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k-TUPLE TOTAL DOMINATION AND MYCIELESKIAN GRAPHS

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ABSTRACT. Let k be a positive integer. A subset S of V(G) in a graph G is a k-tuple total dominating set of G if every vertex of G has at least k neighbors in S. The k-tuple total domination number $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a k-tuple total dominating set of G. In this paper for a given graph G with minimum degree at least k, we find some sharp lower and upper bounds on the k-tuple total domination number of the m-Mycieleskian graph $\mu_m(G)$ of G in terms on k and $\gamma_{\times k,t}(G)$. Specially we give the sharp bounds $\gamma_{\times k,t}(G) + 1$ and $\gamma_{\times k,t}(G) + k$ for $\gamma_{\times k,t}(\mu_1(G))$, and characterize graphs with $\gamma_{\times k,t}(\mu_1(G)) = \gamma_{\times k,t}(G) + 1$.

1. Introduction

In this paper, G = (V, E) is a simple graph with the vertex set V and the edge set E. The order |V| of G is denoted by n = n(G). The open neighborhood and the closed neighborhood of a vertex $v \in V$ are $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the degree of v is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write K_n and C_n for the complete graph and the cycle of order n, respectively, while G[S] and K_{n_1,n_2,\dots,n_p} denote the subgraph induced on G by a vertex set S, and the complete p-partite graph, respectively.

Let $S \subseteq V$ and let k be a positive integer. For each k-element subset $S' \subseteq S$ the (S, k)-private neighborhood $\operatorname{pn}_k(S', S)$ of S' is the set of all vertices $v \in V$ such that $N(v) \cap S = S'$. Further, the open k-boundary $\operatorname{OB}_k(S)$ of S is the set of all vertices v in G such that $v \in \operatorname{pn}_k(S', S)$ for some k-element subset $S' \subseteq S$ [4]. Obviously, $\operatorname{OB}_k(S) = \bigcup_{S'} \operatorname{pn}_k(S', S)$, where S' is a k-element subset of S.

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As we will see, the generalized Mycieleskian graphs, which are also called *cones over graphs* [7], are natural generalization of Mycieleski graphs. If $V(G) = V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and $E(G) = E_0$, then for any integer $m \ge 1$ the *m*-Mycieleskian $\mu_m(G)$ of *G* is the graph with vertex set $V^0 \cup V^1 \cup V^2 \cup$ $\cdots \cup V^m \cup \{u\}$, where $V^i = \{v_j^i \mid v_j^0 \in V^0\}$ is the *i*-th distinct copy of V^0 , for $i = 1, 2, \ldots, m$, and edge set $E_0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} \mid v_j^0 v_{j'}^0 \in E_0\}\right) \cup \{v_j^m u \mid v_j^m \in V^m\}$. The 1-Mycieleskian $\mu_1(G)$ of *G* is the well-studied Mycieleskian of *G*, and denoted simply by $\mu(G)$ or M(G).

For positive integer k, the k-join of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H. We denote the k-join of G to H by $G \circ_k H$.

Domination in graphs is now well-studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2, 3].

In [4], Henning and Kazemi introduced the k-tuple total domination number of a graph. Let k be a positive integer. A subset S of V is a k-tuple total dominating set of G, abbreviated kTDS, if for every vertex $v \in V$, $|N(v) \cap S| \ge k$, that is, S is a kTDS of G if every vertex of V has at least k neighbors in S. The k-tuple total domination number $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a kTDS of G. We remark that a 1-tuple total domination is the well-studied total domination number. Thus, $\gamma_t(G) = \gamma_{\times 1,t}(G)$. For a graph to have a k-tuple total dominating set, its minimum degree is at least k. Since every (k + 1)-tuple total dominating set is also a k-tuple total dominating set, we note that $\gamma_{\times k,t}(G) \le \gamma_{\times (k+1),t}(G)$ for all graphs with minimum degree at least k + 1. A kTDS in a graph G is a minimal kTDS if no proper subset of it is a kTDS in G. A kTDS of cardinality $\gamma_{\times k,t}(G)$ is called a $\gamma_{\times k,t}(G)$ -set. A 2-tuple total dominating set is called a double total dominating set, abbreviated DTDS, and the 2-tuple total domination number is called the double total domination number. The redundancy involved in k-tuple total domination makes it useful in many applications. The references [5, 6] give more information about the k-tuple total domination number of a graph.

In this paper, we study the k-tuple total domination number of the m-Mycieleskian graph of a graph G. We prove that for every positive integers m and k and every graph G with $\delta(G) \ge k$, if $m-1 \cong r \pmod{4}$, where $0 \le r \le 3$, and $r' \cong r+1 \pmod{2}$, then

$$\gamma_{\times k,t}(G) + 1 \le \gamma_{\times k,t}(\mu_m(G)) \le \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + kr' & \text{if } r = 0,3, \\ 2\lceil (m-1)/4 \rceil\gamma_{\times k,t}(G) + kr' & \text{otherwise.} \end{cases}$$

Hence $\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(M(G)) \leq \gamma_{\times k,t}(G) + k$. We also prove that the bounds $\gamma_{\times k,t}(G) + 1$ and $\gamma_{\times k,t}(G) + k$ are sharp and characterize graphs with $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$.

Through of this paper, k is a positive integer. The next results are useful for our investigations.

Proposition 1.1. (Henning, Kazemi [4] 2010) Let G be a graph of order n with $\delta(G) \ge k \ge 1$, and let S be a kTDS in G. Then

- 1. $k+1 \leq \gamma_{\times k,t}(G) \leq n$,
- 2. for every spanning subgraph H of G, $\gamma_{\times k,t}(G) \leq \gamma_{\times k,t}(H)$,
- 3. for every vertex v of degree k, $N_G(v) \subseteq S$.

Proposition 1.2. (Henning, Kazemi [4] 2010) Let G be a graph of order n with $\delta(G) \ge k \ge 1$, and let S be a kTDS in G. Then S is a minimal kTDS of G if and only if for each vertex $v \in S$, there exists a k-element subset $S_v \subseteq S$ such that $v \in S_v$ and $|pn_k(S_v, S)| \ge 1$.

Proposition 1.3. (Henning, Kazemi [4] 2010) Let G be a graph with $\delta(G) \ge k \ge 1$. Then, $\gamma_{\times k,t}(G) = k + 1$ if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph F.

Proposition 1.4. (Henning, Kazemi [5] 2010) Let G be a graph of order n with $\delta(G) \ge k \ge 1$. Then $\gamma_{\times k,t}(G) \ge \lceil kn/\Delta(G) \rceil$.

2. *m*-mycieleskian graphs

In the next theorem we give a lower bound and an upper bound on the k-tuple domination number of the m-Mycieleskian graph $\mu_m(G)$ in terms k and the k-tuple domination number of G. First we state the following lemma which has an easy proof that is left to the reader.

Lemma 2.1. Let G be a graph with $\delta(G) \ge k \ge 1$. Let $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$. If $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G)$, then for every $\gamma_{\times k,t}(\mu_m(G))$ -set S, $u \notin S$, and so m = 1.

Theorem 2.2. Let m and k be two positive integers, and let G be a graph with $\delta(G) \ge k \ge 1$. Then

$$\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(\mu_m(G)) \leq \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) & \text{if } m \cong 0 \pmod{4}, \\ (1 + 2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + k & \text{if } m \cong 1 \pmod{4}, \\ 2\lceil (m-1)/4 \rceil\gamma_{\times k,t}(G) & \text{if } m \cong 2 \pmod{4}, \\ 2\lceil (m-1)/4 \rceil\gamma_{\times k,t}(G) + k & \text{if } m \cong 3 \pmod{4}. \end{cases}$$

Proof. Let $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$. Since G is an induced subgraph of $\mu_m(G)$, $\gamma_{\times k,t}(G) \leq \gamma_{\times k,t}(\mu_m(G))$. If $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G)$, then Lemma 2.1 implies m = 1 and for every $\gamma_{\times k,t}(M(G))$ -set S, $u \notin S$. Since every vertex of V^1 is adjacent to at least k vertices of $S \cap V^0$, we conclude that every vertex of V^0 is adjacent to at least k vertices of $S \cap V^0$. Hence

$$\gamma_{\times k,t}(G) \leq |S \cap V^0| = |S| - |S \cap V^1| \leq \gamma_{\times k,t}(G) - k < \gamma_{\times k,t}(G)$$

a contradiction. Therefore $\gamma_{\times k,t}(\mu_m(G)) \ge \gamma_{\times k,t}(G) + 1$.

Now we prove the other inequality. For an arbitrary $\gamma_{\times k,t}(G)$ -set S, let $S^i = \{v^i \mid v \in S\} \subseteq V^i$ be the *i*-th distinct copy of S when $0 \leq i \leq m$. Let also S_k be an arbitrary subset of V^m of cardinality k. We continue our proof in the following four cases.

Case 0. $m \cong 0 \pmod{4}$.

The set
$$S' = S^0 \cup (\bigcup_{t=1}^{\lfloor (m-1)/4 \rfloor} (S^{4t-1} \cup S^{4t})) \cup (S^{m-1} \cup S^m)$$
 is a kTDS of $\mu_m(G)$ of cardinality
 $(3+2|(m-1)/4|)\gamma_{\times k,t}(G) = (1+2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G).$

Case 1. $m \cong 1 \pmod{4}$.

The set $S' = S^0 \cup (\bigcup_{t=1}^{\lfloor (m-1)/4 \rfloor} (S^{4t-1} \cup S^{4t})) \cup S_k$ is a kTDS of $\mu_m(G)$ of cardinality $(1+2\lfloor (m-1)/4 \rfloor)\gamma_{\times k,t}(G) + k = (1+2\lfloor (m-1)/4 \rfloor)\gamma_{\times k,t}(G) + k.$ 9

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Case 2. $m \cong 2 \pmod{4}$. The set $S' = (\bigcup_{t=1}^{\lfloor (m-1)/4 \rfloor} (S^{4t-3} \cup S^{4t-2})) \cup (S^{m-1} \cup S^m)$ is a kTDS of $\mu_m(G)$ of cardinality $(2 + 2\lfloor (m-1)/4 \rfloor)\gamma_{\times k,t}(G) = 2\lceil (m-1)/4 \rceil\gamma_{\times k,t}(G).$

Case 3. $m \cong 3 \pmod{4}$. The set $S' = (\bigcup_{t=1}^{\lceil (m-1)/4 \rceil} (S^{4t-3} \cup S^{4t-2})) \cup S_k$ is a kTDS of $\mu_m(G)$ of cardinality

 $2\lceil (m-1)/4 \rceil \gamma_{\times k,t}(G) + k.$

Therefore we have proved

$$\gamma_{\times k,t}(\mu_m(G)) \le |S'| = \begin{cases} (1+2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) & \text{if } m \cong 0 \pmod{4}, \\ (1+2\lceil (m-1)/4 \rceil)\gamma_{\times k,t}(G) + k & \text{if } m \cong 1 \pmod{4}, \\ 2\lceil (m-1)/4 \rceil\gamma_{\times k,t}(G) & \text{if } m \cong 2 \pmod{4}, \\ 2\lceil (m-1)/4 \rceil\gamma_{\times k,t}(G) + k & \text{if } m \cong 3 \pmod{4}, \end{cases}$$

and this completes our proof.

Corollary 2.3. Let m be a positive integer, and let G be a graph without isolated vertex. Then

$$\gamma_t(G) + 1 \le \gamma_t(\mu_m(G)) \le \begin{cases} (1 + 2\lceil (m-1)/4 \rceil)\gamma_t(G) & \text{if } m \cong 0 \pmod{4} \\ (1 + 2\lceil (m-1)/4 \rceil)\gamma_t(G) + 1 & \text{if } m \cong 1 \pmod{4} \\ 2\lceil (m-1)/4 \rceil\gamma_t(G) & \text{if } m \cong 2 \pmod{4} \\ 2\lceil (m-1)/4 \rceil\gamma_t(G) + 1 & \text{if } m \cong 3 \pmod{4} \end{cases}$$

3. Mycieleskian graphs

Theorem 2.2 implies the next two theorems when m = 1.

Theorem 3.1. If G is a graph with no isolated vertices, then $\gamma_t(M(G)) = \gamma_t(G) + 1$.

Theorem 3.2. If G is a graph with $\delta(G) \ge k \ge 2$, then $\gamma_{\times k,t}(G) + 1 \le \gamma_{\times k,t}(M(G)) \le \gamma_{\times k,t}(G) + k$.

In the next theorem we give other lower bound for $\gamma_{\times k,t}(M(G))$.

Theorem 3.3. If G is a graph with $\delta(G) \ge k \ge 2$, then

$$\gamma_{\times k,t}(M(G)) \ge \min\{\gamma_{\times k,t}(G) + k, \gamma_{\times (k-1),t}(G) + k+1\}.$$

Proof. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$ and let S be an arbitrary kTDS of M(G). Then $|S \cap V^1| \ge k$. If $u \in S$, then $|S \cap V^0| \ge \gamma_{\times (k-1),t}(G)$. Since $S \cap V^0$ must be a (k-1)TDS of V^1 . If $u \notin S$, then $|S \cap V^0| \ge \gamma_{\times k,t}(G)$. Since $S \cap V^0$ must be a kTDS of V^1 . Therefore

$$\gamma_{\times k,t}(M(G)) \ge \min\{\gamma_{\times k,t}(G) + k, \gamma_{\times (k-1),t}(G) + k+1\}.$$

As an immediately result of Theorems 3.2 and 3.3 we have the following two corollaries.

Corollary 3.4. Let G be a graph with $\delta(G) \ge k \ge 2$. If $\gamma_{\times k,t}(G) = \gamma_{\times (k-1),t}(G) + 1$, then

$$\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k$$

Corollary 3.5. If $n \ge k+1 \ge 3$, then $\gamma_{\times k,t}(M(K_n)) = \gamma_{\times k,t}(K_n) + k = 2k+1$.

Also the following two results show that the upper bound $\gamma_{\times k,t}(G) + k$ in Theorem 3.2 is sharp for some of the complete multipartite graphs, the complete graph K_{k+1} and the k-join $F \circ_k K_{k+1}$, for every graph F.

Proposition 3.6. Let $G = K_{n_1,\dots n_p}$ be a complete *p*-partite graph. If $p \ge k+1 \ge 3$, then

$$\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k$$

Proof. We suppose that V(G) has partition $X_1 \cup X_2 \cup ... \cup X_p$ such that $|X_j| = n_j$ for j = 1, 2, ..., p. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = X_1^i \cup X_2^i \cup ... \cup X_p^i$ and $X_j^i = \{v_j^i \mid v_j \in X_j\}$, for i = 0, 1 and j = 1, 2, ..., p. Let also S be an arbitrary kTDS of M(G). Obviously $|S \cap V^1| \ge k$, and, without less of generality, we may assume that $|S \cap V^1| = k$. Let $S \cap V^1$ be a set which contains only one vertex of every X_i^1 , for $1 \le i \le k$. Thus each vertex of $V^0 - \{v_i^0 \mid v_i^1 \in S \cap V^1\}$ is adjacent to all vertices in $S \cap V^1$. Since each vertex of $S \cap V^1$ must be adjacent to at least k vertices of S, we have $|S \cap (V^0 \cup \{u\})| \ge k$. The assumptions $k \ge 2$ and $|S \cap V^0| \ge k - 1$ imply $S \cap V^0 \ne \emptyset$. We see that there exists an unique index $1 \le j \le k$ such that each vertex of X_j^1 is adjacent to k - 1 vertices of $S \cap (V^0 \cup \{u\})$. Hence $|S \cap (V^0 \cup \{u\})| \ge k + 1$, and so

$$\begin{aligned} \gamma_{\times k,t}(M(G)) &= \min \{ \mid S \mid : \text{ S is a kTDS of } M(G) \} \\ &\geq 2k+1 \\ &= \gamma_{\times k,t}(G) + k. \end{aligned}$$

Now Theorem 3.2 implies $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k$.

Theorem 3.7. Let G be a graph of order $n \ge k+1$ with $\delta(G) \ge k \ge 2$. If G is the k-join $F \circ_k K_{k+1}$, for some graph F, then

$$\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k.$$

Proof. Let G be the k-join $F \circ_k K_{k+1}$, for some graph F. Then Proposition 1.3 implies $\gamma_{\times k,t}(G) = k+1$. Since G is a spanning subgraph of K_n , and hence M(G) is a spanning subgraph of $M(K_n)$, we have

$$\gamma_{\times k,t}(M(G)) \geq \gamma_{\times k,t}(M(K_n))$$

$$= \gamma_{\times k,t}(K_n) + k$$

$$= 2k + 1$$

$$= \gamma_{\times k,t}(G) + k.$$

Now Theorem 3.2 implies $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + k = 2k + 1$.

Theorem 3.1 shows that $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$, where G is a graph with no isolated vertices and k = 1. Here, we give an equivalent condition for $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$, when $k \ge 2$. We recall that a kTDS S is a minimal kTDS if and only if for each vertex $v \in S$, there exists a k-element subset $S_v \subseteq S$ such that $v \in S_v$ and $|pn_k(S_v, S)| \ge 1$.

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Theorem 3.8. Let G be a graph with $\delta(G) \ge k \ge 1$. Then $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$ if and only if k = 1 or $k \ge 2$ and G has a $\gamma_{\times k,t}$ -set S with a k-subset $S' \subseteq S$ such that S - S' is a (k - 1) TDS of G and for every vertex v, $|S_v \cap S'| \le 1$.

Proof. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = \{v_j^i \mid 1 \leq j \leq n\}$ for i = 0, 1. If k = 1, then $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$, by Theorem 3.1. Now let $k \geq 2$. Let S be a $\gamma_{\times k,t}(G)$ -set which contains a k-subset $S' \subseteq S$ with this conditions that S - S' is a (k - 1)TDS of G and for every vertex v, $|S_v \cap S'| \leq 1$. Since

$$D = \{v_j^0 \mid v_j \in S - S'\} \cup \{v_j^1 \mid v_j \in S'\} \cup \{u\}$$

is a kTDS of M(G) of cardinality $\gamma_{\times k,t}(G) + 1$, Theorem 3.2 implies $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$.

Conversely, let $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$ and let $k \ge 2$. If u belongs to a $\gamma_{\times k,t}$ -set of M(G), we have no thing to prove. Thus we assume that u belongs to no $\gamma_{\times k,t}(M(G))$ -set D, and so $|D \cap V^1| \ge k$ and $|D \cap V^0| \ge \gamma_{\times k,t}(G)$. Therefore

$$\gamma_{\times k,t}(G) + 1 = \gamma_{\times k,t}(M(G)) = |D| \ge \gamma_{\times k,t}(G) + k \ge \gamma_{\times k,t}(G) + 2,$$

a contradiction.

The next proposition gives graphs that satisfy in the condition of Theorem 3.8. First, we present the definition of the *Harary graph* [8].

Given $m \leq n$, place *n* vertices 1, 2, ..., *n* around a circle, equally spaced. If *m* is even, form $H_{m,n}$ by making each vertex adjacent to the nearest m/2 vertices in each direction around the circle. If *m* is odd and *n* is even, form $H_{m,n}$ by making each vertex adjacent to the nearest (m-1)/2 vertices in each direction and to the diametrically opposite vertex. In each case, $H_{m,n}$ is *m*-regular. When *m* and *n* are both odd, index the vertices by the integers modulo *n*. Construct $H_{m,n}$ from $H_{m-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$.

Proposition 3.9. If G is a cycle of order at least 3 or the Harary graph $H_{2m,\ell m+1}$, where $\ell \geq 3$ and $m \geq 1$, then

$$\gamma_{\times 2,t}(M(G)) = \gamma_{\times 2,t}(G) + 1.$$

Proof. Let $G = C_n$ be a cycle of order at least 3 with the vertex set $V(C_n) = V^0 = \{v_j \mid 1 \le j \le n\}$ and the edge set $E(C_n) = \{(v_j, v_{j+1}) \mid 1 \le j \le n\}$. Let also $V(M(G)) = V^0 \cup V^1 \cup \{u\}$. Proposition 1.1(3) implies $\gamma_{\times 2,t}(G) = n$. Since $S = (V^0 - \{v_1^0, v_n^0\}) \cup \{v_{n-1}^1, v_n^1, u\}$ is a DTDS of M(G) of cardinality $\gamma_{\times 2,t}(C_n) + 1 = n + 1$, Theorem 2.2 implies $\gamma_{\times 2,t}(M(C_n)) = \gamma_{\times 2,t}(C_n) + 1 = n + 1$.

Now let $G = H_{2m,\ell m+1}$ and let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = \{v_j^i \mid 1 \le j \le \ell m + 1\}$ for $i = 0, 1, V^0 = V(G)$ and $\ell \ge 3$. Since $\{v_{jm+1}^0 \mid 0 \le j \le \lceil (\ell m + 1)/m \rceil - 1 = \ell\}$ is a DTDS of G, we have $\gamma_{\times 2,t}(G) = \lceil (\ell m + 1)/m \rceil = \ell + 1$, by Proposition 1.4. Since also $S = \{v_{im+1}^0 \mid 1 \le i \le \ell - 1\} \cup \{v_1^1, v_{\ell m+1}^1, u\}$ is a DTDS of M(G) of cardinality $\ell + 2 = \gamma_{\times 2,t}(G) + 1$, Theorem 2.2 implies $\gamma_{\times 2,t}(M(G)) = \gamma_{\times 2,t}(G) + 1$.

In the end of paper, the author states the following problem.

Problem: For integers $k, m \ge 1$, characterize graphs G with $\delta(G) \ge k$ satisfy $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G) + 1$ or

$$\gamma_{\times k,t}(\mu_m(G)) = \begin{cases} (1+2\lceil (m-1)/4\rceil)\gamma_{\times k,t}(G) & \text{if } m \cong 0 \pmod{4}, \\ (1+2\lceil (m-1)/4\rceil)\gamma_{\times k,t}(G) + k & \text{if } m \cong 1 \pmod{4}, \\ 2\lceil (m-1)/4\rceil\gamma_{\times k,t}(G) & \text{if } m \cong 2 \pmod{4}, \\ 2\lceil (m-1)/4\rceil\gamma_{\times k,t}(G) + k & \text{if } m \cong 3 \pmod{4}. \end{cases}$$

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