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# NOTE ON EDGE DISTANCE-BALANCED GRAPHS

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ABSTRACT. Edge distance-balanced graphs are graphs in which for every edge $e = uv$ the number of edges closer to vertex $u$ than to vertex $v$ is equal to the number of edges closer to $v$ than to $u$. In this paper, we study this property under some graph operations.

## 1. Introduction

Let $a$ and $b$ be two adjacent vertices of the graph $G$, and $e = ab$ the edge connecting them. For an edge $e = ab$ of a graph $G$, let $n_a^G(e)$ be the number of vertices closer to $a$ than to $b$. In other words, $n_a^G(e) = |\{u \in V(G) | d(u,a) < d(u,b)\}|$. In addition, let $n_0^G(e)$ be the number of vertices with equal distances to $a$ and $b$, i. e., $n_0^G(ab) = |\{u \in V(G) | d(u,a) = d(u,b)\}|$.

A graph $G$ is said to be distance-balanced, if $n_a^G(e) = n_b^G(e)$, for each edge $e = ab \in E(G)$, see [1, 5] for details. These graphs were, at least implicitly, first studied by Handa [4] who is considered distance-balanced partial cubes. The term itself, however, is due to Jerebič et al. [7] who is studied distance-balanced graphs in the framework of various kinds of graph products. Let $G$ be a graph, $e = uv \in E(G)$, $m_u^G(e)$ denotes the number of edges lying closer to the vertex $u$ than the vertex $v$, and $m_v^G(e)$ is defined analogously. Here is our key definition. We call a graph $G$ to be edge distance-balanced, if $m_a^G(e) = m_b^G(e)$ holds for each edge $e = ab \in E(G)$. As examples of edge distance-balanced graphs, we mention the complete graph $K_n$ on $n \geq 2$ vertices and the complete bipartite graph $K_{n,n}$ on $2n$ vertices.

Let $G$ and $H$ be two graphs. The corona product $GoH$ is obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$; and by joining each vertex of the $i^{th}$ copy of $H$ to the $i^{th}$ vertex of $G$,

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$i = 1, 2, \cdots, |V(G)|$ , see [10, 13]. The Cartesian product $G \times H$ of the graphs $G$ and $H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$.

The cluster $G\{H\}$ is obtained by taking one copy of $G$ and $|V(G)|$ copies of a rooted graph $H$, and by identifying the root of the $i^{th}$ copy of $H$ with the $i^{th}$ vertex of $G$, $i = 1, 2, ..., |V(G)|$ [13]. The lexicographic product $G = G[H]$ of graphs $G$ and $H$ with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G) \times V(H)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever ($u_1$ is adjacent to $u_2$) or ($u_1 = u_2$ and $v_1$ is adjacent to $v_2$), see [6, p. 22]. Suppose $G$ is a simple connected graph. Following Yan et al. [12], we define the graphs $S(G)$ and $R(G)$ as follows:

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of $G$. Equivalently, each edge of $G$ is replaced by a path of length 2.

(b) $R(G)$ is obtained from $G$ by adding a new vertex corresponding to each edge of $G$, then joining each new vertex to the end vertices of the corresponding edge.

A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree $k$ is called a $k-$regular graph or regular graph of degree $k$. A triangle-free graph is a graph containing no graph cycles of length three. Our other notations are standard and taken mainly form [2, 8, 9, 11].

## 2. **Main Results**

In this section we study the conditions under which the standard graph products produce an edge distance-balanced graph.

**Theorem 2.1.** Let $G$ and $H$ be edge and vertex distance-balanced graphs. Then $G \times H$ is edge distance-balanced graphs.

**Proof.** Consider the following partition of $E(G \times H)$:

$$A = \{(a, x)(b, y) \in E(G \times H) | ab \in E(G), x = y\}$$
$$B = \{(a, x)(b, y) \in E(G \times H) | a = b, xy \in E(H)\}.$$

We assume that $G$ and $H$ are edge and vertex distance-balanced graphs, and $e \in A$. Notice that

$$m_{(a,x)}^{(G \times H)}(e) = m_a^G(ab)|V(H)| + n_a^G(ab)|E(H)|,$$
$$m_{(b,y)}^{(G \times H)}(e) = m_b^G(ab)|V(H)| + n_b^G(ab)|E(H)|.$$

Since $G$ is edge and vertex distance-balanced, thus we have $n_a^G(ab) = n_b^G(ab)$ and $m_a^G(ab) = m_b^G(ab)$. Therefore, in this case we have $m_{(a,x)}^{(G \times H)}(e) = m_{(b,y)}^{(G \times H)}(e)$. In a similar way we can see that, for every edge $e$ of $B$, we have $m_{(a,x)}^{(G \times H)}(e) = m_{(b,y)}^{(G \times H)}(e)$. Then $G \times H$ is edge distance-balanced graphs.     $\square$

A graph $G$ is called nontrivial if $|V(G)| > 1$.

**Theorem 2.2.** The corona product of two arbitrary, nontrivial and connected graphs is not edge distance-balanced.

**Proof.** Let $G$ and $H$ be nontrivial connected graphs. Assume that $uv = e \in E(GoH)$ such that $u \in V(G)$ and $v \in V(H_i)$, where $H_i$, $1 \leq i \leq |V(G)|$ is the $i^{th}$ copy of $H$. Thus,

$$
\begin{aligned}
m_u^{GoH}(e) &= (|V(G)| - 1)|E(H)| + |E(G)| + |V(G)||V(H)| - 1 \\
&+ |\{f \in E(H)|d_H(v, f) \geqslant 2\}|
\end{aligned}
$$

and $m_v^{GoH}(e) = deg_H(v)$. Therefore, $m_u^{GoH}(e) \neq m_v^{GoH}(e)$ and so $GoH$ is not edge distance-balanced.
□

**Theorem 2.3.** The cluster of two arbitrary, nontrivial and connected graphs is not edge distance-balanced.

**Proof.** Let $G$ and $H$ be nontrivial connected graphs. Assume that $e = uv \in E(G\{H\})$ such that $u$ is the root of the $i^{th}$ copy of $H$ and $u \neq v \in V(H_i)$. Thus,

$$
m_u^{G\{H\}}(e) = |E(H)|(|V(G)| - 1) + |E(G)| + m_u^H(e)
$$

and $m_v^{G\{H\}}(e) = m_v^H(e)$. Therefore, $m_u^{G\{H\}}(e) \neq m_v^{G\{H\}}(e)$ and so $G\{H\}$ is not edge distance-balanced.
□

**Theorem 2.4.** Let $G$ and $H$ be connected graphs. Then $G[H]$ is edge distance-balanced if $G$ is nontrivial, edge and vertex distance-balanced and $H$ is triangle-free and regular.

**Proof.** Suppose that $G$ is nontrivial, edge and vertex distance-balanced and $H$ is triangle-free and regular. Consider the following partition of $E(G[H])$.

$$
\begin{aligned}
A &= \{(a, x)(b, y) \in E(G[H])|ab \in E(G) and\ x, y \in V(H)\}, \\
B &= \{(a, x)(b, y) \in E(G[H])|a = b \in V(G), xy \in E(H)\}.
\end{aligned}
$$

Let $e = (a, x)(b, y) \in A$. According to the definition of the lexicographic product, it is clear that

$$
\begin{aligned}
m_{(a,x)}^{G[H]}(e) - m_{(b,y)}^{G[H]}(e) &= (m_a^G(ab) - m_b^G(ab))|V(H)|^2 \\
&+ (n_a^G(ab) - n_b^G(ab))|E(H)| \\
&+ |\{f \in E(H) \mid d_H(y, f) \geqslant 2\}| \\
&- |\{f \in E(H) \mid d_H(x, f) \geqslant 2\}|.
\end{aligned}
$$

Since $G$ is edge and vertex distance-balanced, then $m_a^G(ab) = m_b^G(ab)$ and $n_a^G(ab) = n_b^G(ab)$ and since $H$ is triangle-free and regular one can see that $|\{f \in E(H) \mid d_H(y, f) \geqslant 2\}| = |\{f \in E(H) \mid d_H(x, f) \geqslant 2\}|$. It follows that $m_{(a,x)}^{G[H]}(e) = m_{(b,y)}^{G[H]}(e)$. We now assume that $e = (a, x)(b, y) \in B$. It follows from the edge structure of $G[H]$ that $m_{(a,x)}^{G[H]}(e) = m_{(b,y)}^{G[H]}(e)$, if $H$ is triangle-free and regular. Therefore, for each $e = (a, x)(b, y) \in E(G[H])$, we have $m_{(a,x)}^{G[H]}(e) = m_{(b,y)}^{G[H]}(e)$ and thus $G[H]$ is edge distance-balanced.
□

**Theorem 2.5.** Let $G$ be a nontrivial connected graph. Then $R(G)$ is edge distance-balanced if and only if $G$ is a path with $|V(G)| = 2$.

**Proof.** Let $G$ be a path with $|V(G)| = 2$. Then it is clear that $R(G)$ is edge distance-balanced. Conversely, we assume that $R(G)$ is an edge distance-balanced graph, where $G$ be a graph with $|V(G)| > 2$. Then, there is at least an edge $uv = e$ of $G$ such that $u$ is the end vertex of $e$ with $deg_G(u) > 1$ or $v$ is the end vertex of $e$ with $deg_G(v) > 1$. Without loss of generality, we my assume that $u$ is the end vertex of $e$ with $deg_G(u) > 1$. Also, we assume that $x$ is a new vertex corresponding to edge $e$ of $G$. Then, $m_x^{R(G)}(xu) = 1$ and $m_u^{R(G)}(xu) > 1$. Thus $m_x^{R(G)}(xu) \neq m_u^{R(G)}(xu)$. Therefore $R(G), |V(G)| > 2$, is not an edge distance-balanced graph and hence $G$ is a path with $|V(G)| = 2$. $\square$

**Theorem 2.6.** Let $G$ be a nontrivial connected graph with a pendant. Then $S(G)$ is not edge distance-balanced.

**Proof.** Suppose $x$ is a pendent vertex and $u$ is the new vertex such that $u$ and $x$ are adjacent in $S(G)$. Then $m_x^{S(G)}(ux) = 0$ and $m_u^{S(G)}(ux) \geq 1$, proving the result. $\square$

Suppose $G$ and $H$ are graphs with disjoint vertex sets. Following Doslic [3], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of $G$ and $H$ by vertices $y$ and $z$, $(G \cdot H)(y; z)$, is defined by identifying the vertices $y$ and $z$ in the union of $G$ and $H$. Similarly, a link of $G$ and $H$ by vertices $y$ and $z$ is defined as the graph $(G \sim H)(y; z)$ obtained by joining $y$ and $z$ by an edge in the union of these graphs.

**Theorem 2.7.** Suppose $G$ and $H$ are rooted graphs with respect to the rooted vertices of $a$ and $b$, respectively. The graph $(G \cdot H)(a; b)$ is edge distance-balanced if and only if for each $e = uv \in E(G)$ and $f = xy \in E(H)$ the following conditions are satisfied:

$$(2.1) \qquad m_u^G(e) - m_v^G(e) = \begin{cases} |E(H)| & if \ d(v,a) < d(u,a) \\ 0 & if \ d(v,a) = d(u,a) \end{cases},$$

$$(2.2) \qquad m_x^H(f) - m_y^H(f) = \begin{cases} |E(G)| & if \ d(y,b) < d(x,b) \\ 0 & if \ d(y,b) = d(x,b) \end{cases}.$$

**Proof.** In the graph $(G \cdot H)(a; b)$, we put $r = a = b$. We partition edges of $(G \cdot H)(a; b)$ into the following two subsets:

$$A = \{e = uv \in E(G.H) | d(v,r) < d(u,r)\},$$
$$B = \{e = uv \in E(G.H) | d(v,r) = d(u,r)\}.$$

We first assume that $(G \cdot H)(a; b)$ is edge distance-balanced. Suppose $e = uv$ is an arbitrary edge of $G$. Then $e \in A$ or $e \in B$ and not both. If $e \in A$ then by the hypothesis $m_u^{G.H}(e) = m_v^{G.H}(e)$. On the other hand by the definition of splice, $m_v^{G.H}(e) = m_v^G(e) + |E(H)|$ and $m_u^{G.H}(e) = m_u^G(e)$. Thus, $m_u^G(e) = m_v^G(e) + |E(H)|$ and so $m_u^G(e) - m_v^G(e) = |E(H)|$. Next we assume that $e \in B$. Again by the hypothesis $m_u^{G.H}(e) = m_v^{G.H}(e)$ and by definition of splice we have, $m_v^{G.H}(e) = m_v^G(e)$ and

$m_u^{G.H}(e) = m_u^G(e)$. This implies that $m_u^G(e) = m_v^G(e)$. Therefore, the equation (1) is satisfied. In a similar way we can see that, for every edge $e$ of $H$ the equation (2) is satisfied.

Conversely, suppose that Eqs. (1,2) are satisfied and $e = uv \in A$ is arbitrary. Then $e \in E(G)$ or $e \in E(H)$ and not both. If $e \in E(G)$ then $m_u^{G.H}(e) = m_u^G(e)$ and $m_v^{G.H}(e) = m_v^G(e) + |E(H)|$. This implies that $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^G(e) - (m_v^G(e) + |E(H)|)$. Since $m_u^G(e) - m_v^G(e) = |E(H)|$, $m_u^{G.H}(e) - m_v^{G.H}(e) = 0$, as desired. Suppose that $e \in E(H)$. Then $m_u^{G.H}(e) = m_u^H(e)$ and $m_v^{G.H}(e) = m_v^H(e) + |E(G)|$, so $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^H(e) - (m_v^H(e) + |E(G)|)$. But by the hypothesis, $m_u^H(e) - m_v^H(e) = |E(G)|$, so $m_u^{G.H}(e) - m_v^{G.H}(e) = 0$. We now assume that $e \in B$ is arbitrary. If $e \in E(G)$ then by $m_u^{G.H}(e) = m_u^G(e)$ and $m_v^{G.H}(e) = m_v^G(e)$ we have $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^G(e) - m_v^G(e) = 0$. If $e \in E(H)$ then by $m_u^{G.H}(e) = m_u^H(e)$ and $m_v^{G.H}(e) = m_v^H(e)$ we have $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^H(e) - m_v^H(e) = 0$. Therefore, for every edge $e = uv \in B$, $m_u^{G.H}(e) = m_v^{G.H}(e)$ and for every edge $e = uv \in E(G \cdot H)$, $m_u^{G.H}(e) = m_v^{G.H}(e)$. This completes the proof. $\square$

**Corollary 2.8.** Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then

$$(G_1 \cdot G_2 \cdot \cdots \cdot G_n)(r_1; r_2; \cdots ; r_n)$$

is edge distance-balanced if and only if for each $i$, $1 \leq i \leq n$, and for each $e = uv \in E(G_i)$ the following system of equations are satisfied:

$$m_u^{G_i}(e) - m_v^{G_i}(e) = \begin{cases} \sum_{j=1, j \neq i}^n |E(G_j)| & if \ d(v, r_i) < d(u, r_i) \\ 0 & if \ d(v, r_i) = d(u, r_i) \end{cases}.$$

**Proof.** Induct on $n$. $\square$

**Theorem 2.9.** Suppose $G$ and $H$ are rooted graphs with respect to the rooted vertices of $a$ and $b$, respectively. The graph $(G \sim H)(a; b)$ is edge distance-balanced if and only if $|E(G)| = |E(H)|$ and for each $e = uv \in E(G)$ and $f = xy \in E(H)$ the following conditions are satisfied:

$$m_u^G(e) - m_v^G(e) = \begin{cases} |E(H)| + 1 & if \ d(v, a) < d(u, a) \\ 0 & if \ d(v, a) = d(u, a) \end{cases},$$

$$m_x^H(f) - m_y^H(f) = \begin{cases} |E(G)| + 1 & if \ d(y, b) < d(x, b) \\ 0 & if \ d(y, b) = d(x, b) \end{cases}.$$

**Proof.** The proof is similar to Theorem 2.7 and so omitted. $\square$

**Corollary 2.10.** Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then $(G_1 \sim G_2 \sim \cdots \sim G_n)(r_1; r_2; \cdots ; r_n)$ is edge distance-balanced if and only if for each $i$, $1 \leq i \leq n$, $|E(G_i)| = |E(G_1)|$ and for each $e = uv \in E(G_i)$ the following system of equations are satisfied:

$$m_u^{G_i}(e) - m_v^{G_i}(e) = \begin{cases} \sum_{j=1, j \neq i}^n |E(G_j)| + \binom{n}{2} & if \ d(v, r_i) < d(u, r_i) \\ 0 & if \ d(v, r_i) = d(u, r_i) \end{cases}.$$

**Proof.** Induct on $n$. $\square$

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