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## ON THE RANKS OF FISCHER GROUP $Fi'_{24}$ AND THE BABY MONSTER GROUP $\mathbb{B}$

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**ABSTRACT.** If  $G$  is a finite group and  $X$  a conjugacy class of elements of  $G$ , then we define  $\text{rank}(G:X)$  to be the minimum number of elements of  $X$  generating  $G$ . In the present article, we determine the ranks for the Fischer's simple group  $Fi'_{24}$  and the baby monster group  $\mathbb{B}$ .

### 1. Introduction

Group generations have played a significant role in solving outstanding problems in representation theory, topology, geometry and number theory. Generation of a group by suitable subsets has been studied since the origins of group theory. This paper is intended as a sequel to the author's earlier work on the determination of ranks for the sporadic simple groups. In a series of articles [1, 2, 3, 4, 5, 6, 7], the authors determined the ranks for the sporadic simple groups  $J_1, J_2, J_3, J_4, HS, McL, He, Co_1, Co_2, Co_3, O'N, Ly$  and  $Fi_{22}$ . We encourage the reader to consult [10] for summary of results on generation of sporadic groups by minimal sets of conjugate elements. The motivation for this study is outlined in these papers and we encourage the reader to consult [1, 2] and [7] for background material as well as basic computational techniques.

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Suppose that  $G$  is a finite group and  $X \subseteq G$ . We denote the rank of  $X$  in  $G$  by  $rank(G:X)$ , the minimum number of elements of  $X$  generating  $G$ . This paper focuses on the determination of  $rank(G:X)$  where  $X$  is a conjugacy class of  $G$  and  $G$  is a sporadic simple group. In the present article, we completely determine the ranks for Fischer's largest sporadic simple group  $Fi'_{24}$  and the Fischer's Baby Monster group  $\mathbb{B}$ . In particular, we prove the following two theorems.

**Theorem 3.5 .** *Let  $Fi'_{24}$  be the Fischer's largest sporadic simple group. Then*

- (i)  $rank(Fi'_{24} : nX) = 3$ , if  $nX \in \{2A, 2B, 3A, 3B\}$ ,
- (ii)  $rank(Fi'_{24} : nX) = 2$ , if  $nX \notin \{1A, 2A, 2B, 3A, 3B\}$ .

**Theorem 4.5 .** *Let  $\mathbb{B}$  be the Fischer's Baby Monster group. Then*

- (i)  $rank(\mathbb{B} : 2A) = 4$ ,
- (ii)  $rank(\mathbb{B} : nX) = 3$ , if  $nX \in \{2B, 2C, 2D\}$ ,
- (iii)  $rank(\mathbb{B} : nX) = 2$ , if  $nX \notin \{1A, 2A, 2B, 2C, 2D\}$ .

## 2. Preliminaries

Throughout this paper we use the same notion as in [1, 2, 7]. If  $G$  is a finite group,  $C_1, C_2, C_3$  conjugacy classes of the elements of  $G$ , and  $g_3$  is a fixed representative of  $C_3$ , then  $\Delta_G(C_1, C_2, C_3)$  denotes the number of distinct ordered pairs  $(g_1, g_2) \in (C_1 \times C_2)$  such that  $g_1g_2 = g_3$ . It is well known that  $\Delta_G(C_1, C_2, C_3)$  is *structure constant* for the conjugacy classes  $C_1, C_2, C_3$  and can easily be computed from the character table of  $G$  by the following formula

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1||C_2|}{|G|} \sum_{i=1}^m \frac{\chi_i(g_1)\chi_i(g_2)\overline{\chi_i(g_3)}}{\chi_i(1_G)}$$

where  $\chi_1, \chi_2, \dots, \chi_m$  are the irreducible complex characters of  $G$ . Also, let  $\Delta_G^*(C_1, C_2, C_3)$  denotes the number of distinct ordered pairs  $(g_1, g_2) \in (C_1 \times C_2)$  such that  $g_1g_2 = g_3$  and  $G = \langle g_1, g_2 \rangle$ . If  $\Delta_G^*(C_1, C_2, C_3) > 0$ , then we say that  $G$  is  $(C_1, C_2, C_3)$ -generated. If  $H$  is any subgroup of  $G$  containing the fixed element  $g_3 \in C_3$ , then  $\Sigma_H(C_1, C_2, C_3)$  denotes the number of distinct ordered pairs  $(g_1, g_2) \in (C_1 \times C_2)$  such that  $g_1g_2 = g_3$  and  $\langle g_1, g_2 \rangle \leq H$  where  $\Sigma_H(C_1, C_2, C_3)$  is obtained by summing the structure constants  $\Delta_H(c_1, c_2, c_3)$  of  $H$  over all  $H$ -conjugacy classes  $c_1, c_2$  satisfying  $c_i \subseteq H \cap C_i$  for  $1 \leq i \leq 2$ .

Similarly, if  $C_1, C_2, C_3, C_4$  are conjugacy classes of  $G$  and  $g_4$  a fixed representative of  $C_4$  then  $\Delta_G(C_1, C_2, C_3, C_4)$  denote the number of distinct triples  $(g_1, g_2, g_3)$  with  $g_i \in C_i$   $1 \leq i \leq 3$  such that  $g_1g_2g_3 = g_4$ . This number can be computed with formula

$$\Delta_G(C_1, C_2, C_3, C_4) = \frac{|C_1||C_2||C_3|}{|G|} \sum_{i=1}^m \frac{\chi_i(g_1)\chi_i(g_2)\chi_i(g_3)\overline{\chi_i(g_4)}}{(\chi_i(1_G))^2}$$

where  $\chi_1, \chi_2, \dots, \chi_m$  are the irreducible complex characters of  $G$ .

The ATLAS [8] serves as a valuable source of information and we use its notation for conjugacy classes, maximal subgroups, character tables, permutation characters, etc. A general conjugacy class of elements of order  $n$  in  $G$  is denoted by  $nX$ . For example,  $2A$  represents the first conjugacy class of involutions in a group  $G$ .

**Lemma 2.1.** [7] *Let  $G$  be a finite simple group such that  $G$  is  $(lX, mY, nZ)$ -generated. Then  $G$  is  $(\underbrace{lX, lX, \dots, lX}_{m\text{-times}}, (nZ)^m)$ -generated.*

**Corollary 2.2.** *Let  $G$  be a finite simple group such that  $G$  is  $(lX, mY, nZ)$ -generated, then  $\text{rank}(G : lX) \leq m$ .*

*Proof.* Immediately follows from Lemma 2.1. □

**Lemma 2.3.** *Conder et al. [9] Let  $G$  be a simple  $(2X, mY, nZ)$ -generated group. Then  $G$  is  $(mY, mY, (nZ)^2)$ -generated.*

**Theorem 2.4.** [14] *Let  $G$  be a finite group and  $H$  a subgroup of  $G$  containing a fixed element  $x$  such that  $\gcd(o(x), [N_G(H):H]) = 1$ . Then the number  $h$  of conjugates of  $H$  containing  $x$  is  $\chi_H(x)$ , where  $\chi_H$  is the permutation character of  $G$  with action on the conjugates of  $H$ . In particular,*

$$h = \sum_{i=1}^m \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where  $x_1, \dots, x_m$  are representatives of the  $N_G(H)$ -conjugacy classes that fuse to the  $G$ -class  $[x]_G$ .

The following result in certain situations is very effective at establishing non-generations.

**Lemma 2.5.** [9] *Let  $G$  be a finite centerless group and suppose  $lX, mY, nZ$  are  $G$ -conjugacy classes for which  $\Delta^*(G) = \Delta_G^*(lX, mY, nZ) < |C_G(z)|, z \in nZ$ . Then  $\Delta^*(G) = 0$  and therefore  $G$  is not  $(lX, mY, nZ)$ -generated.*

### 3. Ranks of $Fi'_{24}$

A group  $G$  is called a 3-transposition group if it is generated by a conjugacy class  $D$  of involutions in  $G$  such that  $o(de) \leq 3$  for all  $d, e \in D$ . The conjugacy class  $D$  is called a class of conjugate 3-transpositions. B. Fischer [11] introduced and investigated 3-transposition groups. He classified all finite 3-transposition groups with no non-trivial normal soluble subgroups. In the process of classifying the 3-transposition groups, Fischer discovered three new groups  $Fi_{22}$ ,  $Fi_{23}$  and  $Fi_{24}$  with 3510, 31671 and 306936 transpositions respectively. Of these, the first two groups are simple, while the third contains a simple normal subgroup  $Fi'_{24}$  of index 2 (consisting of the products of evenly many transpositions). The group  $Fi'_{24}$  has order

$$1255205709190661721292800 = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29.$$

The group  $Fi'_{24}$  has 108 conjugacy classes of its elements in total including two classes of involutions and five classes of elements of order 3, namely  $2A, 2B, 3A, 3B, 3C, 3D$  and  $3E$  as represented in ATLAS [8]. Linton and Wilson [16] classified all the maximal subgroups of  $Fi'_{24}$  and its automorphism group  $Fi_{24}$  which are available in the ATLAS[8]. We will use these maximal subgroups and the permutation characters of  $Fi'_{24}$  on the conjugates (right cosets) of the maximal subgroups listed in the above theorem extensively.

Next, we compute rank for each conjugacy class of  $Fi'_{24}$ . It is well known that each sporadic simple group can be generated by three involutions. In the following lemma we show that the group  $Fi'_{24}$  can be generated by three conjugate involutions  $a, b, c \in 2X$ , where  $X \in \{A, B\}$  such that  $abc \in 23A$ .

**Lemma 3.1.** *Let  $2A$  and  $2B$  be two conjugacy classes of involutions in the Fischer group  $Fi'_{24}$ . Then  $\text{rank}(Fi'_{24} : 2X) = 3$ , where  $X \in \{A, B\}$ .*

*Proof.* We consider the triples  $(2A, 3E, 23A)$  and  $(2B, 3E, 23A)$ . Simple computation shows that the structure constants  $\Delta_{Fi'_{24}}(2A, 3E, 23A)=138$  and  $\Delta_{Fi'_{24}}(2B, 3E, 23A)=199962$ . From the maximal subgroups of  $Fi'_{24}$  (see [8]), we observe that, up to isomorphism,  $Fi_{23}$  and  $2^{11}:M_{24}$  are the only maximal subgroup of  $Fi'_{24}$  with order divisible by 23. However,  $Fi'_{24}$ -class  $3E$  fails to meet  $Fi_{23}$ . Further, a fixed  $z \in 23A$  is contained in precisely a unique conjugate copy of  $2^{11}:M_{24}$ . Since  $\Sigma_{2^{11}:M_{24}}(2A, 3E, 23A) = 46$  and  $\Sigma_{2^{11}:M_{24}}(2B, 3A, 23A) = 506$ , we have

$$\begin{aligned} \Delta_{Fi'_{24}}^*(2A, 3E, 23A) &\geq \Delta_{Fi'_{24}}(2A, 3E, 23A) - \Sigma_{2^{11}:M_{24}}(2A, 3E, 23A) \\ &= 138 - 46 > 0, \\ \Delta_{Fi'_{24}}^*(2B, 3E, 23A) &\geq \Delta_{Fi'_{24}}(2B, 3E, 23A) - \Sigma_{2^{11}:M_{24}}(2B, 3E, 23A) \\ &= 199962 - 506 > 0. \end{aligned}$$

Hence,  $Fi'_{24}$  is  $(2X, 3E, 23A)$ -generated, where  $X \in \{A, B\}$ . By applying Corollary 2.2, we have  $\text{rank}(Fi'_{24} : 2X) \leq 3$ ,  $X \in \{A, B\}$ . But  $\text{rank}(Fi'_{24} : 2X) = 2$  is not possible since if there are  $x, y \in 2X$  such that  $Fi'_{24} = \langle x, y \rangle$ , then  $Fi'_{24} = D_{2n}$  where  $o(xy) = n$ . This concluded that  $\text{rank}(Fi'_{24} : 2X) = 3$ , where  $X \in \{A, B\}$ . □

**Lemma 3.2.** *The Fischer group  $Fi'_{24}$  is not  $(3X, 3X, nY)$ -generated for any integer  $n$ , where  $X \in \{A, B\}$ .*

*Proof.* It is well known that if  $G$  is  $(l, m, n)$ -generated simple group then  $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ . It follows that in  $(3X, 3X, nY)$ - generations of  $Fi'_{24}$ , we have  $n > 3$ .

Let  $P = \{4A, 5A, 6A, 6B, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 8B, 9A, 9B, 9C, 9E, 9F, 10A, 12A, 12B, 12E, 12H, 13A, 15A, 15C, 18B\}$ . For each  $nY \notin P$ , non-generation of the triple  $(3X, 3X, nY)$  follows immediately since the structure constant  $\Delta_{Fi'_{24}}(3X, 3X, nY) = 0$ . Further, computation reveals that for

each  $nY \in P$ , we have

$$\Delta_{Fi'_{24}}(3X, 3X, nY) < |C_{Fi'_{24}}(nY)|$$

and by Lemma 2.5, we obtain  $\Delta_{Fi'_{24}}^*(3X, 3X, nY) = 0$ . Hence, the Fischer group  $Fi'_{24}$  is not  $(3X, 3X, nY)$ -generated for any  $n$ . This completes the proof.  $\square$

**Lemma 3.3.** *Let  $3X$  denote a conjugacy class of element of order 3 in the Fischer's simple group  $Fi'_{24}$ . Then*

- (i)  $\text{rank}(Fi'_{24} : 3X) = 3$  if  $X \in \{A, B\}$
- (ii)  $\text{rank}(Fi'_{24} : 3X) = 2$  if  $X \in \{C, D, E\}$ .

*Proof.* We treat both cases separately.

(i) From the Lemma 3.2, we know that  $Fi'_{24}$  is not  $(3X, 3X, tZ)$ -generated, where  $X \in \{A, B\}$  and  $tZ \in Fi'_{24}$ . Thus,  $\text{rank}(Fi'_{24} : 3X) > 2$  for  $X \in \{A, B\}$ . Next, we show that  $\text{rank}(Fi'_{24} : 3X) = 3$  where  $X \in \{A, B\}$ . Consider the quadruple  $(3X, 3X, 3X, 23A)$ . As mentioned earlier, up to isomorphisms,  $Fi_{23}$  and  $2^{11}:M_{24}$  are the only maximal subgroups of  $Fi'_{24}$  that may admit  $(2X, 2X, 2X, 23A)$ -generation. However, from the fusion map of  $2^{11}:M_{24}$  into the group  $Fi'_{24}$ , we observe that  $2^{11}:M_{24}$  does not meet classes  $3A$  and  $3B$  of  $Fi'_{24}$ . In particular,

$$2^{11}:M_{24} \cap 3A = \emptyset = 3A \cap 2^{11}:M_{24}.$$

So, clearly any proper  $(3X, 3X, 3X, 23A)$ -generated subgroup of  $Fi'_{24}$  must lie in  $Fi_{23}$ . By looking at the fusion map of  $Fi_{23}$  into  $Fi'_{24}$ , we compute  $\Delta_{Fi'_{24}}(3A, 3A, 3A, 23A) = 16148783$ ,  $\Delta_{Fi'_{24}}(3B, 3B, 3B, 23A) = 110561539051$ ,  $\Sigma_{Fi_{23}}(3A, 3A, 3A, 23A) = 5484067$ ,  $\Sigma_{Fi_{23}}(3B, 3B, 3B, 23A) = 3766015009$ . Since a fixed element of order 23 is contained in a unique conjugate subgroup of  $Fi_{23}$ , we have

$$\begin{aligned} \Delta_{Fi'_{24}}^*(3A, 3A, 3A, 23A) &\geq \Delta_{Fi'_{24}}(3A, 3A, 3A, 23A) - \Sigma_{Fi_{23}}(3A, 3A, 3A, 23A) \\ &= 16148783 - 5884067 > 0, \end{aligned}$$

$$\begin{aligned} \Delta_{Fi'_{24}}^*(3B, 3B, 3B, 23A) &\geq \Delta_{Fi'_{24}}(3B, 3B, 3B, 23A) - \Sigma_{Fi_{23}}(3B, 3B, 3B, 23A) \\ &= 110561539051 - 3766015009 > 0, \end{aligned}$$

proving the generation of  $Fi'_{24}$  by the  $(3X, 3X, 3X, 23A)$ . Thus,  $\text{rank}(Fi'_{24} : 3X) = 3$  for  $X \in \{A, B\}$  since  $\text{rank}(Fi'_{24} : 3X) > 2$ .

(ii) Consider the triple  $(2B, 3Y, 29A)$ , where  $Y \in \{C, D, E\}$ . We calculate the structure constants

$$\Delta_{Fi'_{24}}(2B, 3C, 29A) = 261, \Delta_{Fi'_{24}}(2B, 3D, 29A) = 47096, \Delta_{Fi'_{24}}(2B, 3E, 29A) = 205001.$$

From the list of maximal subgroups (see ATLAS [8]), we observe that, up to isomorphisms, 29:14 is the only maximal subgroup of  $F_i'_{24}$  with order divisible by 29. However,  $29:14 \cap 3Y = \emptyset$  for  $Y \in \{C, D, E\}$ . Thus no proper subgroup of  $F_i'_{24}$  is  $(2B, 3Y, 29A)$ -generated and hence

$$\Delta_{F_i'_{24}}^*(2B, 3Y, 29A) = \Delta_{F_i'_{24}}(2B, 3Y, 29A) > 0.$$

It follows that  $F_i'_{24}$  is  $(2B, 3Y, 29A)$ -generated for  $Y \in \{C, D, E\}$  and by Lemma 2.3,  $F_i'_{24}$  is  $(3Y, 3Y, (29A)^2)$ -generated where  $Y \in \{C, D, E\}$ . This concludes that  $\text{rank}(F_i'_{24}:3Y) = 2$  for  $Y \in \{C, D, E\}$ .  $\square$

**Lemma 3.4.** *Let  $nX$  be a conjugacy class of the Fischer's sporadic simple group  $F_i'_{24}$  such that  $n \geq 4$ , then  $\text{rank}(F_i'_{24} : nX) = 2$ .*

*Proof.* Set  $T = \{7B, 14B, 29A, 29B\}$ . First we consider the triple  $(nX, nX, 29A)$  with  $nX \notin T$  and  $n \geq 4$ . By looking at the maximal subgroups of the group  $F_i'_{24}$ , we observe that the only maximal subgroup that may contain  $(nX, nX, 29A)$ -generated proper subgroups, up to isomorphisms, is 29:14. However, each  $F_i'_{24}$ -class  $nX$  in the triple does not meet the maximal subgroup 29:14. That is,  $29:14 \cap nX = \emptyset$ . Since, the structure constant  $\Delta_{F_i'_{24}}(nX, nX, 29A) > 0$  we obtain

$$\Delta_{F_i'_{24}}^*(nX, nX, 29A) = \Delta_{F_i'_{24}}(nX, nX, 29A) > 0,$$

proving generation of  $F_i'_{24}$  by the triple  $(nX, nX, 29A)$  with  $n \geq 4$  and  $nX \notin T$ .

Next, we consider the triple  $(nX, nX, 29A)$  for  $nX \in T$ . From the fusion map of the maximal subgroup 29:14 into the group  $F_i'_{24}$ , we obtain that all the classes of order 7 in 29:14 fuses to 7B-class of  $F_i'_{24}$ , all the classes of order 14 in 29:14 fuses to 14B-class of  $F_i'_{24}$ , and the classes 29A and 29B in 29:14 fuses to 29A and 29B classes of  $F_i'_{24}$ , respectively. Further, we compute  $\Delta_{F_i'_{24}}(7B, 7B, 29A) = 296363124652705632$ ,  $\Delta_{F_i'_{24}}(14B, 14B, 29A) = 7115678621348544239968$ ,  $\Delta_{F_i'_{24}}(29A, 29A, 29A) = 1492487887225908548178$ ,  $\Delta_{F_i'_{24}}(7B, 7B, 29A) = 1492487886962906026578$ ,  $\Sigma_{29:14}(7B, 7B, 29A) = 174 = \Sigma_{29:14}(14B, 14B, 29A)$ ,  $\Sigma_{29:14}(29A, 29A, 29A) = 6$  and  $\Sigma_{29:14}(29B, 29B, 29A) = 7$ . Since a fixed element of order 29 in  $F_i'_{24}$  is contained in a unique conjugate class of 29:14, we calculate

$$\begin{aligned} \Delta_{Fi'_{24}}^*(7B, 7B, 29A) &\geq \Delta_{Fi'_{24}}(7B, 7B, 29A) - \Sigma_{29:14}(7B, 7B, 29A) \\ &= 296363124652705632 - 174 > 0, \\ \Delta_{Fi'_{24}}^*(14B, 14B, 29A) &\geq \Delta_{Fi'_{24}}(14B, 14B, 29A) - \Sigma_{29:14}(14B, 14B, 29A) \\ &= 7115678621348544239968 - 174 > 0, \\ \Delta_{Fi'_{24}}^*(29A, 29A, 29A) &\geq \Delta_{Fi'_{24}}(29A, 29A, 29A) - \Sigma_{29:14}(29A, 29A, 29A) \\ &= 1492487887225908548178 - 6 > 0, \\ \Delta_{Fi'_{24}}^*(29B, 29B, 29A) &\geq \Delta_{Fi'_{24}}(29B, 29B, 29A) - \Sigma_{29:14}(29B, 29B, 29A) \\ &= 1492487886962906026578 - 7 > 0. \end{aligned}$$

Thus,  $Fi'_{24}$  is  $(nX, nX, 29A)$ -generated for  $nX \in T$  and hence  $\text{rank}(Fi'_{24}:nX) = 2$  for each conjugacy class  $nX \in Fi'_{24}$  with  $n \geq 4$ . This completes the lemma.  $\square$

We summarize our results of this section in the following theorem:

**Theorem 3.5.** *Let  $Fi'_{24}$  be the Fischer's largest sporadic simple group and  $nX$  be a conjugacy class of  $Fi'_{24}$ . Then*

- (i)  $\text{rank}(Fi'_{24} : nX) = 3$ , if  $nX \in \{2A, 2B, 3A, 3B\}$
- (ii)  $\text{rank}(Fi'_{24} : nX) = 2$ , if  $nX \notin \{1A, 2A, 2B, 3A, 3B\}$

*Proof.* The proof follows from the lemmas proved in this section.  $\square$

#### 4. Ranks of the Baby Monster group $\mathbb{B}$

The baby monster group, also known as Fischer's baby monster group, is the second-largest sporadic group and is denoted by  $\mathbb{B}$ . The Fischer's baby monster group is a  $\{3, 4\}$ -transposition simple group of order

$$4, 154, 781, 481, 226, 191, 177, 580, 544, 000, 000 = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$$

It was first discovered by B. Fischer [12], who himself determined much of its properties including  $\{3, 4\}$ -transposition properties. J.S. Leon and C.C. Sims [15] proved its existence and uniqueness by constructed the group  $\mathbb{B}$  as a permutation group on 13571955 points. D.C. Hunt [13] was the first to compute its character table. Later, R.A. Wilson constructed it in a new way, studied its subgroup structure, classified much of its maximal subgroups along with its application (see [18, 19]). The character table of the group  $\mathbb{B}$  and its all known properties are now available in [8] and [17]. The baby monster group  $\mathbb{B}$  has exactly 30 conjugacy classes of maximal subgroups and 184 conjugacy classes of its elements including four classes of involutions  $2A, 2B, 2C$  and  $2D$  as represented in ATLAS ([8]).

Next, we compute the rank for each conjugacy class of the group  $\mathbb{B}$ . In the following lemma we show that the baby monster group  $\mathbb{B}$  can be generated by three conjugate involutions from its conjugacy class  $2A$ .

**Lemma 4.1.** *The Baby Monster sporadic group  $\mathbb{B}$  is not  $(2A, 2A, 2A, nX)$ -generated for any integer  $n$ .*

*Proof.* Let  $T_1 = \{2A, 2D, 4A, 4B, 4C, 4D, 4F, 4G, 4H, 4I, 6A, 6B, 6D, 6E, 6H, 8A, 8B, 8C, 8F, 12A, 12B, 12E, 14A\}$ . For each  $nX \notin T_1$ , the non-generation of the group  $\mathbb{B}$  by the quadruples  $(2A, 2A, 2A, nX)$  follows immediately since the structure constant  $\Delta_{\mathbb{B}}(2A, 2A, 2A, nX) = 0$  for each  $nX \notin T_1$ . Next, suppose  $nX \in T_1$  and consider the triple  $(2A, 2A, 2A, nX)$ . Our computation reveals that for each  $nX \in T_1$ , we obtain

$$\Delta_{\mathbb{B}}(2A, 2A, 2A, nX) < |C_{\mathbb{B}}(nX)|.$$

Now by an application of lemma 2.5, we have  $\Delta_{\mathbb{B}}^*(2A, 2A, 2A, nX) = 0$ . Hence the group  $\mathbb{B}$  is not  $(2A, 2A, 2A, nX)$ -generated for any  $n$ . This completes the proof.  $\square$

**Lemma 4.2.**  $\text{rank}(\mathbb{B} : 2A) = 4$ .

*Proof.* Direct computation shows that  $\Delta_{\mathbb{B}}(2A, 2A, 2A, 2A, 47A) = 4049097$ . The only maximal subgroups of group  $\mathbb{B}$  with order divisible by 47, up to isomorphism, is  $47:23$ . Since the group  $47:23$  has odd order, it contains no involutions. This proves that  $\mathbb{B}$  has no proper subgroup of type  $(2A, 2A, 2A, 2A, 47A)$  and so in this case, we have

$$\Delta_{\mathbb{B}}^*(2A, 2A, 2A, 2A, 47A) = \Delta_{\mathbb{B}}(2A, 2A, 2A, 2A, 47A) = 4049097.$$

Thus, the group  $\mathbb{B}$  is  $(2A, 2A, 2A, 2A, 47A)$ -generated and therefore by Corollary 2.2, we have  $\text{rank}(\mathbb{B} : 2A) \leq 4$ . Since  $\text{rank}(\mathbb{B} : 2A) > 2$  (as we argue in Lemma 3.1, that two involutions can not generate a simple group), the result follows by considering the previous lemma.  $\square$

**Lemma 4.3.** *If  $nX \in \{2B, 2C, 2D\}$  then  $\text{rank}(\mathbb{B} : nX) = 3$ .*

*Proof.* First we prove that the Baby Monster group  $\mathbb{B}$  is  $(2X, 3Y, 23A)$ -generated, for  $(X, Y) \in \{(B, B), (C, B), (D, A)\}$ . We treat each case separately.

*Case  $(2B, 3B, 23A)$ :* The maximal subgroups of the group  $\mathbb{B}$  with orders divisible by 23 and having non-empty intersection with the classes in this triple are isomorphic to  $2^{1+22}.Co_2$  and  $Fi_{23}$ . We calculate that  $\Delta_{\mathbb{B}}(2B, 3B, 23A) = 92$ ,  $\Sigma_{2^{1+22}.Co_2}(2B, 3B, 23A) = 0$ ,  $\Sigma_{Fi_{23}}(2B, 3B, 23A) = 23$ . Further, a fixed element of order 23 in  $\mathbb{B}$  is contained in precisely unique conjugate of  $2^{1+22}.Co_2$  and two conjugate copies of  $Fi_{23}$ . Hence

$$\begin{aligned} \Delta_{\mathbb{B}}^*(2B, 3B, 23A) &\geq \Delta_{\mathbb{B}}(2B, 3B, 23A) - 2\Sigma_{Fi_{23}}(2B, 3B, 23A) \\ &= 92 - 2(23) > 0, \end{aligned}$$



proving the generation of  $\mathbb{B}$  by the triple  $(2B, 3B, 23A)$ .

*Case  $(2C, 3B, 23A)$ :* Amongst the maximal subgroups of  $\mathbb{B}$  with order divisible by  $2 \times 3 \times 23$ , the only maximal subgroup with non-empty intersection with all the classes in this triple is isomorphic to  $2^{1+22}.Co_2$ . We compute

$$\begin{aligned} \Delta_{\mathbb{B}}^*(2C, 3B, 23A) &\geq \Delta_{\mathbb{B}}(2C, 3B, 23A) - \Sigma_{2^{1+22}.Co_2}(2C, 3B, 23A) \\ &= 2503504 - 4416 > 0. \end{aligned}$$

This shows that the group  $\mathbb{B}$  is  $(2C, 3B, 23A)$ -generated.

*Case  $(2D, 3A, 23A)$ :* We calculate the structure constant  $\Delta_{\mathbb{B}}(2D, 3A, 23A) = 598$ . The  $(2D, 3A, 23A)$ -generated proper subgroups of  $\mathbb{B}$  are contained in the maximal subgroups isomorphic to  $2^{1+22}.Co_2$  and  $Fi_{23}$ . As mentioned above, a fixed element  $z \in \mathbb{B}$  of order 23 is contained in precisely a unique conjugate of  $2^{1+22}.Co_2$  and two conjugate copies of  $Fi_{23}$ . Thus

$$\begin{aligned} \Delta_{\mathbb{B}}^*(2D, 3A, 23A) &\geq \Delta_{\mathbb{B}}(2D, 3A, 23A) - \Sigma_{2^{1+22}.Co_2}(2D, 3A, 23A) - 2\Sigma_{Fi_{23}}(2D, 3A, 23A), \\ &= 598 - 138 - 2(161) > 0, \end{aligned}$$

and generation of  $\mathbb{B}$  by the triple  $(2B, 3B, 23A)$  follows.

Thus by applying Corollary 2.2 on the triples  $(2B, 3B, 23A)$ -,  $(2C, 3B, 23A)$ -, and  $(2D, 3A, 23A)$ -generations of  $\mathbb{B}$ , we obtain that  $\text{rank}(\mathbb{B} : 2X) = 2$  where  $X \in \{B, C, D\}$ . This completes the proof.  $\square$

**Lemma 4.4.** *Let  $nX$  be a conjugacy class in the Baby Monster group  $\mathbb{B}$  such that  $nX \notin \{1A, 2A, 2B, 2C, 2D\}$ . Then  $\text{rank}(\mathbb{B} : nX) = 2$ .*

*Proof.* Set  $W = \{23A, 23B, 47A, 47B\}$ . Consider the triple  $(nX, nX, 47A)$  with  $nX \notin W$ . Amongst the maximal subgroups of the group  $\mathbb{B}$  with order divisible by 47, the only maximal subgroup that may admit  $(nX, nX, 47A)$ -generated proper subgroups is isomorphic to  $47:23$ . However, each non-involution  $\mathbb{B}$ -class  $nX \notin W$  does not meet the maximal subgroup  $47:23$ . That is,  $47:23 \cap nX = \emptyset$ . Since, the structure constant  $\Delta_{\mathbb{B}}(nX, nX, 47A) > 0$  we obtain

$$\Delta_{\mathbb{B}}^*(nX, nX, 47A) = \Delta_{\mathbb{B}}(nX, nX, 47A) > 0.$$

Hence, the group  $\mathbb{B}$  is  $(nX, nX, 47A)$ -generated with  $nX \notin W$

Now we consider the triple  $(nX, nX, 47A)$  for  $nX \in W$ . By looking at the fusion map of the maximal subgroup  $47:23$  into the group  $\mathbb{B}$ , we observe that  $23A, 23B, 23C, 23D, 23F, 23H, 23I, 23L, 23M, 23P, 23R$  classes in maximal subgroup  $47:23$  fuses to  $23A$ -class of  $\mathbb{B}$ , while the classes  $23E, 23G, 23J, 23K, 23N$ ,

23O, 23Q, 23S, 23T, 23U, 23V in 47:23 fuses to 23B-class of  $\mathbb{B}$ . The 47:23-classes 47A and 47B fuses to 47A and 47B classes of  $\mathbb{B}$ , respectively. We compute

$$\begin{aligned} \Delta_{\mathbb{B}}(23A, 23A, 47A) &= 1963507316354951733772778733568, \\ \Delta_{\mathbb{B}}(23B, 23B, 47A) &= 1963507316354951733772778733568, \\ \Delta_{\mathbb{B}}(47A, 47A, 47A) &= 1880823140432881168431474063924, \\ \Delta_{\mathbb{B}}(47B, 47B, 47A) &= 1880823141406367549686998153780. \end{aligned}$$

Further, we calculate

$$\Sigma_{47:23}(23A, 23A, 47A) = 0 = \Sigma_{47:23}(23B, 23B, 47A),$$

$\Sigma_{\mathbb{B}}(47A, 47A, 47A) = 11$  and  $\Sigma_{47:23}(47B, 47B, 47A) = 12$ . Since a fixed element of order 47 in  $\mathbb{B}$  is contained in precisely a unique conjugates of the group 47:23, we have

$$\begin{aligned} \Delta_{\mathbb{B}}^*(23A, 23A, 47A) &= \Delta_{\mathbb{B}}(23A, 23A, 47A) - \Sigma_{47:23}(23A, 23A, 47A) \\ &= 1963507316354951733772778733568 - 0 > 0, \\ \Delta_{\mathbb{B}}^*(23B, 23B, 47A) &= \Delta_{\mathbb{B}}(23B, 23B, 47A) - \Sigma_{29:14}(23B, 23B, 47A) \\ &= 1963507316354951733772778733568 - 0 > 0, \\ \Delta_{\mathbb{B}}^*(47A, 47A, 47A) &\geq \Delta_{\mathbb{B}}(47A, 47A, 47A) - \Sigma_{29:14}(47A, 47A, 47A) \\ &= 1880823140432881168431474063924 - 11 > 0, \\ \Delta_{\mathbb{B}}^*(47B, 47B, 47A) &\geq \Delta_{\mathbb{B}}(47B, 47B, 47A) - \Sigma_{29:14}(47B, 47B, 47A) \\ &= 1880823141406367549686998153780 - 12 > 0, \end{aligned}$$

showing that the group  $\mathbb{B}$  is  $(nX, nX, 47A)$ -generated for  $nX \in T$  and hence together with the above case, we have  $\text{rank}(\mathbb{B} : nX) = 2$  for each conjugacy class of  $nX \in \mathbb{B}$  such that  $nX \notin \{1A, 2A, 2B, 2C, 2D\}$ . This completes the lemma.  $\square$

We now summarize the above results of this section in the following theorem.

**Theorem 4.5 .** *Let  $\mathbb{B}$  be the Fischer's Baby Monster group and  $nX$  a conjugacy class of elements of  $\mathbb{B}$ . Then*

- (i)  $\text{rank}(\mathbb{B} : 2A) = 4$ .
- (ii)  $\text{rank}(\mathbb{B} : nX) = 3$  if  $nX \in \{2B, 2C, 2D\}$
- (iii)  $\text{rank}(\mathbb{B} : nX) = 2$  if  $nX \notin \{1A, 2A, 2B, 2C, 2D\}$

*Proof.* Proof follows from Lemmas 4.1 to 4.4.  $\square$

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