



www.theoryofgroups.ir

International Journal of Group Theory
 ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
 Vol. 8 No. 1 (2019), pp. 1-9.
 © 2019 University of Isfahan



www.ui.ac.ir

ON NONINNER AUTOMORPHISMS OF FINITE p -GROUPS THAT FIX THE CENTER ELEMENTWISE

S. MOHSEN GHORAISHI

Communicated by Ali Reza Jamali

ABSTRACT. In this paper we show that every finite nonabelian p -group G in which the Frattini subgroup $\Phi(G)$ has order $\leq p^5$ admits a noninner automorphism of order p leaving the center $Z(G)$ elementwise fixed. As a consequence it follows that the order of a possible counterexample to the conjecture of Berkovich is at least p^8 .

1. Introduction

One of the important conjectures in studying p -groups is the conjecture of Berkovich that states every finite nonabelian p -group G has at least one noninner automorphism of order p (See [16, Problem 4.13]). The conjecture was established for various classes of finite p -groups. Some of them are gathered in the following.

Remark 1.1. *Let G be a finite nonabelian p -group. If G satisfies one of the following conditions, then it admits a noninner automorphism of order p .*

- (1) *The nilpotency class of G is ≤ 3 [2, 6, 15],*
- (2) *The coclass of G is 2 [4], or $p \neq 3$ and the coclass of G is 3 [19],*
- (3) *$C_G(Z(\Phi(G))) \neq \Phi(G)$ [10],*
- (4) *G is regular [10, 20],*
- (5) *$G/Z(G)$ is powerful [1],*

MSC(2010): Primary: 20D45; Secondary: 20D15.

Keywords: p -Groups, automorphisms, noninner automorphisms.

Received: 19 November 2017, Accepted: 13 January 2018.

<http://dx.doi.org/10.22108/ijgt.2018.108082.1457>

- (6) G is 2-generator p -group with abelian Frattini subgroup [7],
- (7) The commutator subgroup of G is cyclic [14].

For the other results see [13] and the references therein. In most of the above mentioned results, the noninner automorphism of order p can be chosen so that it fixes pointwise either the center $Z(G)$ or the Frattini subgroup $\Phi(G)$ of G . Schmid proved that every finite nonabelian p -group G admits a noninner automorphism of p -power order leaving the center $Z(G)$ elementwise fixed [21]. Thus it is reasonable to look for noninner automorphisms of order p that fix the the center $Z(G)$ elementwise.

Remark 1.2. *It was observed in [4, 5] that if a finite nonabelian p -group G satisfies one of the following conditions, then it admits a noninner automorphism of order p that fixes pointwise the center $Z(G)$ of G .*

- (1) G is regular,
- (2) G is nilpotent of class 2,
- (3) The commutator subgroup of G is cyclic,
- (4) $G/Z(G)$ is powerful,
- (5) The coclass of G is 2.

The main result of this paper is the following.

Theorem 1.3. *Let p be any prime and let G be a finite nonabelian p -group such that the Frattini subgroup of G has order $\leq p^5$. Then G admits a noninner automorphism of order p , leaving the center $Z(G)$ elementwise fixed.*

Theorem 1.3, has the following consequence.

Corollary 1.4. *Let p be any prime and let G be a finite nonabelian p -group of order $\leq p^7$. Then G has a noninner automorphism of order p leaving the center $Z(G)$ elementwise fixed.*

Corollary 1.4 improves the result of Bodnarchuk and O. S. Pylyavska. They proved the conjecture of Berkovich in the case when G is a finite p -group of order p^n , $n \leq 6$, and $p \geq 5$ [8].

2. Proof of the main result

Throughout the section, G is a finite nonabelian p -group. By $Z(G)$, $Z_i(G)$, $\gamma_i(G)$, and $C_G(X)$ we denote the center of G , the i th term of the upper central series of G , the i th term of lower central series of G , and the centralizer of a subset X of G , respectively. If G is of order p^n and of nilpotency class $cl(G)$, then G is called to be of coclass $cc(G) = n - cl(G)$. For positive integer n , G^n denotes the subgroup $\langle x^n \mid x \in G \rangle$ and $\Omega_1(G)$ stands for subgroup $\langle x \in G \mid x^p = 1 \rangle$. If $x \in G$ and α is an automorphism of G , then $[x, \alpha]$, denotes $x^{-1}\alpha(x)$. We use the following results.

Theorem 2.1. [9, Theorem 3.2] *Let $G = \langle a, b \rangle$ be a metabelian, two-generated group. Then the following are equivalent:*

<http://dx.doi.org/10.22108/ijgt.2018.108082.1457>

- (1) For all $u, v \in \gamma_2(G)$, there is an automorphism of G that maps a to au and b to bv ;
- (2) G is nilpotent.

Remark 2.2. Let G be a finite nonabelian p -group. If G has no noninner automorphism of order p leaving the $\Phi(G)$ elementwise fixed, then by [12], we have

$$(2.1) \quad Z_2^*(G) \leq C_G(Z_2^*(G)) = \Phi(G),$$

where $Z_2^*(G) = \{a \in Z_2(G) \mid a^p \in Z(G)\}$. Condition (2.1) implies that $C_G(Z(\Phi(G))) = \Phi(G)$ and

$$(2.2) \quad Z_2^*(G) \leq Z(\Phi(G)).$$

Moreover, it follows from [1, Lemma 2.2] that

$$(2.3) \quad d\left(\frac{Z_2(G)}{Z(G)}\right) = d(G)d(Z(G)).$$

Remark 2.3. Let G be a finite nonabelian p -group. Then it is easy to see that $G = AH$ for some subgroups A and H such that $A \leq Z(G)$ and $Z(H) \leq \Phi(H)$. If H has a noninner automorphism of order p leaving the center $Z(H)$ elementwise, then G has a noninner automorphism of order p which fixes both $Z(H)$ and A elementwise (See [5, Lemma 2.1] and [10, Remark 4]).

Lemma 2.4. Let G be a finite p -group with cyclic center.

- (1) If $cl(G) \geq 4$ and $\gamma_2(G) \leq G^{2p}\gamma_3(G)Z_2^*(G)$, then G is powerful.
- (2) If $\gamma_2(G)$ is abelian noncyclic, then there is a noncentral element $u \in \gamma_2(G) \cap Z_2(G)$ of order p .

Proof. (1) Note that $G^{2p} = \begin{cases} G^p & p > 2, \\ G^4 & p = 2 \end{cases}$. If $\gamma_2(G) \leq G^{2p}\gamma_3(G)Z_2^*(G)$, then

$$\gamma_3(G) \leq [G^{2p}\gamma_3(G)Z_2^*(G), G] \leq G^{2p}\gamma_4(G)\Omega_1(Z(G)).$$

Since $\Omega_1(Z(G)) \leq \gamma_4(G)$, we get $\gamma_3(G) \leq G^{2p}\gamma_4(G)$. By reiterating this argument we get $\gamma_2(G) \leq G^{2p}$, and therefore G is powerful.

(2) Suppose that $\gamma_2(G)$ is abelian noncyclic. Then $\frac{\Omega_1(\gamma_2(G))Z(G)}{Z(G)}$ is a nontrivial normal subgroup of $\frac{G}{Z(G)}$. Thus there exists an element $u \in (\Omega_1(\gamma_2(G)) \cap Z_2(G)) \setminus Z(G)$. □

Remark 2.5. Let G be a finite nonabelian p -group such that $Z(G)$ is cyclic and $C_G(\Phi(G)) = Z(\Phi(G))$. Then $Z_2^*(G)$ is abelian, since $Z_2^*(G) \leq C_G(\Phi(G))$. To prove the main result we use the following arguments frequently.

- (1) If $d(Z_2^*(G)) \geq 2$ then $Z_2^*(G)$ has a noncentral element u of order p (Indeed, let t be an element of maximal order in $Z_2^*(G)$. Then $Z_2^*(G) = \langle t \rangle \times B$, for some nontrivial subgroup B of $Z_2^*(G)$. If $\exp(Z_2^*(G)) > \exp(Z(G))$, then $Z(G) = \langle t^p \rangle$ and if $\exp(Z_2^*(G)) = \exp(Z(G))$, then we may assume that $\langle t \rangle = Z(G)$. In both cases B is a nontrivial elementary abelian subgroup, since $B^p \leq Z(G)$). Now consider the mapping $\varphi_u : G \rightarrow \Omega_1(Z(G))$, given by $x \mapsto [x, u]$, for all $x \in G$. Then φ_u is a homomorphism and $\ker(\varphi_u) = C_G(u)$ is a maximal subgroup of G . Let

$M = C_G(u)$ and $g \in G \setminus M$. If p is odd, then $(gu)^p = g^p$. It is well-known that the mapping $g \mapsto gu, m \mapsto m$, for all $m \in M$, extends to an automorphism σ of G of order p leaving M elementwise fixed. Suppose that for some $h \in G, \sigma = \theta_h$, the inner automorphism induced by h . Since $g^{-1}g^\sigma \in Z_2(G)$, for all $g \in G$, and σ fixes the $\Phi(G)$ elementwise, we get that $h \in Z_3(G) \setminus Z_2(G)$ and $h \in Z(\Phi(G))$. Therefore, $h \in (Z_3(G) \cap Z(\Phi(G))) \setminus Z_2(G)$.

- (2) If $d(Z_2^*(G)) \geq 3$ then $Z_2^*(G)$ has an elementary abelian subgroup $\langle u, v \rangle$ of order p^2 such that $\langle u, v \rangle \cap Z(G) = 1$. Set $M = C_G(u)$ and $N = C_G(v)$. Then M and N are distinct maximal subgroups of G . Let $a \in M \setminus N$ and $b \in N \setminus M$. If $p = 2$, then let α be the mapping given by $a \mapsto au, b \mapsto bv$ and $m \mapsto m$, for all $m \in M \cap N$, and if $p > 2$, then let β , and γ be the mappings given by $a \mapsto av, m \mapsto m$, for all $m \in M$, and $b \mapsto bu, n \mapsto n$, for all $n \in N$. Then α, β and γ extend to automorphisms of G of order p that fix $M \cap N, M$, and N elementwise, respectively (See [3, Lemma 2.1]). If α, β , and γ are inner, then $\alpha = \theta_w, \beta = \theta_x$ and $\gamma = \theta_y$, for some $w, x, y \in G$. Similar to argument (1) we must have $w, x, y \in (Z_3(G) \cap Z(\Phi(G))) \setminus Z_2(G)$.

The following well-known fact can be easily proved by induction.

Lemma 2.6. *Let G be a group of nilpotency class 3. If $x, y \in G$, then for each positive integer n ,*

- (1) $[y, x^n] = [y, x]^n [y, x, x]^{\binom{n}{2}}$.
- (2) *Moreover, if $[y, x, y] = 1$, then $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}} [y, x, x]^{\binom{n}{3}}$.*

Lemma 2.7. *Let G be a finite p -group with abelian Frattini subgroup and nilpotency class 3. Then,*

- (1) $\gamma_3(G)$ *is elementary abelian.*
- (2) $\Phi(G) \leq Z_2^*(G)$.
- (3) *if $G = \langle a, b \rangle$ is two generated and $\gamma_2(G)$ is noncyclic, then $\gamma_2(G)$ is elementary abelian.*

Proof. Let $x, y, z \in G$. Then it follows from Lemma 2.6 that

$$[x, y, z]^p = [x^p, y, z] = [x, y, z^p] = [[x, y]^p, z] = 1.$$

It implies that $\gamma_3(G)$ has exponent $p, G^p \leq Z_2(G)$, and $\gamma_2(G) \leq Z_2^*(G)$, since $\gamma_2(G) \leq Z_2(G)$. Moreover, for each $x, y \in G$,

$$[x^{p^2}, y] = [x, y]^{p^2} [x, y, x]^{\binom{p^2}{2}} = 1.$$

This shows that $G^p \leq Z_2^*(G)$. As $\gamma_3(G)$ is abelian and $\Phi(G) = \gamma_2(G)G^p$, items (a) and (b) follow. Now suppose that $G = \langle a, b \rangle$ is two generated and $\gamma_2(G)$ is noncyclic. Then G has a noncentral element $u \in \gamma_2(G) \cap Z_2(G)$ of order p , by Lemma 2.4. But $\gamma_2(G) = \langle [a, b], \gamma_3(G) \rangle$. Hence $u = [a, b]^i c$ for some integer i and $c \in \gamma_3(G)$. Since $[a, b]^p \in Z(G)$ we must have $\gcd(i, p) = 1$. So $[a, b]$ is of order p . Therefore, $\gamma_2(G)$ is elementary abelian and item (c) follows. □

Remark 2.8. *Let G be a finite p -group. If α is an automorphism of G of p -power order, then the group $\langle \alpha \rangle$ acts on $Z(G)$. Therefore, $|C_{Z(G)}(\alpha)| \geq p$ (See for instance [17, Section 5.4.1]). In particular, if $Z(G)$ has order p , then α fixes $Z(G)$ pointwise.*

Lemma 2.9. *Let G be a finite p -group with cyclic center. If $Z(G) \leq \Phi(G) \leq Z_2(G)$, then G has a noninner automorphism of order p leaving the center $Z(G)$ elementwise fixed.*

Proof. Let G be a counterexample to the lemma. Then G has no noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Therefore G fulfils conditions (2.1) - (2.3) of Remark 2.2. If either $p \geq 3$, or $p = 2$ and $d(Z_2^*(G)) \geq 3$, then Remark 2.5 gives a contradiction. Thus we may assume that $p = 2$ and $d(Z_2^*(G)) = 2$. Hence, $d(G) = 2$ and $Z(G) = \Phi(Z_2^*(G)) = (Z_2^*(G))^2$, since by (2.3), $d(Z_2^*(G)/Z(G)) = 2$. Clearly $cl(G) \leq 3$. But by Remark 1.2, we may assume that $cl(G) = 3$. Now it follows from Lemma 2.7 that $Z_2^*(G) = \Phi(G)$.

Let $G = \langle a, b \rangle$. Then $\Phi(G) = \langle a^2, b^2, \gamma_2(G) \rangle$. By Lemma 2.7, $\gamma_2(G)$ is elementary abelian. Hence, $Z(G) = \langle a^4, b^4 \rangle$. Without loss we may assume that $Z(G) = \langle a^4 \rangle$. Thus $\Phi(G) = \langle a^2 \rangle \times \langle [a, b] \rangle$. Clearly $a^2 \notin Z(G)$. Then $[a^2, b] = [a, b]^2[a, b, a] = [a, b, a] \neq 1$. We may suppose that $[a, b, b] = 1$. For if $[a, b, b] \neq 1$, then $[a, ab, ab] = [a, b, a][a, b, b] = 1$. Replacing b , by ab , we have $[a, b, b] = 1$. Next, we have $[a, b^2] = [a, b]^2[a, b, b] = 1$. Thus $b^2 \in Z(G)$, and hence $b^2 = a^{4i}$, for some integer i . Replacing b by ba^{-2i} , we may assume that $b^2 \in \Omega_1(Z(G))$. Let $2^m = |Z(G)|$. By Remark 2.8, we may suppose that $m \geq 2$. Hence $b^2 = a^{2^{m+1}j}$, for some integer j . Now, replacing b by $ba^{-2^m j}$ we have $b^2 = 1$. Therefore, $G = \langle a, b \rangle$ and $Z_2(G) = \Phi(G) = \langle a^2 \rangle \times \langle [a, b] \rangle$, $a^{2^{m+2}} = 1 = b^2$, $[a^4, b] = 1$, and $[a, b]^2 = 1$. Let $x_1 = a$ and $x_2 = b$. Then G has the following power-commutator presentation.

$$G = \langle x_1, x_2, x_3, x_4, x_5 \mid x_3 = [x_2, x_1], x_4 = x_1^2, x_5 = x_4^2, x_3^2 = 1, x_5^{2^m} = 1, x_2^2 = 1, [x_5, x_2] = 1, [x_1, x_2, x_1] = x_5^{2^{m-1}} \rangle.$$

It is straightforward to check that this presentation is consistent (See [22, page 424]). Now, let $y_1 = x_1x_2, y_2 = x_2, y_3 = x_3, y_4 = x_3x_4, y_5 = x_5$. Then

$$\begin{aligned} [y_2, y_1] &= [x_1x_2, x_1] = [x_2, x_1] = x_3 = y_3, \\ y_1^2 &= (x_1x_1)^2 = x_1^2x_2^2[x_2, x_1] = x_4x_3 = y_4, \\ y_4^2 &= (x_3x_4)^2 = x_3^2x_4^2 = x_5 = y_5, \\ [y_5, y_2] &= [x_5, x_2] = 1, \\ [y_1, y_2, y_1] &= [x_1x_2, x_2, x_1x_2] = [x_1, x_2, x_1] = x_5^{2^{m-1}} = y_5^{2^{m-1}}. \end{aligned}$$

Thus by von Dyck's Theorem, the mapping $\alpha : x_i \mapsto y_i, 1 \leq i \leq 5$ is an automorphism of G . Clearly α is noninner of order 2, and $\alpha(x_1^4) = (x_1x_2)^4 = x_1^4$. Therefore α fixes the center elementwise. \square

Proof of Theorem 1.3. Let G be a counterexample of the theorem. By Remark 2.3 we may assume that $Z(G) \leq \Phi(G)$. Thus G has no noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Hence G satisfies conditions (2.1) - (2.3) of Remark 2.2. Then

$$(2.4) \quad p^5 \geq |Z(\Phi(G))| \geq |Z_2^*(G)| \geq p^{d(G)d(Z(G))} |Z(G)| \geq p^{d(G)d(Z(G))+d(Z(G))}.$$

Therefore $Z(G)$ is cyclic of order $\leq p^3$. Moreover, by Remark 1.2, we may assume that $\gamma_2(G)$ is not cyclic. Now, we consider three cases: (a) $Z(G) \cong \mathbb{Z}_{p^3}$, (b) $Z(G) \cong \mathbb{Z}_{p^2}$, and (c) $Z(G) \cong \mathbb{Z}_p$.

Case (a): Let $Z(G) \cong \mathbb{Z}_{p^3}$. Then $Z_2^*(G) = Z(\Phi(G)) = \Phi(G)$ is of order p^5 , by (2.4). Now Lemma 2.9 gives a contradiction in this case.

Case (b): Let $Z(G) \cong \mathbb{Z}_{p^2}$. By Lemma 2.9 we may assume that $Z_2^*(G) \not\leq \Phi(G)$. Then it follows from (2.4) that $|Z_2^*(G)| = p^4$, $d(G) = 2$, and $|G| = p^7$. Then $cl(G) \leq 4$, as $|Z_2^*(G)| = p^4$. Thus we may assume by Remark 1.2 that $cl(G) = 3$ or $cl(G) = 4$. If $cl(G) = 3$, then it follows from Lemma 2.7, that $Z_2^*(G) = \Phi(G)$, a contradiction. Hence $cl(G) = 4$ and G has the following upper central series;

$$1 < Z_1(G) < Z_2(G) < Z_3(G) < Z_4(G) = G,$$

where $|Z_1(G)| = p^2$, $|Z_2(G)/Z_1(G)| = p^2$, and $|Z_3(G)/Z_2(G)| = p$.

By Remark 1.2, we may assume that $\gamma_2(G)$ is not cyclic. Thus it follows from Lemma 2.4 that there exists a noncentral element u of order p in $\gamma_2(G) \cap Z_2(G)$. If $u \notin G^{2p}\gamma_3(G)$, then $u = [a, b]^i c$ for some $c \in G^{2p}\gamma_3(G)$ and $1 \leq i \leq p - 1$, since $\gamma_2(G) = \langle [a, b], \gamma_3(G) \rangle$. This implies that $\gamma_2(G) \leq G^{2p}\gamma_3(G)Z_2^*(G)$. Hence G is powerful, by Lemma 2.4, and then Remark 1.2 yields a contradiction. Therefore $u \in G^{2p}\gamma_3(G)$. Now, it follows from Theorem 2.1 that there are automorphisms α and α' of G such that $\alpha : a \mapsto au, b \mapsto b$ and $\alpha' : a \mapsto a, b \mapsto bu$. Clearly α and α' have order p , as they fix $G^{2p}\gamma_3(G)$ elementwise. Then $\alpha = \theta_s$ and $\alpha' = \theta_{s'}$, for some $s, s' \in Z_3(G) \setminus Z_2(G)$. As $s' = s^i t$ for some integer $1 \leq i \leq p - 1$ and some $t \in Z_2(G)$ we get

$$1 = [a, \alpha'] = [a, s'] = [a, s^i t] = [a, s]^i [a, t] = u^i [a, t].$$

Thus $u \in Z(G)$, a contradiction.

Case (c): Let $|Z(G)| = p$, then it suffices to show that G admits a noninner automorphism of order p , by Remark 2.8. Suppose that $\Phi(G)$ is nonabelian. It follows from (2.4) that $Z_2^*(G) = Z(\Phi(G))$ is of order p^3 , and $d(G) = 2$. Thus G has order p^7 . Hence we have either $cl(G) \leq 3$ or $cc(G) \leq 3$. By [1, Corollary 2.4], G is not of maximal class, and by parts (1) and (2) of Remark 1.1 we have $p = 3$ and $\text{coclass } G$ is 3. Now $d(Z_2^*(G)) \geq 2$ and hence argument (1) of Remark 2.5 gives a contradiction. Therefore, $\Phi(G)$ is abelian. By part (6) of Remark 1.1 we may assume that $d(G) \geq 3$, and by Lemma 2.9, we have $\Phi(G) \not\leq Z_2(G)$. Thus (2.4), implies that

$$d(G) = 3, |Z_2^*(G)| = p^4, |\Phi(G)| = p^5, \text{ and } |G| = p^8.$$

Then the argument (2) of Remark 2.5 holds.

Now, let $g \in Z_2(G)$. For every $h \in G$ we have $[g^p, h] = [g, h]^p = 1$, as $Z(G)$ is of order p . Hence

$$(2.5) \quad Z_2(G) = Z_2^*(G).$$

Since $Z(\frac{G}{Z_2(G)}) \cap \frac{\Phi(G)}{Z_2(G)}$ is nontrivial, we get $\frac{Z_3(G)}{Z_2(G)} \cap \frac{\Phi(G)}{Z_2(G)}$ is cyclic of order p . Suppose that p is odd. Use the same notations and terminology as in the argument (2) of Remark 2.5. Then $y = x^i t$, for

some integer $0 < i < p$ and $t \in Z_2(G)$. Thus

$$u = [b, \gamma] = [b, y] = [b, x^i t] = [b, x]^i [b, t] = [b, \beta]^i [b, t] = [b, t] \in Z(G),$$

that contradicts the choice of u . Hence we have $p = 2$. If $G^4 \gamma_3(G) \not\leq Z_2^*(G)$, then $\gamma_2(G) \leq \Phi(G) = G^4 \gamma_3(G) Z_2^*(G)$. Thus by Lemma 2.4, G is powerful and part (5) of Remark 1.1 gives a contradiction. Therefore,

$$(2.6) \quad G^4 \gamma_3(G) \leq Z_2^*(G).$$

Since $G/Z_2^*(G)$ is nonabelian, we get $\gamma_2(G) \not\leq Z_2^*(G)$. Hence $\Phi(G) = \gamma_2(G) Z_2^*(G)$ and $|\gamma_2(G/Z_2^*(G))| = 2$. Therefore, $G/Z_2^*(G)$ is of nilpotency class 2.

It is easy to see that if H is finite 2-group such that H' is of order 2 and $d(H) = 3$, then $Z(H) \not\leq \Phi(H)$. Therefore there is some $c \in G \setminus \Phi(G)$ such that $c Z_2^*(G) \in Z(G/Z_2^*(G))$. Thus $[c, G] \subseteq Z_2^*(G)$.

Recall w from the argument (2) of Remark 2.5. We have $w = ts$, for some $t \in \gamma_2(G) \setminus Z_2^*(G)$ and $s \in Z_2^*(G)$. Thus $u = [a, ts] = [a, t][a, s]$ and $v = [b, ts] = [b, t][b, s]$. Then $u, v \in \gamma_3(G)$. This shows that $\Omega_1(Z_2(G)) = \gamma_3(G)$.

We know that $[\gamma_i(G), Z_j(G)] \leq Z_{j-i}$, for each $j \geq i$ (See for instance [23, Corollary 2, page 20]). Hence $[\gamma_3(G), c] = 1$. This shows that $c \in M \cap N$. Thus $G = \langle a, b, c \rangle$. Since $\gamma_2(G) = \langle [a, b], [a, c], [b, c], \gamma_3(G) \rangle$ and $[a, c], [b, c] \in Z_2^*(G)$, it follows that $\Phi(G)/Z_2^*(G) = \langle [a, b] Z_2^*(G) \rangle$.

Note that $[c^2, a] = [c, a]^2 [c, a, c] \in Z(G)$, since $[c, a] \in Z_2^*(G)$. Similarly $[c^2, b] \in Z(G)$. Hence $c^2 \in Z_2(G)$. Set $N = \langle c, Z_2(G) \rangle$. Then $|N| = 2^5$, $N \trianglelefteq G$ and G/N is nonabelian of order 8. Let $\bar{G} = G/N$ and $\bar{g} = Ng$, for $g \in G$. We have $[N, \gamma_3(G)] = 1$. Thus \bar{G} acts by conjugation on $\gamma_3(G)$. Therefore $\gamma_3(G)$ can be considered as a \bar{G} -module. If f is a derivation from $\bar{G} \rightarrow \gamma_3(G)$, then the mapping $\alpha_f : G \rightarrow G$, given by $\alpha_f(x) = xf(\bar{x})$, is an automorphism of G of order 2. We consider the following two cases.

- If $\bar{a}^2 = \bar{b}^2 = \overline{[a, b]}$, then

$$\bar{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^4 = \bar{1}, \bar{a}^2 = \bar{b}^2, [\bar{a}, \bar{b}] = \bar{a}^2 \rangle \cong Q_8.$$

So every element of \bar{G} can be written in the form $\bar{a}^i \bar{b}^j$ for some $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1\}$. Define the mapping $f : \bar{G} \rightarrow \gamma_3(G)$, by $f(\bar{a}^i \bar{b}^j) = v^{j+1+a+\dots+a^{i-1}} u^{1+b+\dots b^{j-1}}$. Then

$$f(\bar{a}^i \bar{b}^j \bar{a}^k \bar{b}^l) = f(\bar{a}^{i+k+2jk} \bar{b}^{j+l}) = v^{j+l+1+a+\dots+a^{i+k+2jk-1}} u^{1+b+\dots b^{j+l-1}}.$$

On the other hand,

$$\begin{aligned} f(\bar{a}^i \bar{b}^j)^{a^k b^l} f(\bar{a}^k \bar{b}^l) &= (v^{j+1+a+\dots+a^{i-1}} u^{1+b+\dots b^{j-1}})^{a^k b^l} v^{l+1+a+\dots+a^{k-1}} u^{1+b+\dots b^{l-1}} \\ &= v^{ja^k+a^k+\dots+a^{i+k-1}} u^{b^l+b^{l+1}+\dots b^{j+l-1}} v^{l+1+a+\dots+a^{k-1}} u^{1+b+\dots b^{l-1}} \\ &= v^{j+l+1+a+\dots+a^{i+k-1}} u^{1+b+\dots b^{j+l-1}} [v, a]^{jk}. \end{aligned}$$

Since $v^{a^{i+k}+\dots+a^{i+k+2jk-1}} = [v, a]^{jk}$, f is a derivation. By assumption $\alpha_f = \theta_{w_0}$, for some $w_0 \in G$. Since $[G, \alpha_f] \leq Z_2(G)$ and α_f is identity on N , we have $w_0 \in Z_3(G) \setminus Z_2(G)$ and $w_0 \in C_G(N) \leq C_G(Z_2^*(G)) = \Phi(G)$. Hence $w_0 = wt$, for some $t \in Z_2(G)$. Now,

$$uv = [b, \alpha_f] = [b, w_0] = [b, wt] = [b, w][b, t] = v[b, t]$$

This implies that $u \in Z(G)$, a contradiction.

• If $\bar{a}^2 = \bar{1}$ or $\bar{a}^2 = \bar{1}$, then by replacement may assume that $\bar{a}^2 = \overline{[a, b]}$ and $\bar{b}^2 = \bar{1}$. Indeed, if $\bar{b}^2 = \overline{[a, b]}$, and $\bar{a}^2 = \bar{1}$, then we replace a and b by each other, and u and v by each other, and if $\bar{a}^2 = \bar{b}^2 = \bar{1}$, then we replace a by ab and u by uv . Therefore we have

$$\bar{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^4 = \bar{b}^2 = \bar{1}, [\bar{a}, \bar{b}] = \bar{a}^2 \rangle \cong D_8,$$

and each element of \bar{G} is written in the form $\bar{a}^i \bar{b}^j$ for some $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1\}$. In this case the mapping $f : \bar{G} \rightarrow \gamma_3(G)$, defined by $f(\bar{a}^i \bar{b}^j) = v^{j+1+a+\dots+a^{i-1}}$ is a derivation. Then α_f is an automorphism of G of order 2. Thus $\alpha_f = \theta_{w_0}$ is inner. Hence $w_0 = wt$, for some $t \in Z_2(G)$.

$$v = [a, \alpha_f] = [a, w_0] = [a, wt] = [a, w][a, t] = u[a, t]$$

It implies that $uv \in Z(G)$, a contradiction. This completes the proof. □

REFERENCES

- [1] A. Abdollahi, Powerful p -groups have noninner automorphisms of order p and some cohomology, *J. Algebra*, **323** (2010) 779–789.
- [2] A. Abdollahi, Finite p -groups of class 2 have noninner automorphisms of order p , *J. Algebra*, **312** (2007) 876–879.
- [3] A. Abdollahi and S. M. Ghoraiishi, On noninner 2-automorphisms of finite 2-groups, *Bull. Aust. Math. Soc.*, **90** (2014) 227–231.
- [4] A. Abdollahi, S. M. Ghoraiishi, Y. Guerboussa, M. Reguiat and B. Wilkens, Noninner automorphisms of order p for finite p -groups of coclass 2, *J. Group Theory*, **17** (2014) 267–272.
- [5] A. Abdollahi and S. M. Ghoraiishi, Noninner automorphisms of finite p -groups leaving the center elementwise fixed, *Int. J. Group Theory*, **2** no. 4 (2013) 17–20.
- [6] A. Abdollahi, S. M. Ghoraiishi and B. Wilkens, Finite p -groups of class 3 have noninner automorphisms of order p , *Beitr. Algebra Geom.*, **54** (2013) 363–381.
- [7] A. Abdollahi and S. M. Ghoraiishi, On noninner automorphisms of 2-generator finite p -groups, *Comm. Algebra*, **48** (2017) 3636–3642.
- [8] L. Y. Bodnarchuk and O. S. Pylyavska, On the existence of a noninner automorphism of order p for p -groups, *Ukrainian Math. J.*, **53** (2001) 1771–1783.
- [9] A. Caranti and C. M. Scoppola, Endomorphisms of two-generated metabelian groups that induce the identity modulo the derived subgroup, *Arch. Math. (Basel)*, **56** (1991) 218–227.
- [10] M. Deaconescu and G. Silberberg, Noninner automorphisms of order p of finite p -groups, *J. Algebra*, **250** (2002) 283–287.
- [11] W. Gaschütz, Nichtabelsche p -Gruppen besitzen äussere p -Automorphismen, *J. Algebra*, **4** (1966) 1–2.

<http://dx.doi.org/10.22108/ijgt.2018.108082.1457>

- [12] S. M. Ghoraishi, On noninner automorphisms of finite nonabelian p -groups, *Bull. Aust. Math. Soc.*, **89** (2014) 202–209.
- [13] S. M. Ghoraishi, On noninner automorphisms of finite p -groups that fix the Frattini subgroup elementwise, *J. Algebra Appl.*, **17** (2018) 1850137.
- [14] A. R. Jamali, M. Viseh, On the existence of noninner automorphisms of order two in finite 2-groups, *Bull. Aust. Math. Soc.*, **87** (2013) 278–287.
- [15] H. Liebeck, Outer automorphisms in nilpotent p -groups of class 2, *J. London Math. Soc.*, **40** (1965) 268–275.
- [16] V. D. Mazurov, E. I. Khukhro and eds, *Unsolved problems in group theory*, The Kourovka Notebook, no. 16, Russian Academy of Sciences, Siberian Division, Institute of Mathematics, Novosibirsk, 2006.
- [17] D.S. Malik, J. M. Mordeson and M. K. Sen, *Fundamentals of Abstract Algebra*, The McGraw-Hill. Companies, Inc., New York, 1997.
- [18] O. Müller, On p -automorphisms of finite p -groups, *Arch. Math. (Basel)*, **32** (1979) 533–538.
- [19] M. Ruscitti, L. Legarreta and M. K. Yadav, Non-inner automorphisms of order p in finite p -groups of coclass 3, *Monatsh. Math.*, **183** (2017) 679–697.
- [20] P. Schmid, A cohomological property of regular p -groups, *Math. Z.*, **175** (1980) 1–3.
- [21] P. Schmid, Normal p -subgroups in the group of outer automorphisms of a finite p -group, *Math. Z.*, **147** (1976) 271–277.
- [22] C. C. Sims, *Computation with finitely presented groups*, Cambridge University Press, 1994.
- [23] M. Suzuki, *Group Theory II*, New York, Springer-Verlag, 1986.

S. Mohsen Ghoraishi

Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Email: m.ghoraishi@scu.ac.ir