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k-TUPLE TOTAL DOMINATION AND MYCIELESKIAN GRAPHS

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Abstract. Let k be a positive integer. A subset S of $V(G)$ in a graph G is a k-tuple total dominating set of G if every vertex of G has at least k neighbors in S. The k-tuple total domination number $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a k-tuple total dominating set of G. In this paper for a given graph G with minimum degree at least k, we find some sharp lower and upper bounds on the k-tuple total domination number of the m-Mycieleskian graph $\mu_m(G)$ of G in terms on k and $\gamma_{\times k,t}(G)$. Specially we give the sharp bounds $\gamma_{\times k,t}(G)+1$ and $\gamma_{\times k,t}(G)+k$ for $\gamma_{\times k,t}(\mu_1(G))$, and characterize graphs with $\gamma_{\times k,t}(\mu_1(G)) = \gamma_{\times k,t}(G)+1$.

1. Introduction

In this paper, $G = (V, E)$ is a simple graph with the *vertex set* V and the *edge set* E. The *order* $\mid V \mid$ of G is denoted by $n = n(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the *degree* of v is $\deg_G(v) =\mid N_G(v) \mid$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write K_n and C_n for the *complete graph* and the *cycle* of order n, respectively, while $G[S]$ and $K_{n_1,n_2,\ldots,n_p}$ denote the *subgraph induced* on G by a vertex set S, and the complete p-partite graph, respectively.

Let $S \subseteq V$ and let k be a positive integer. For each k-element subset $S' \subseteq S$ the *(S,k)-private neighborhood* $\mathrm{pn}_k(S', S)$ of S' is the set of all vertices $v \in V$ such that $N(v) \cap S = S'$. Further, the *open k-boundary* $\mathrm{OB}_k(S)$ of S is the set of all vertices v in G such that $v \in \mathrm{pn}_k(S', S)$ for some k-element subset $S' \subseteq S$ [4]. Obviously, $\mathrm{OB}_k(S) = \bigcup_{S'} \mathrm{pn}_k(S', S)$, where S' is a k-element subset of S.

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As we will see, the generalized Mycieleskian graphs, which are also called *cones over graphs* [7], are natural generalization of Mycieleski graphs. If $V(G) = V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and $E(G) = E_0$, then for any integer $m \geq 1$ the *m-Mycieleskian* $\mu_m(G)$ of G is the graph with vertex set $V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$, where $V^i = \{v_j^i \mid v_j^0 \in V^0\}$ is the i-th distinct copy of V^0, for $i = 1, 2, \ldots, m$, and edge set $E_0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} \mid v_j^0 v_{j'}^0 \in E_0\}\right) \cup \{v_j^m u \mid v_j^m \in V^m\}$. The 1-*Mycieleskian* $\mu_1(G)$ of G is the well-studied *Mycieleskian* of G, and denoted simply by $\mu(G)$ or $M(G)$.

For positive integer k, the *k-join* of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H. We denote the k-join of G to H by $G \circ_k H$.

Domination in graphs is now well-studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2, 3].

In [4], Henning and Kazemi introduced the k-tuple total domination number of a graph. Let k be a positive integer. A subset S of V is a *k-tuple total dominating set* of G, abbreviated kTDS, if for every vertex $v \in V$, $\mid N(v) \cap S \mid \geq k$, that is, S is a kTDS of G if every vertex of V has at least k neighbors in S. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a kTDS of G. We remark that a 1-tuple total domination is the well-studied *total domination number*. Thus, $\gamma_t(G) = \gamma_{\times 1,t}(G)$. For a graph to have a k-tuple total dominating set, its minimum degree is at least k. Since every $(k+1)$-tuple total dominating set is also a k-tuple total dominating set, we note that $\gamma_{\times k,t}(G) \leq \gamma_{\times (k+1),t}(G)$ for all graphs with minimum degree at least $k+1$. A kTDS in a graph G is a *minimal* kTDS if no proper subset of it is a kTDS in G. A kTDS of cardinality $\gamma_{\times k,t}(G)$ is called a $\gamma_{\times k,t}(G)$-*set*. A 2-tuple total dominating set is called a *double total dominating set*, abbreviated DTDS, and the 2-tuple total domination number is called the *double total domination number*. The redundancy involved in k-tuple total domination makes it useful in many applications. The references [5, 6] give more information about the k-tuple total domination number of a graph.

In this paper, we study the k-tuple total domination number of the m-Mycieleskian graph of a graph G. We prove that for every positive integers m and k and every graph G with $\delta(G) \geq k$, if $m - 1 \cong r \pmod 4$, where $0 \leq r \leq 3$, and $r' \cong r + 1 \pmod 2$, then

$$\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(\mu_m(G)) \leq \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + kr' & \text{if } r = 0, 3, \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) + kr' & \text{otherwise.} \end{cases}$$

Hence $\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(M(G)) \leq \gamma_{\times k,t}(G) + k$. We also prove that the bounds $\gamma_{\times k,t}(G) + 1$ and $\gamma_{\times k,t}(G) + k$ are sharp and characterize graphs with $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$.

Through of this paper, k is a positive integer. The next results are useful for our investigations.

Proposition 1.1. (**Henning, Kazemi** [4] **2010**) *Let G be a graph of order n with $\delta(G) \geq k \geq 1$, and let S be a kTDS in G. Then*

1. $k + 1 \leq \gamma_{\times k,t}(G) \leq n$,
2. *for every spanning subgraph H of G,* $\gamma_{\times k,t}(G) \leq \gamma_{\times k,t}(H)$,
3. *for every vertex v of degree k,* $N_G(v) \subseteq S$.

Proposition 1.2. (**Henning, Kazemi** [4] **2010**) *Let G be a graph of order n with $\delta(G) \geq k \geq 1$, and let S be a kTDS in G. Then S is a minimal kTDS of G if and only if for each vertex $v \in S$, there exists a k-element subset $S_v \subseteq S$ such that $v \in S_v$ and $|\mathrm{pn}_k(S_v, S)| \geq 1$.*

Proposition 1.3. (**Henning, Kazemi** [4] **2010**) *Let G be a graph with $\delta(G) \geq k \geq 1$. Then, $\gamma_{\times k,t}(G) = k+1$ if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph F.*

Proposition 1.4. (**Henning, Kazemi** [5] **2010**) *Let G be a graph of order n with $\delta(G) \geq k \geq 1$. Then $\gamma_{\times k,t}(G) \geq \lceil kn/\Delta(G) \rceil$.*

2. m-mycieleskian graphs

In the next theorem we give a lower bound and an upper bound on the k-tuple domination number of the m-Mycieleskian graph $\mu_m(G)$ in terms k and the k-tuple domination number of G. First we state the following lemma which has an easy proof that is left to the reader.

Lemma 2.1. *Let G be a graph with $\delta(G) \geq k \geq 1$. Let $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$. If $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G)$, then for every $\gamma_{\times k,t}(\mu_m(G))$-set S, $u \notin S$, and so $m = 1$.*

Theorem 2.2. *Let m and k be two positive integers, and let G be a graph with $\delta(G) \geq k \geq 1$. Then*

$$\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(\mu_m(G)) \leq \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) & \text{if } m \cong 0 \ (mod\ 4), \\ (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + k & \text{if } m \cong 1 \ (mod\ 4), \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) & \text{if } m \cong 2 \ (mod\ 4), \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) + k & \text{if } m \cong 3 \ (mod\ 4). \end{cases}$$

Proof. Let $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$. Since G is an induced subgraph of $\mu_m(G)$, $\gamma_{\times k,t}(G) \leq \gamma_{\times k,t}(\mu_m(G))$. If $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G)$, then Lemma 2.1 implies $m = 1$ and for every $\gamma_{\times k,t}(M(G))$-set S, $u \notin S$. Since every vertex of V^1 is adjacent to at least k vertices of $S \cap V^0$, we conclude that every vertex of V^0 is adjacent to at least k vertices of $S \cap V^0$. Hence

$$\gamma_{\times k,t}(G) \leq\mid S \cap V^0 \mid=\mid S \mid - \mid S \cap V^1 \mid\leq \gamma_{\times k,t}(G) - k < \gamma_{\times k,t}(G),$$

a contradiction. Therefore $\gamma_{\times k,t}(\mu_m(G)) \geq \gamma_{\times k,t}(G) + 1$.

Now we prove the other inequality. For an arbitrary $\gamma_{\times k,t}(G)$-set S, let $S^i = \{v^i \mid v \in S\} \subseteq V^i$ be the i-th distinct copy of S when $0 \leq i \leq m$. Let also S_k be an arbitrary subset of V^m of cardinality k. We continue our proof in the following four cases.

Case 0. $m \cong 0$ (mod 4).
The set $S' = S^0 \cup (\bigcup_{t=1}^{\lfloor (m-1)/4 \rfloor} (S^{4t-1} \cup S^{4t})) \cup (S^{m-1} \cup S^m)$ is a kTDS of $\mu_m(G)$ of cardinality

$$(3 + 2\lfloor (m-1)/4 \rfloor)\gamma_{\times k,t}(G) = (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G).$$

Case 1. $m \cong 1$ (mod 4).
The set $S' = S^0 \cup (\bigcup_{t=1}^{\lfloor (m-1)/4 \rfloor} (S^{4t-1} \cup S^{4t})) \cup S_k$ is a kTDS of $\mu_m(G)$ of cardinality

$$(1 + 2\lfloor (m-1)/4 \rfloor)\gamma_{\times k,t}(G) + k = (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + k.$$

Case 2. $m \cong 2 \pmod 4$.
The set $S' = (\bigcup_{t=1}^{\lfloor (m-1)/4 \rfloor}(S^{4t-3} \cup S^{4t-2})) \cup (S^{m-1} \cup S^m)$ is a kTDS of $\mu_m(G)$ of cardinality

$$(2 + 2\lfloor (m-1)/4 \rfloor)\gamma_{\times k,t}(G) = 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G).$$

Case 3. $m \cong 3 \pmod 4$.
The set $S' = (\bigcup_{t=1}^{\lceil (m-1)/4 \rceil}(S^{4t-3} \cup S^{4t-2})) \cup S_k$ is a kTDS of $\mu_m(G)$ of cardinality

$$2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) + k.$$

Therefore we have proved

$$\gamma_{\times k,t}(\mu_m(G)) \leq \mid S' \mid = \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) & \text{if } m \cong 0 \pmod 4, \\ (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + k & \text{if } m \cong 1 \pmod 4, \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) & \text{if } m \cong 2 \pmod 4, \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) + k & \text{if } m \cong 3 \pmod 4, \end{cases}$$

and this completes our proof. □

Corollary 2.3. *Let m be a positive integer, and let G be a graph without isolated vertex. Then*

$$\gamma_t(G) + 1 \leq \gamma_t(\mu_m(G)) \leq \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_t(G) & \text{if } m \cong 0 \pmod 4, \\ (1 + 2\lceil (m-1)/4 \rceil)\gamma_t(G) + 1 & \text{if } m \cong 1 \pmod 4, \\ 2\lceil (m-1)/4 \rceil \gamma_t(G) & \text{if } m \cong 2 \pmod 4, \\ 2\lceil (m-1)/4 \rceil \gamma_t(G) + 1 & \text{if } m \cong 3 \pmod 4. \end{cases}$$

3. Mycieleskian graphs

Theorem 2.2 implies the next two theorems when $m = 1$.

Theorem 3.1. *If G is a graph with no isolated vertices, then $\gamma_t(M(G)) = \gamma_t(G) + 1$.*

Theorem 3.2. *If G is a graph with $\delta(G) \geq k \geq 2$, then $\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(M(G)) \leq \gamma_{\times k,t}(G) + k$.*

In the next theorem we give other lower bound for $\gamma_{\times k,t}(M(G))$.

Theorem 3.3. *If G is a graph with $\delta(G) \geq k \geq 2$, then*

$$\gamma_{\times k,t}(M(G)) \geq \min\{\gamma_{\times k,t}(G) + k, \gamma_{\times (k-1),t}(G) + k + 1\}.$$

Proof. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$ and let S be an arbitrary kTDS of $M(G)$. Then $\mid S \cap V^1 \mid \geq k$. If $u \in S$, then $\mid S \cap V^0 \mid \geq \gamma_{\times (k-1),t}(G)$. Since $S \cap V^0$ must be a (k−1)TDS of V^1. If $u \notin S$, then $\mid S \cap V^0 \mid \geq \gamma_{\times k,t}(G)$. Since $S \cap V^0$ must be a kTDS of V^1. Therefore

$$\gamma_{\times k,t}(M(G)) \geq \min\{\gamma_{\times k,t}(G) + k, \gamma_{\times (k-1),t}(G) + k + 1\}.$$

□

As an immediately result of Theorems 3.2 and 3.3 we have the following two corollaries.

Corollary 3.4. *Let G be a graph with $\delta(G) \geq k \geq 2$. If $\gamma_{\times k,t}(G) = \gamma_{\times (k-1),t}(G) + 1$, then*

$$\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k.$$

Corollary 3.5. *If $n \geq k+1 \geq 3$, then $\gamma_{\times k,t}(M(K_n)) = \gamma_{\times k,t}(K_n) + k = 2k+1$.*

Also the following two results show that the upper bound $\gamma_{\times k,t}(G) + k$ in Theorem 3.2 is sharp for some of the complete multipartite graphs, the complete graph K_{k+1} and the k-join $F \circ_k K_{k+1}$, for every graph F.

Proposition 3.6. *Let $G = K_{n_1,\ldots n_p}$ be a complete p-partite graph. If $p \geq k+1 \geq 3$, then*

$$\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k.$$

Proof. We suppose that $V(G)$ has partition $X_1 \cup X_2 \cup \ldots \cup X_p$ such that $\mid X_j \mid = n_j$ for $j = 1, 2, \ldots, p$. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = X_1^i \cup X_2^i \cup \ldots \cup X_p^i$ and $X_j^i = \{v_j^i \mid v_j \in X_j\}$, for $i = 0, 1$ and $j = 1, 2, \ldots, p$. Let also S be an arbitrary kTDS of $M(G)$. Obviously $\mid S \cap V^1 \mid \geq k$, and, without less of generality, we may assume that $\mid S \cap V^1 \mid = k$. Let $S \cap V^1$ be a set which contains only one vertex of every X_i^1, for $1 \leq i \leq k$. Thus each vertex of $V^0 - \{v_i^0 \mid v_i^1 \in S \cap V^1\}$ is adjacent to all vertices in $S \cap V^1$. Since each vertex of $S \cap V^1$ must be adjacent to at least k vertices of S, we have $\mid S \cap (V^0 \cup \{u\}) \mid \geq k$. The assumptions $k \geq 2$ and $\mid S \cap V^0 \mid \geq k-1$ imply $S \cap V^0 \neq \emptyset$. We see that there exists an unique index $1 \leq j \leq k$ such that each vertex of X_j^1 is adjacent to $k-1$ vertices of $S \cap (V^0 \cup \{u\})$. Hence $\mid S \cap (V^0 \cup \{u\}) \mid \geq k+1$, and so

$$\begin{aligned} \gamma_{\times k,t}(M(G)) &= \min \{\mid S \mid \ : \text{ S is a kTDS of } M(G)\} \\ &\geq 2k+1 \\ &= \gamma_{\times k,t}(G) + k. \end{aligned}$$

Now Theorem 3.2 implies $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k$. □

Theorem 3.7. *Let G be a graph of order $n \geq k+1$ with $\delta(G) \geq k \geq 2$. If G is the k-join $F \circ_k K_{k+1}$, for some graph F, then*

$$\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k.$$

Proof. Let G be the k-join $F \circ_k K_{k+1}$, for some graph F. Then Proposition 1.3 implies $\gamma_{\times k,t}(G) = k+1$. Since G is a spanning subgraph of K_n, and hence $M(G)$ is a spanning subgraph of $M(K_n)$, we have

$$\begin{aligned} \gamma_{\times k,t}(M(G)) &\geq \gamma_{\times k,t}(M(K_n)) \\ &= \gamma_{\times k,t}(K_n) + k \\ &= 2k+1 \\ &= \gamma_{\times k,t}(G) + k. \end{aligned}$$

Now Theorem 3.2 implies $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k = 2k+1$. □

Theorem 3.1 shows that $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$, where G is a graph with no isolated vertices and $k = 1$. Here, we give an equivalent condition for $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$, when $k \geq 2$. We recall that a kTDS S is a minimal kTDS if and only if for each vertex $v \in S$, there exists a k-element subset $S_v \subseteq S$ such that $v \in S_v$ and $|\text{pn}_k(S_v, S)| \geq 1$.

Theorem 3.8. *Let G be a graph with $\delta(G) \geq k \geq 1$. Then $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$ if and only if $k = 1$ or $k \geq 2$ and G has a $\gamma_{\times k,t}$-set S with a k-subset $S' \subseteq S$ such that $S - S'$ is a $(k-1)$TDS of G and for every vertex v, $\mid S_v \cap S' \mid \leq 1$.*

Proof. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = \{v_j^i \mid 1 \leq j \leq n\}$ for $i = 0, 1$. If $k = 1$, then $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$, by Theorem 3.1. Now let $k \geq 2$. Let S be a $\gamma_{\times k,t}(G)$-set which contains a k-subset $S' \subseteq S$ with this conditions that $S - S'$ is a $(k-1)$TDS of G and for every vertex v, $\mid S_v \cap S' \mid \leq 1$. Since

$$D = \{v_j^0 \mid v_j \in S - S'\} \cup \{v_j^1 \mid v_j \in S'\} \cup \{u\}$$

is a kTDS of $M(G)$ of cardinality $\gamma_{\times k,t}(G) + 1$, Theorem 3.2 implies $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$.

Conversely, let $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$ and let $k \geq 2$. If u belongs to a $\gamma_{\times k,t}$-set of $M(G)$, we have no thing to prove. Thus we assume that u belongs to no $\gamma_{\times k,t}(M(G))$-set D, and so $\mid D \cap V^1 \mid \geq k$ and $\mid D \cap V^0 \mid \geq \gamma_{\times k,t}(G)$. Therefore

$$\gamma_{\times k,t}(G) + 1 = \gamma_{\times k,t}(M(G)) = \mid D \mid \geq \gamma_{\times k,t}(G) + k \geq \gamma_{\times k,t}(G) + 2,$$

a contradiction. □

The next proposition gives graphs that satisfy in the condition of Theorem 3.8. First, we present the definition of the *Harary graph* [8].

Given $m \leq n$, place n vertices 1, 2, ..., n around a circle, equally spaced. If m is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $m/2$ vertices in each direction around the circle. If m is odd and n is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $(m-1)/2$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{m,n}$ is m-regular. When m and n are both odd, index the vertices by the integers modulo n. Construct $H_{m,n}$ from $H_{m-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$.

Proposition 3.9. *If G is a cycle of order at least 3 or the Harary graph $H_{2m,\ell m+1}$, where $\ell \geq 3$ and $m \geq 1$, then*

$$\gamma_{\times 2,t}(M(G)) = \gamma_{\times 2,t}(G) + 1.$$

Proof. Let $G = C_n$ be a cycle of order at least 3 with the vertex set $V(C_n) = V^0 = \{v_j \mid 1 \leq j \leq n\}$ and the edge set $E(C_n) = \{(v_j, v_{j+1}) \mid 1 \leq j \leq n\}$. Let also $V(M(G)) = V^0 \cup V^1 \cup \{u\}$. Proposition 1.1(3) implies $\gamma_{\times 2,t}(G) = n$. Since $S = (V^0 - \{v_1^0, v_n^0\}) \cup \{v_{n-1}^1, v_n^1, u\}$ is a DTDS of $M(G)$ of cardinality $\gamma_{\times 2,t}(C_n) + 1 = n + 1$, Theorem 2.2 implies $\gamma_{\times 2,t}(M(C_n)) = \gamma_{\times 2,t}(C_n) + 1 = n + 1$.

Now let $G = H_{2m,\ell m+1}$ and let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = \{v_j^i \mid 1 \leq j \leq \ell m + 1\}$ for $i = 0, 1$, $V^0 = V(G)$ and $\ell \geq 3$. Since $\{v_{jm+1}^0 \mid 0 \leq j \leq \lceil(\ell m + 1)/m\rceil - 1 = \ell\}$ is a DTDS of G, we have $\gamma_{\times 2,t}(G) = \lceil(\ell m + 1)/m\rceil = \ell + 1$, by Proposition 1.4. Since also $S = \{v_{im+1}^0 \mid 1 \leq i \leq \ell - 1\} \cup \{v_1^1, v_{\ell m+1}^1, u\}$ is a DTDS of $M(G)$ of cardinality $\ell + 2 = \gamma_{\times 2,t}(G) + 1$, Theorem 2.2 implies $\gamma_{\times 2,t}(M(G)) = \gamma_{\times 2,t}(G) + 1$. □

In the end of paper, the author states the following problem.

Problem: For integers $k, m \geq 1$, characterize graphs G with $\delta(G) \geq k$ satisfy $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G) + 1$ or

$$\gamma_{\times k,t}(\mu_m(G)) = \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) & \text{if } m \cong 0 \pmod 4, \\ (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + k & \text{if } m \cong 1 \pmod 4, \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) & \text{if } m \cong 2 \pmod 4, \\ 2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) + k & \text{if } m \cong 3 \pmod 4. \end{cases}$$

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