cdot 5 \cdot 7 ||T|$. Therefore $2^3 \cdot 3^2 \cdot 5 \cdot 7 ||C_G(N)|$. Hence $5||C_G(N)|$ and so there is an element $a \in C_G(N)$ such that o(a) = 5 and since 7||N|, there is an element of order 7, say b in N. $o(ab) = 5 \cdot 7$, because ab = ba. But $5 \approx 7$ in $\Gamma(G)$ and this is a contradiction. Thus $C_G(N) = 1$.

It implies that

$$G \cong \frac{G}{1} = \frac{G}{C_G(N)} \hookrightarrow Aut(N).$$

Therefore

$$9! = |G| ||Aut(N)|.$$

So we proved that N is a simple group with

$$2^{3} \cdot 3^{2} \cdot 5 \cdot 7 ||N|,$$

9! = 2⁷ \cdot 3⁴ \cdot 5 \cdot 7 ||Aut(N)|

and $|N||^{2^7} \cdot 3^4 \cdot 5 \cdot 7$. By table 1 of [5] ,we conclude that $N \cong \mathbb{A}_9$. But

$$G \hookrightarrow Aut(N),$$
$$|G| = |\mathbb{S}_9|$$
$$Aut(N) \cong Aut(\mathbb{A}_9)$$
$$\cong \mathbb{S}_9.$$

Hence $G \cong \mathbb{S}_9$.

and

Lemma 2.8. Let T be a finite group and $T \cong S_1 \times S_2 \times \cdots \times S_t$, where S_is are isomorphic simple groups, $1 \leq i \leq t$. Let T contain a copy of the alternating group \mathbb{A}_{n-3} , $n \geq 16$ and |T||n!. Then T is a simple group.

Proof. Without loss of generality we may assume that

$$T = S_1 \times S_2 \times \cdots \times S_t.$$

Suppose that $\pi_1: S_1 \times S_2 \times \cdots \times S_t \to S_1 \times 1 \times \cdots \times 1$ is defined by

$$\pi_1(s_1, s_2, \dots, s_t) = (s_1, 1, \dots, 1)$$

and K is a subgroup of T isomorphic to \mathbb{A}_{n-3} . Set

$$\overline{S_1} = S_1 \times 1 \times \dots \times 1$$

and

$$\overline{S_2 \times \cdots \times S_t} = 1 \times S_2 \times \cdots \times S_t.$$

Now we consider the following three cases.

Case 1) $K \cap \overline{S_1} = K \cap \overline{S_2 \times \cdots \times S_t} = 1.$

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In this case $\phi: K \to \pi_1(K)$ defined by $\phi(k) = \pi_1(k)$ for all $k \in K$ is an isomorphism from K onto $\pi_1(K)$. This means that $K \cong \pi_1(K)$. Thus we have

$$\mathbb{A}_{n-3} \cong K \cong \pi_1(K) \le \overline{S_1} \cong S_1$$

and so

$$\frac{(n-3)!}{2} = |\mathbb{A}_{n-3}| \big| |S_1|.$$

But S_i s are isomorphic simple groups, $1 \le i \le t$ and thus

$$\frac{(n-3)!}{2} = |\mathbb{A}_{n-3}| ||S_i|,$$
$$1 \le i \le t.$$

Therefore

$$\left[\frac{(n-3)!}{2}\right]^t ||T|$$

and since |T| | n!, we obtain $\left[\frac{(n-3)!}{2}\right]^t | n!$. But

$$\left[\frac{(n-3)!}{2}\right]^2 \nmid n!$$

for $n \ge 16$ and so t = 1 and T is a simple group.

Case 2) $K \cap \overline{S_1} \neq 1$ Since $\overline{S_1} \leq T$, we have

$$1 \neq K \cap \overline{S_1} \trianglelefteq K \cong \mathbb{A}_{n-3},$$

which implies that $K \cap \overline{S_1} = K$ and so

$$\mathbb{A}_{n-3} \cong K \leq \overline{S_1} \cong S_1.$$

Now similar argument as in Case (1) shows that T is a simple group .

Case 3) $K \cap \overline{S_2 \times \cdots \times S_t} \neq 1$ Since

$$\overline{S_2 \times \cdots \times S_t} \trianglelefteq T,$$

we have

$$1 \neq K \cap \overline{S_2 \times \cdots \times S_t} \trianglelefteq K \cong \mathbb{A}_{n-3},$$

which implies that

$$K \cap \overline{S_2 \times \dots \times S_t} = K$$

and so

$$A_{n-3} \cong K \le \overline{S_2 \times \dots \times S_t} \cong S_2 \times \dots \times S_t.$$

Thus \mathbb{A}_{n-3} is embedded in $S_2 \times \cdots \times S_t$. By repeating above argument for

$$T_i = S_i \times \dots \times S_t,$$
$$2 \le i \le t,$$

we conclude that T is a simple group.

Lemma 2.9. Let a, b be two natural numbers. Then:

1) $a^{b}.b! \leq (ab)!$ and $a^{0}.0! = (a.0)!$ 2) If $a \geq 4$, then $a^{b-1}.b! \leq (a(b-1))!$ 3) $3^{b-1}b! \leq (3b-3)!$ 4) If $b \geq 3$, then $2^{b-1}b! \leq (2b-2)!$ 5) If $b \geq 5$, then $2^{b-2}b! \leq 2(2b-4)!$ 6) If $b \geq 4$, then $3^{b-2}b! \leq 2(3b-6)!$

Proof. 1) We prove Lemma 2.9 part 1 by induction on *b*. If b = 1, then clearly (1) holds. Suppose that $a^k k! \leq (ak)!$. We prove that $a^{k+1}(k+1)! \leq (ak+a)!$. By induction hypothesis

$$a^{k+1}(k+1)! \le (ak)!a(k+1).$$

But clearly

$$(ak)!a(k+1) \le (ak+a)!$$

and so

$$a^{k+1}(k+1)! \le (ak+a)!$$

and this completes the proof of (1).

2) We prove part 2 by induction on b. If b = 1, then clearly (2) holds. Suppose that

$$a^{k-1}k! \le (a(k-1))!$$

for $k \ge 1$ and $a \ge 4$. We prove that

$$a^k(k+1)! \le (ak)!.$$

By induction hypothesis,

$$a^{k}(k+1)! \le (a(k-1))!a(k+1).$$

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But since $ak \ge 4$, we have $ak - 1 \ge 3$ and so

$$(ak)(ak-1)\cdots(ak-a+1) \ge ak+a.$$

Thus $(ak)! \ge (ak - a)!a(k + 1)$. Hence

$$a^k(k+1)! \le (ak)!$$

and this completes the proof of (2).

3) We prove this part by induction on b too. If
$$b = 1$$
, then (3) clearly holds. Suppose that

$$3^{k-1}k! \le (3k-3)!$$

We prove that

$$3^k(k+1)! \le (3k)!.$$

By induction hypothesis we obtain

$$3^{k}(k+1)! \le (3k-3)!3(k+1).$$

It is easy to know that

$$k + 1 \le k(3k - 1)(3k - 2)$$

for $k \geq 1$. Thus

$$(3k-3)!3(k+1) \le (3k)!$$

and so

 $3^k(k+1)! \le (3k)!$

and this completes the proof of (3).

4) We prove part (4) by induction on b. If b = 3, then clearly (4) holds. Suppose that

$$2^{k-1}k! \le (2k-2)!$$

for $k \geq 3$. We prove that

$$2^k(k+1)! \le (2k)!.$$

By induction hypothesis we obtain $2^k(k+1)! \leq (2k-2)!2(k+1)$. It is easy to see that $k+1 \leq k(2k-1)$ for $k \geq 3$. Thus

$$(2k-2)!2(k+1) \le (2k)!$$

and so

$$2^k(k+1)! \le (2k)!$$

and this completes the proof of (4).

5) We prove this part by induction on b. If b = 5, then (5) clearly holds. Suppose that

$$2^{k-2}k! \le 2(2k-4)!$$

for $k \geq 5$. We prove that

$$2^{k-1}(k+1)! \le 2(2k-2)!.$$

By induction hypothesis

$$2^{k-1}(k+1)! \le 2(2k-4)!2(k+1).$$

But since

$$k^2 - 3k + 1 \ge 0$$

for $k \geq 5$, we have

$$k+1 \le (k-1)(2k-3)$$

and so

$$2(2k-4)!2(k+1) \le 2(2k-2)!.$$

Hence

$$2^{k-1}(k+1)! \le 2(2k-2)!$$

and this completes the proof of (5).

6) We prove (6) by induction on b too. If b = 4, then (6) clearly holds. Suppose that

$$3^{k-2}k! \le 2(3k-6)!$$

for $k \geq 4$. We prove that

$$3^{k-1}(k+1)! \le 2(3k-3)!.$$

By induction hypothesis

$$3^{k-1}(k+1)! \le 2(3k-6)!3(k+1)$$

It is easy to see that

$$3(k+1) \le (3k-3)(3k-4)(3k-5)$$

for $k\geq 4$ and so

$$2(3k-6)!3(k+1) \le 2(3k-3)!$$

for $k \geq 4$. Hence

$$3^{k-1}(k+1)! \le 2(3k-3)!$$

and this completes the proof of (6).

Lemma 2.10. Let $a \ge 0$, $b \ge 0$ be two integers. Then $a!b! \le (a+b)!$.

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Proof. If $a \ge 1$, $b \ge 1$, then since

$$a + b > b,$$

 $a + b - 1 > b - 1, \dots,$
 $a + 1 > 1.$

we have

$$(a+b)(a+b-1)\cdots(a+1) > b!$$

-

and so

$$(a+b)!$$

= $(a+b)(a+b-1)\cdots(a+1)a! > b!a!.$

If a = 0 or b = 0, then clearly a!b! = (a + b)!.

Lemma 2.11. Let a_1, a_2, \ldots, a_m be integers with $a_i \ge 0, 1 \le i \le m$. Then $a_1!a_2!\cdots a_m! \le (a_1 + \cdots + a_m)!$.

Proof. We prove Lemma by induction on m. If m = 1, then clearly Lemma holds. Assume that

$$a_1!a_2!\cdots a_k!$$

$$\leq (a_1+a_2+\cdots+a_k)!.$$

We prove that

$$a_1!a_2!\cdots a_k!a_{k+1}!$$

 $\leq (a_1+a_2+\cdots+a_k+a_{k+1})!.$

By induction hypothesis

$$a_1!a_2!\cdots a_k!a_{k+1}!$$

 $\leq (a_1+a_2+\cdots+a_k)!a_{k+1}!$

But by Lemma 2.10 we have

$$(a_1 + \dots + a_k)!a_{k+1}!$$

 $\leq (a_1 + a_2 + \dots + a_k + a_{k+1})!.$

Thus

$$a_1!a_2!\cdots a_{k+1}!$$

 $\leq (a_1+a_2+\cdots+a_{k+1})!$

Lemma 2.12. Let l, m, n be three natural numbers with $n \ge 13$. Then the following holds.

- 1) If there exists a m-cycle, $m \ge 4$ in a cycle type of $x \in \mathbb{S}_n$, then $|C_{\mathbb{S}_n}(x)| \le m(n-m)!$
- 2) If there exists two l-cycles in a cycle type of $x \in S_n$, where l = 2 or l = 3, then $|C_{S_n}(x)| \le l^2 2! (n-2l)!$
- 3) If there exist a 2-cycle and a 3-cycle in a cycle type of $x \in S_n$, then $|C_{S_n}(x)| \le 2.3.(n-5)!$.

Proof. 1) Assume that $x \in \mathbb{S}_n$ is a permutation of type

$$1^{\alpha_1} \cdot 2^{\alpha_2} \cdots m^{\alpha_m} \cdots n^{\alpha_n},$$

where $\alpha_i \ge 0, 1 \le i \le n$. By assumption $\alpha_m \ge 1$. Thus

$$|C_{\mathbb{S}_n}(x)| =$$

1^{\alpha_1}\alpha_1!\dots m^{\alpha_m}\alpha_m!\dots n^{\alpha_n}\alpha_n!,

where $\alpha_m \geq 1$. By Lemma 2.9 part 1 and 2 we conclude that

$$|C_{\mathbb{S}_n}(x)| \le \alpha_1!(2\alpha_2)!\cdots m(m(\alpha_m-1))!\cdots(n\alpha_n)!$$

and so by Lemma 2.11, we have

$$C_{\mathbb{S}_n}(x)|$$

$$\leq m(\alpha_1 + 2\alpha_2 + \dots + m(\alpha_m - 1) + \dots + n\alpha_n)!$$

$$= m(n - m)!$$

and this completes the proof of (1).

2) Assume that $x \in \mathbb{S}_n$ is a permutation of type

$$1^{\alpha_1}\cdots 2^{\alpha_2}\cdots n^{\alpha_n}$$

where $\alpha_i \ge 0, 1 \le i \le n$. By assumption $\alpha_l \ge 2$, where l = 2 or l = 3. First suppose that l = 2. We have

$$|C_{\mathbb{S}_n}(x)|$$

= $1^{\alpha_1} \alpha_1 ! 2^{\alpha_2} \alpha_2 ! \cdots n^{\alpha_n} \alpha_n ! .$

If $\alpha_2 \geq 5$, then by Lemma 2.9 part 5 and 1 we conclude that

$$|C_{\mathbb{S}_n}(x)| \le \alpha_1! 2^3 (2\alpha_2 - 4)! \cdots (n\alpha_n)!$$

and so by Lemma 2.11 we have

$$\begin{aligned} |C_{\mathbb{S}_n}(x)| \\ &\leq 2^3(\alpha_1 + 2\alpha_2 - 4 + \dots + n\alpha_n)! \\ &= 2^3(n-4)! = 2^2 2!(n-4)!. \end{aligned}$$

If $\alpha_2 = 2$, then

$$|C_{\mathbb{S}_n}(x)|$$

= 1^{\alpha_1} \alpha_1! 2² \cdot 2! \dots n^{\alpha_n} \alpha_n!.

By part 1 of Lemma 2.9 and Lemma 2.11 we conclude that

$$\begin{aligned} &|C_{\mathbb{S}_n}(x)| \\ &\leq 2^2 \cdot 2! \alpha_1 ! (3\alpha_3)! \cdots (n\alpha_n)! \\ &\leq 2^2 \cdot 2! (n-4)!. \end{aligned}$$

If $\alpha_2 = 3$ or $\alpha_2 = 4$, then similar argument as case $\alpha_2 = 2$ shows us that

$$|C_{\mathbb{S}_n}(x)| \le 2^3 3! (n-6)!$$

or

$$|C_{\mathbb{S}_n}(x)| \le 2^4 4! (n-8)!$$

respectively and since

$$2^{3}3!(n-6)!$$

 $\leq 2^{2}.2!(n-4)!$

and

$$2^{4}4!(n-8)!$$

$$\leq 2^{2}2!(n-4)!$$

for $n \ge 13$, we have

$$|C_{\mathbb{S}_n}(x)| \le 2^2 2! (n-4)!$$

in this case too.

Now suppose that l = 3. If $\alpha_3 \ge 4$, then by Lemma 2.9 part 6 and 1 we have

$$|C_{\mathbb{S}_n}(x)|$$

 $\leq \alpha_1!(2\alpha_2)!3^22(3\alpha_3-6)!\cdots(n\alpha_n)!$

and so by Lemma 2.11 we have

$$|C_{\mathbb{S}_n}(x)|$$

$$\leq 3^2 \cdot 2! (\alpha_1 + 2\alpha_2 + 3\alpha_3 - 6 + \dots + n\alpha_n)!$$

$$= 3^2 \cdot 2! (n-6)!.$$

If $\alpha_3 = 2$, then

$$|C_{\mathbb{S}_n}(x)|$$

= 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^22!\dots n^{\alpha_n}\alpha_n!

By Lemma 2.9 part 1 and Lemma 2.11 we conclude that

$$|C_{\mathbb{S}_n}(x)| \le 3^2 2! \alpha_1! (2\alpha_2)! (4\alpha_4)! \cdots (n\alpha_n)! \le 3^2 2! (n-6)!.$$

If $\alpha_3 = 3$, then similar argument as case $\alpha_3 = 2$ shows us that

$$|C_{\mathbb{S}_n}(x)| \le 3^3 \cdot 3! \cdot (n-9)!$$

and since

$$3^{3}3!(n-9)! \le 3^{2}2!(n-6)$$

for $n \geq 13$, we have

$$|C_{\mathbb{S}_n}(x)| \le 3^2 2! (n-6)!$$

in this case too and so the proof of (2) is complete.

3) Again assume that $x \in \mathbb{S}_n$ is a permutation of type

$$1^{\alpha_1} \cdot 2^{\alpha_2} \cdots n^{\alpha_n},$$

where $\alpha_i \ge 0, 1 \le i \le n$. By assumption $\alpha_2 \ge 1$ and $\alpha_3 \ge 1$. We have

$$|C_{\mathbb{S}_n}(x)|$$

= $1^{\alpha_1}\alpha_1! 2^{\alpha_2}\alpha_2! 3^{\alpha_3}\alpha_3! \cdots n^{\alpha_n}\alpha_n!.$

If $\alpha_2 \geq 3$, then by Lemma 2.9 part 4,3 and 1 we have

$$|C_{\mathbb{S}_n}(x)|$$

 $\leq \alpha_1! 2(2\alpha_2 - 2)! 3(3\alpha_3 - 3)! \cdots (n\alpha_n)!$

and so by Lemma 2.11

$$|C_{\mathbb{S}_n}(x)| \le 2.3.(\alpha_1 + 2\alpha_2 - 2 + 3\alpha_3 - 3 + \dots + n\alpha_n)! = 2.3.(n-5)!.$$

If $\alpha_2 = 1$, then we have

$$|C_{\mathbb{S}_n}(x)|$$

= 1^{\alpha_1}\alpha_1!.2.3^{\alpha_3}\alpha_3!\dots n^\alpha_n \alpha_n!.

By Lemma 2.9 part 1 and 3 we have

$$|C_{\mathbb{S}_n}(x)|$$

$$\leq \alpha_1! \cdot 2 \cdot 3 \cdot (3\alpha_3 - 3)! \cdots (n\alpha_n)!$$

and so by Lemma 2.11

$$|C_{\mathbb{S}_n}(x)|$$

$$\leq 2 \cdot 3 \cdot (\alpha_1 + 3\alpha_3 - 3 + \dots + n\alpha_n)!$$

$$= 2 \cdot 3 \cdot (n-5)!.$$

If $\alpha_2 = 2$, then similar argument as case $\alpha_2 = 1$ shows us that

$$\begin{aligned} |C_{\mathbb{S}_n}(x)| \\ &\leq \alpha_1! 2^2 \cdot 2! \cdot 3 \cdot (3\alpha_3 - 3)! \cdots (n\alpha_n)! \\ &\leq 2^2 \cdot 2! \cdot 3(\alpha_1 + 3\alpha_3 - 3 + \dots + n\alpha_n)! \\ &= 2^2 \cdot 2! \cdot 3(n - 7)! \end{aligned}$$

and since

$$2^{2}.2!.3(n-7)!$$

 $\leq 2.3(n-5)!$

for $n \geq 13$, we have

$$|C_{\mathbb{S}_n}(x)| \le 2.3(n-5)!$$

in this case too and the proof of (3) is complete.

Lemma 2.13. Let l, k be two natural numbers with l > 1 and 1 < l + k < n - 1, where $n \ge 13$ is a natural number. Then l(n-l)! > (l+k)(n-l-k)!

Proof. We prove Lemma 2.12 by induction on k. If k = 1, then since n - l > 2, l > 1, we have l(n - l) > l + 1 and so

$$l(n-l)! > (l+1)(n-l-1)!.$$

Thus the lemma holds whenever k = 1. Suppose that if

$$1 < l + k < n - 1$$
,
 $l > 1$,

then

$$l(n-l)! > (l+k)(n-l-k)!$$

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We prove the lemma for k + 1. Suppose that

$$1 < l + k + 1 < n - 1$$

 $l > 1$.

Since

$$(n-l-k) > 2$$
$$l+k > 1,$$

we have

$$(l+k)(n-l-k)$$

> 2(l+k) > l+k+1

and so

Thus by induction hypothesis we conclude that

$$l(n-l)!$$

> $(l+k+1)(n-l-k-1)!.$

Hence the lemma is proved.

Lemma 2.14. Let l, m, n be three natural numbers with $l > 1, n \ge 13, m \ne n$ and $l \le m$. Then $l(n-l)! \ge m(n-m)!$

Proof. If l = m, then clearly Lemma holds. If l < m and 1 < m < n - 1, then since l > 1, Lemma 2.14 concluded from Lemma 2.13. But if l < m and m = n - 1, then we have

$$m(n-m)!$$

= $(n-1)1!$
= $n-1$.

We have $(n-1) < (n-2)^2$ for $n \ge 13$ and since 1 < n-2 < n-1 by above argument for all $1 < l \le n-2$ we have

$$l(n-l)! \ge (n-2)2!.$$

Hence l(n-l)! > n-1, also if l = n-1, clearly

$$l(n-l)! \ge n-1.$$

So the proof is complete.

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$$(l+k)(n-l-k)$$

> 2(l+k) > l+k+1
 $(l+k)(n-l-k)!$

$$> (l+k+1)(n-l-k-1)!.$$

Lemma 2.15. If $x \in \mathbb{S}_n$ and $|x^{\mathbb{S}_n}| \leq n(n-1)(n-2)$, where $x^{\mathbb{S}_n}$ is the conjugacy class of \mathbb{S}_n , $n \geq 13$ containing x. Then x = 1, x is a 2-cycle or x is a 3-cycle and $|x^{\mathbb{S}_n}| = 1$, $|x^{\mathbb{S}_n}| = \frac{n(n-1)}{2}$ or $|x^{\mathbb{S}_n}| = \frac{n(n-1)(n-2)}{3}$.

Proof. Suppose that $|x^{\mathbb{S}_n}| \leq n(n-1)(n-2)$. Then

$$|C_{\mathbb{S}_n}(x)|$$

 $\geq \frac{n!}{n(n-1)(n-2)} = (n-3)!.$

If there exists a *m*-cycle, $m \ge 4$ in a cycle type of *x*, then by Lemma 2.12 part 1

 $|C_{\mathbb{S}_n}(x)| \le m(n-m)!$

and by Lemma 2.14 we conclude that if $m \neq n$, then

$$m(n-m)! \le 4(n-4)!.$$

But if m = n, then m(n - m)! = n. It is easy to know that

$$n < 4(n-4)!$$

for $n \ge 13$. Therefore if there exists a *m*-cycle, $m \ge 4$ in a cycle type of *x*, then

$$|C_{\mathbb{S}_n}(x)| \le 4(n-4)!.$$

But we have $|C_{\mathbb{S}_n}(x)| \ge (n-3)!$ and so

$$(n-3)! \le 4(n-4)!,$$

which is a contradiction, because $n \ge 13$. Thus there is no *m*-cycle, $m \ge 4$ in a cycle type of *x*. If there exist two 2-cycles or two 3-cycles in a cycle type of *x*, then by Lemma 2.12 part 2 we conclude that

$$|C_{\mathbb{S}_n}(x)| \le 2^2 2! (n-4)!$$

or

$$|C_{\mathbb{S}_n}(x)| \le 3^2 2! (n-6)!$$

respectively and so

$$(n-3)! \le 2^2 2! (n-4)!$$

or

$$(n-3)! \le 3^2 2! (n-6)!,$$

which is a contradiction, because $n \ge 13$. Also if there exists a 3-cycle and a 2-cycle in a cycle type of x, then by Lemma 2.12 part 3 we conclude that

$$|C_{\mathbb{S}_n}(x)| \le 2.3.(n-5)!$$

and so

$$(n-3)! \le 2.3.(n-5)!,$$

which is a contradiction with $n \ge 13$. Thus x = 1 or x is a 2-cycle or x is a 3-cycle. Hence $|x^{\mathbb{S}_n}| = 1$ or $|x^{\mathbb{S}_n}| = \frac{n(n-1)}{2}$ or $x^{\mathbb{S}_n} = \frac{n(n-1)(n-2)}{3}$.

Lemma 2.16. Let x be an element of \mathbb{S}_n , $n \ge 13$. If $|C_{\mathbb{S}_n}(x)| = 3(n-3)!$, then x is a 3-cycle.

Proof. If $|C_{\mathbb{S}_n}(x)| = 3(n-3)!$, then

$$|C_{\mathbb{S}_n}(x)| \ge (n-3)!$$

and so by Lemma 2.15 we conclude that x = 1 or x is a 2-cycle or x is a 3-cycle. But if x = 1 or x is a 2-cycle, then clearly

$$|C_{\mathbb{S}_n}(x)| \neq 3(n-3)!.$$

 $(n! \neq 3(n-3)!$ and $2(n-2)! \neq 3(n-3)!)$ and so x is a 3-cycle.

3. Main result

In this section we will prove our main result.

Theorem 3.1. Let G be a finite group with $\nabla(G) \cong \nabla(\mathbb{S}_n)$, where \mathbb{S}_n is the symmetric group of degree n and $n \ge 3$, then $G \cong \mathbb{S}_n$.

Proof. By Lemma 2.1, we have $|G| = |\mathbb{S}_n|$. Since $\nabla(G) \cong \nabla(\mathbb{S}_n)$,

$$|G - Z(G)|$$

= $|\mathbb{S}_n - Z(\mathbb{S}_n)| = |\mathbb{S}_n| - 1$

and so |Z(G)| = 1.

By Lemmas 2.6 and 2.7 we may assume that $n \ge 16$. Without loss of generality we can assume that $\varphi : \mathbb{S}_n \to G$ and $\varphi(1) = 1$, where φ is an isomorphism from $\nabla(\mathbb{S}_n)$ to $\nabla(G)$.

Now we prove the theorem by induction on n, where $n \ge 16$. If n = 16, then theorem holds by Lemma 2.7. Suppose the theorem is true for all m < n and assume that $n \ge 16$. We will prove that the result is valid for S_n .

Set

$$A = \{ \alpha \in \mathbb{S}_n | (i)\alpha = i, i = 4, 5, \dots, n \}.$$

Clearly

 $A \leq \mathbb{S}_n,$ $A \cong \mathbb{S}_3.$

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By Lemma 2.2 we have

$$\nabla(C_{\mathbb{S}_n}(A)) \cong \nabla(C_G(\varphi(A)))$$

and since $C_{\mathbb{S}_n}(A) \cong \mathbb{S}_{n-3}$, we have

$$\nabla(\mathbb{S}_{n-3}) \cong \nabla(C_G(\varphi(A))).$$

Thus by induction hypothesis $C_G(\varphi(A)) \cong \mathbb{S}_{n-3}$. Therefore G has a subgroup isomorphic to \mathbb{S}_{n-3} i.e. $C_G(\varphi(A))$.

Let $H = C_G(\varphi(A))$. Now we assume that N is an arbitrary minimal normal subgroup of G. We will prove that N is a simple group and that

$$\mathbb{A}_{n-3} \hookrightarrow N \cap P$$

for all subgroups P of G isomorphic to \mathbb{S}_{n-3} . In particular N contains all even permutations of P, for all

$$P \le G,$$
$$P \cong \mathbb{S}_{n-3}.$$

Let P be an arbitrary subgroup of G isomorphic to \mathbb{S}_{n-3} . We have $N \cap P \leq P$. We assert that $N \cap P \neq 1$. If $N \cap P = 1$, then we have

$$|NP| = |N||P|||G| = n!.$$

Thus

$$|N|.(n-3)!|n!,$$

since |P| = (n-3)!. This implies that |N||n(n-1)(n-2). Moreover N is a union of conjugacy classes of G and the size of conjugacy class of G containing x is equal to the size of conjugacy class of \mathbb{S}_n containing $\varphi^{-1}(x)$ for all $x \in G - \{1\}$.

By Lemma 2.15 we see that all conjugacy class sizes less than n(n-1)(n-2) in \mathbb{S}_n , $n \ge 16$ are 1, $\frac{n(n-1)}{2}$ and $\frac{n(n-1)(n-2)}{3}$.

Let y be an arbitrary element of $N - \{1\}$. Thus the size of the conjugacy class of G containing y and so the size of conjugacy class of \mathbb{S}_n containing $\varphi^{-1}(y)$ is equal to $\frac{n(n-1)}{2}$ or $\frac{n(n-1)(n-2)}{3}$. Also by Lemma 2.15 $\varphi^{-1}(y)$ is a 2-cycle or $\varphi^{-1}(y)$ is a 3-cycle.

In any case there exists a subgroup of \mathbb{S}_n , say E isomorphic to \mathbb{S}_3 such that $\varphi^{-1}(y) \in C_{\mathbb{S}_n}(E)$ and

$$C_{\mathbb{S}_n}(E) \cong \mathbb{S}_{n-3}.$$

So $y \in C_G(\varphi(E))$, also we know that $y \in N - \{1\}$. Therefore

$$y \in N \cap C_G(\varphi(E))$$

and

$$N \cap C_G(\varphi(E)) \neq 1.$$

By Lemma 2.2

$$\nabla(\mathbb{S}_{n-3}) \cong \nabla(C_{\mathbb{S}_n}(E))$$
$$\cong \nabla(C_G(\varphi(E)))$$

and so by induction hypothesis

$$C_G(\varphi(E)) \cong \mathbb{S}_{n-3}$$

Since

$$1 \neq N \cap C_G(\varphi(E))$$
$$\trianglelefteq C_G(\varphi(E)) \cong \mathbb{S}_{n-3},$$

we conclude that

$$\mathbb{A}_{n-3} \hookrightarrow N \cap C_G(\varphi(E)).$$

Set $R = N \cap C_G(\varphi(E))$. Therefore

Since $P \cap N = 1$,

$$\subseteq P\cap N=1$$

and so $P \cap R = 1$. Thus |PR| = |P||R|. On the other hand |P| = (n-3)! and

$$\frac{(n-3)!}{2} ||R|.$$

 So

$$\frac{[(n-3)!]^2}{2} ||P||R| = |PR|.$$

But since $PR \subseteq G$, we have

$$|PR| \le |G| = n!.$$

 So

$$\frac{[(n-3)!]^2}{2} \le n!,$$

which is a contradiction, since we assumed that $n \ge 16$. Hence $P \cap N \ne 1$ for all subgroup P of G isomorphic to \mathbb{S}_{n-3} . In particular $N \cap H \ne 1$. Also

$$1 \neq N \cap P \trianglelefteq P \cong \mathbb{S}_{n-3}$$

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implies that

$$\mathbb{A}_{n-3} \hookrightarrow N \cap P$$

for all $P \leq G$, $P \cong \mathbb{S}_{n-3}$.

Since N is a minimal normal subgroup of G, N is a direct product of isomorphic simple group, say

$$N \cong S_1 \times \cdots \times S_t,$$

where S_i 's are isomorphic simple groups, $1 \le i \le t$. Also since

$$\mathbb{A}_{n-3} \hookrightarrow N \cap H,$$

 $\mathbb{A}_{n-3} \hookrightarrow N$. Thus by Lemma 2.8 N is a simple group.

Next set

$$B = \{\beta \in \mathbb{S}_n | (i)\beta = i, i = 1, 2, \dots, n-3\}.$$

 $B \leq \mathbb{S}_n,$

 $B \cong \mathbb{S}_3$

 $\cong \mathbb{S}_n$

-3-

Clearly

and

It is easy to see that

$$C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B) \cong \mathbb{S}_{n-6}.$$

(B)

By Lemma 2.2 we have

$$\nabla(\mathbb{S}_{n-6})$$

$$\cong \nabla(C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B))$$

$$= \nabla(C_{\mathbb{S}_n}(A \cup B))$$

$$\cong \nabla(C_G(\varphi(A \cup B)))$$

$$= \nabla(C_G(\varphi(A) \cup \varphi(B)))$$

$$= \nabla(C_G(\varphi(A)) \cap C_G(\varphi(B)))$$

and so by induction hypothesis

$$C_G(\varphi(A)) \cap C_G(\varphi(B)) \cong \mathbb{S}_{n-6}.$$

Similarly $C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$. By above argument

$$\mathbb{A}_{n-3} \hookrightarrow N \cap C_G(\varphi(A))$$

and

$$\mathbb{A}_{n-3} \hookrightarrow N \cap C_G(\varphi(B)).$$

Let

$$L \le N \cap C_G(\varphi(A)),$$
$$K \le N \cap C_G(\varphi(B))$$

and $L \cong K \cong \mathbb{A}_{n-3}$. We have

$$L \cap K$$

$$\leq N \cap C_G(\varphi(A)) \cap C_G(\varphi(B))$$

$$\leq C_G(\varphi(A)) \cap C_G(\varphi(B))$$

$$\cong \mathbb{S}_{n-6}.$$
im.

Now we will prove the following claim.

Claim: $L \cap K \neq C_G(\varphi(A)) \cap C_G(\varphi(B))$

Suppose by way of contradiction, that

$$L \cap K = C_G(\varphi(A)) \cap C_G(\varphi(B)).$$

Assume that $a = (1 \ 2 \ 3) \in \mathbb{S}_n$. Clearly $a \in C_{\mathbb{S}_n}(B)$. Since

$$|C_{\mathbb{S}_n}(B) \cap C_{\mathbb{S}_n}(a)|$$

= $|C_{C_{\mathbb{S}_n}(B)}(a)| = 3(n-6)!$

we conclude that

$$|C_G(\varphi(B)) \cap C_G(\varphi(a))|$$

= $|C_{C_G(\varphi(B))}(\varphi(a))| = 3(n-6)!.$

But

$$C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$$

and by Lemma 2.16 if

$$y \in S_{n-3},$$

 $|C_{S_{n-3}}(y)| = 3(n-6)!,$
 $n \ge 16,$

then y is a 3-cycle. Thus $\varphi(a)$ is a 3-cycle in $C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$. Therefore $\varphi(a)$ is an even permutation in $C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$ and so

$$\varphi(a) \in K \cong \mathbb{A}_{n-3}$$

(Note that $K \leq C_G(\varphi(B))$). Also we have

$$C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B) \subseteq C_{\mathbb{S}_n}(a),$$

 \mathbf{SO}

$$L \cap K = C_G(\varphi(A)) \cap C_G(\varphi(B)) \le C_G(\varphi(a)),$$

which implies that

$$C_G(\varphi(A)) \cap C_G(\varphi(B))$$

$$\leq C_G(\varphi(a)) \cap K = C_K(\varphi(a))$$

On the other hand $\langle \varphi(a) \rangle \leq C_K(\varphi(a))$. Since

$$C_{\mathbb{S}_n}(C_{\mathbb{S}_n}(a)) \cap C_{\mathbb{S}_n}(A)$$
$$\cap C_{\mathbb{S}_n}(B) = 1,$$

we have

$$\begin{split} \langle \varphi(a) \rangle &\cap C_G(\varphi(A)) \cap C_G(\varphi(B)) \\ &\subseteq C_G(C_G(\varphi(a))) \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) = 1 \end{split}$$

and so

$$\langle \varphi(a) \rangle \cap C_G(\varphi(A))$$

 $\cap C_G(\varphi(B)) = 1.$

Therefore

$$|\langle \varphi(a) \rangle C_G(\varphi(A)) \cap C_G(\varphi(B))|$$

= $|\langle \varphi(a) \rangle || C_G(\varphi(A)) \cap C_G(\varphi(B))| = 3(n-6)!.$

Moreover since a commutes with all elements of $C_{\mathbb{S}_n}(A) \cap C_{\mathbb{S}_n}(B)$, $\varphi(a)$ commutes with all elements of

$$C_G(\varphi(A)) \cap C_G(\varphi(B)).$$

 So

$$\langle \varphi(a) \rangle C_G(\varphi(A))$$

 $\cap C_G(\varphi(B)) \leq G.$

But we have

$$\langle \varphi(a) \rangle \le C_K(\varphi(a)),$$

 $C_G(\varphi(A)) \cap C_G(\varphi(B)) \le C_K(\varphi(a))$

and thus

$$\langle \varphi(a) \rangle C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_K(\varphi(a)).$$

Hence

 $3(n-6)! ||C_K(\varphi(a))|,$

where $K \cong \mathbb{A}_{n-3}$. But This is impossible, because $\varphi(a)$ is a 3-cycle in K and so

$$|C_K(\varphi(a))| = |C_{\mathbb{A}_{n-3}}(\varphi(a))|$$
$$= 3 \cdot \frac{(n-6)!}{2}.$$

Hence

$$L \cap K \neq C_G(\varphi(A)) \cap C_G(\varphi(B))$$

and the claim is proved. For the order of N we will prove the followings:

1. $|N| > \frac{n!}{4}$

We know that $L, K \leq N$ and $|L| = |K| = \frac{(n-3)!}{2}$. Also

$$L \cap K \lneq C_G(\varphi(A)) \cap C_G(\varphi(B))$$

and so

$$|L \cap K| \le \frac{|C_G(\varphi(A)) \cap C_G(\varphi(B))|}{2}$$

From $L, K \leq N$, we deduce that $LK \leq N$. Thus

$$\begin{split} |N| &\geq |LK| \\ &= \frac{|L||K|}{|L \cap k|} \\ &\geq \frac{|L||K|}{\frac{|C_G(\varphi(A)) \cap C_G(\varphi(B))|}{2}} \\ &= \frac{\frac{(n-3)!}{2} \frac{(n-3)!}{2}}{\frac{(n-6)!}{2}}. \end{split}$$

On the other hand

$$\frac{\frac{(n-3)!}{2}\frac{(n-3)!}{2}}{\frac{(n-6)!}{2}} > \frac{n!}{4}$$

for $n \ge 16$. Thus $|N| > \frac{n!}{4}$.

2.
$$|N| \neq \frac{n!}{3}$$

We know that

$$C_G(\varphi(A)) \cong C_G(\varphi(B)) \cong \mathbb{S}_{n-3}$$

and

$$N \cap C_G(\varphi(A)) \neq 1$$

and

 $N \cap C_G(\varphi(B)) \neq 1.$

If $C_G(\varphi(A)) \leq N$ and $C_G(\varphi(B)) \leq N$, then

$$C_G(\varphi(A))C_G(\varphi(B)) \subseteq N.$$

Thus

$$\begin{split} |N| &\geq |C_G(\varphi(A))C_G(\varphi(B))| \\ &= \frac{|C_G(\varphi(A))||C_G(\varphi(B))|}{|C_G(\varphi(A)) \cap C_G(\varphi(B))|} \\ &= \frac{(n-3)!(n-3)!}{(n-6)!}. \end{split}$$

But

$$\frac{(n-3)!(n-3)!}{(n-6)!} > \frac{n!}{2}$$

for $n \ge 16$, which implies that |N| = |G| and since N is an arbitrary minimal normal subgroup of G, we conclude that G is a simple group. By assumption $\nabla(G) \cong \nabla(\mathbb{S}_n)$ and [6] we have $G \cong \mathbb{S}_n$, so \mathbb{S}_n must be a simple group too, which is a contradiction.

Hence

$$N \cap C_G(\varphi(A)) \neq C_G(\varphi(A))$$

or

$$N \cap C_G(\varphi(B)) \neq C_G(\varphi(B)).$$

Suppose that

$$N \cap C_G(\varphi(A)) \neq C_G(\varphi(A)).$$

We know that

$$1 \neq N \cap C_G(\varphi(A))$$
$$\leq C_G(\varphi(A)) \cong \mathbb{S}_{n-3}.$$

Therefore

$$|N \cap C_G(\varphi(A))| = |\mathbb{A}_{n-3}| = \frac{(n-3)!}{2}$$

and so we have

$$|NC_G(\varphi(A))|$$

$$= \frac{|N||C_G(\varphi(A))|}{|N \cap C_G(\varphi(A))|}$$

$$= \frac{|N|(n-3)!}{\frac{(n-3)!}{2}} = 2|N|.$$

$$A)) \leq G$$
 Thus

Moreover $N \trianglelefteq G$ implies that $NC_G(\varphi(A)) \le G$. Thus

$$|NC_G(\varphi(A))|=2|N|\big||G|=n!.$$

Now if $|N| = \frac{n!}{3}$, then we have $\frac{2n!}{3}|n!$, a contradiction. This shows that $|N| \neq \frac{n!}{3}$.

3. $|N| = \frac{n!}{2}$

From $|N| > \frac{n!}{4}$ and |N|||G| = n!, we conclude that |N| is equal to one of $\frac{n!}{3}$, $\frac{n!}{2}$ or n!. By $2 |N| \neq \frac{n!}{3}$. If |N| = |G| = n!, then G is a simple group, since N is an arbitrary minimal normal subgroup of G. By assumption $\nabla(G) \cong \nabla(\mathbb{S}_n)$. Now since G is a simple group, by [6] $G \cong \mathbb{S}_n$. So \mathbb{S}_n must be a simple group too, a contradiction. Hence $|N| = \frac{n!}{2}$.

From $|N| = \frac{n!}{2}$, simplicity of N and by corollary 2.4, $N \cong \mathbb{A}_n$. We assert that $C_G(N) = 1$. Otherwise there is a minimal normal subgroup of G, say M such that $M \leq C_G(N)$. We proved that all minimal normal subgroups of G are isomorphic to \mathbb{A}_n . Thus $M \cong \mathbb{A}_n$ and since

$$N \cap C_G(N) = Z(N) = 1,$$

 $M \cap N = 1.$

On the other hand $MN \leq G$ and so

$$|MN| = |M||N|||G|.$$

It follows that $\left(\frac{n!}{2}\right)^2 ||G| = n!$, a contradiction. Hence $C_G(N) = 1$ and so

$$G \cong \frac{G}{1}$$
$$= \frac{G}{C_G(N)} \hookrightarrow Aut(N)$$

and since for $n \ge 16$,

$$Aut(N) \cong Aut(\mathbb{A}_n) \cong \mathbb{S}_n,$$

we conclude that G is embedded into \mathbb{S}_n . But $|G| = |\mathbb{S}_n|$ and so $G \cong \mathbb{S}_n$.

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