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MINIMAL, VERTEX MINIMAL AND COMMONALITY MINIMAL CN-DOMINATING GRAPHS

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Abstract. We define minimal CN-dominating graph $\mathbf{MCN}(G)$, commonality minimal CN-dominating graph $\mathbf{CMCN}(G)$ and vertex minimal CN-dominating graph $\mathbf{M_vCN}(G)$, characterizations are given for graph G for which the newly defined graphs are connected. Further serval new results are developed relating to these graphs.

1. Introduction

All the graphs considered here are finite and undirected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighbourhoods of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a neighbourhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set of a graph G. A neighbourhood set $S \subseteq V$ is a minimal neighbourhood set, if $S - v$ for all $v \in S$, is not a neighbourhood set of G.

For terminology and notations not specifically defined here we refer reader to [2]. For more details about domination number and neighbourhood number and their related parameters, we refer to [3],

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[4], and [9].

Let G be simple graph $G = (V, E)$ with vertex set

$$V(G) = \{v_1, v_2, \dots, v_n\}.$$

For $i \neq j$, the common neighborhood of the the vertices v_i and v_j, denoted by $\Gamma(v_i, v_j)$, is the set of vertices, different from v_i and v_j, which are adjacent to both v_i and v_j. A subset D of V is called common neighbourhood dominating set (CN-dominating set) if for every $v \in V - D$ there exist a vertex $u \in D$ such that $uv \in E(G)$ and $|\Gamma(u, v)| \geq 1$, where $|\Gamma(u, v)|$ is the number of common neighbourhood between the vertices u and v. The minimum cardinality of such CN-dominating set denoted by $\gamma_{cn}(G)$ and is called common neighbourhood domination number (CN-domination number) of G. It is clear that CN-domination number is defined for any graph. A common neighbourhood dominating set D is said to be minimal common neighbourhood dominating set if no proper subset of D is common neighbourhood dominating set. A minimal common neighbourhood dominating set D of maximum cardinality is called Γ_{cn}-set and its cardinality is denoted by Γ_{cn}. Let $u \in V$. The CN-neighbourhood of u denoted by $N_{cn}(u)$ is defined as $N_{cn}(u) = \{v \in N(u) : |\Gamma(u, v)| \geq 1\}$. The cardinality of $N_{cn}(u)$is denoted by $d_{cn}(u)$ in G, and $N_{cn}[u] = N_{cn}(u) \cup \{u\}$. The maximum and minimum common neighbourhood degree of a vertex in G are denoted respectively by $\Delta_{cn}(G)$ and $\delta_{cn}(G)$. That is $\Delta_{cn}(G) = max_{u \in V}|N_{cn}(u)|$, $\delta_{cn}(G) = min_{u \in V}|N_{cn}(u)|$. A subset S of V is called a common neighbourhood independent set (CN-independent set), if for every $u \in S, v \notin N_{cn}(u)$ for all $v \in S - \{u\}$. It is clear that every independent set is CN-independent set. An CN-independent set S is called maximal if any vertex set properly containing S is not CN-independent set. The maximum cardinality of CN-independent set is denoted by β_{cn}, and the lower CN-independence number i_{cn} is the minimum cardinality of the CN-maximal independent set. An edge $e = uv \in E(G)$ is said to be common neighbourhood edge(CN-edge) if $|\Gamma(u, v)| \geq 1$. A subset S of V is called common neighbourhood vertex covering (CN-vertex covering) of G if for CN-edge $e = uv$ either $u \in S$ or $v \in S$. The minimum cordiality of CN-vertex covering of G is called the CN-covering number of G and denoted by $\alpha_{cn}(G)$. For more details about CN-dominating set see [1].

A graph G is strongly regular with parameters (n, k, λ, μ) whenever G is regular of degree k, every pair of adjacent vertices has λ common neighbors, and every pair of distinct nonadjacent vertices has μ common neighbors.

Let S be a finite set and $F = \{S_1, S_2, \dots, S_n\}$ be a partition of S. Then the intersection graphs $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \phi$. Kulli and Janakiram introduced many classes of intersection graphs in the field of domination theory see [5-8].
In this paper, we define CN-minimal dominating graph, vertex CN-minimal dominating graph and

commonality minimal CN-dominating graph, some fundamental and interesting results results of these graphs are established.

2. Minimal CN-Dominating Graphs

Definition 2.1. *Let $G = (V, E)$ be graph. The minimal CN-dominating graph of G is denoted by* $\mathbf{MCN}(G)$ *is defined on the family of all minimal CN-dominating set of G, the vertex set is the CN-minimal dominating sets and any two vertices are adjacent if their intersection is not empty.*

Theorem 2.2. *Let G be a graph. The* $\mathbf{MCN}(G)$ *is complete graph if and only if G contains at least one CN-isolated vertex.*

Proof. Let u be CN-isolated vertex in G. Then u is in every minimal CN-dominating set of G. Hence every two vertices in $\mathbf{MCN}(G)$ are adjacent, thus $\mathbf{MCN}(G)$ is complete.
Conversely, suppose $\mathbf{MCN}(G)$ is complete graph and G has no CN-isolated vertex. Assume D be a minimal CN-dominating of G. Then $V - D$ contains a minimal CN-dominating set $D^{'}$. Then D and $D^{'}$ are two nonadjacent vertices in $\mathbf{MCN}(G)$, a contradiction. Hence G has CN-isolated vertex. □

A line graph $L(G)$ (also called an interchange graph or edge graph) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. And the lattice graph $L_{m,n}$ is the line graph of the complete bipartite graph $K_{m,n}$, and any lattice graph is strongly regular graph. From the next theorem we can get infinite family of lattice graphs.

Theorem 2.3. *Let G_1 and G_2 be any two complete graphs with same number of vertices p. Let G be the graph which obtained from the two complete graph by joining each vertex in G_1 by at most one vertex in G_2 and vice versa. Then* $\mathbf{MCN}(G)$ *is strongly regular graph of parameters $(p^2, 2(p-1), p-2, 2)$.*

Proof. Let the vertices of G_1 and G_2 be

$$v_1, v_2, \ldots, v_p \text{ and } v_{p+1}, v_{p+2}, \ldots, v_{2p}$$

respectively. Then the minimal CN-dominating sets of G are of the form $\{v_i, v_j\}$ where $i = 1, 2, \ldots, p$ and $j = p+1, p+2, \ldots, 2p$. Hence there are p^2 vertices in $\mathbf{MCN}(G)$. It is clear for any vertex $u = \{v_i, v_j\}$ in $\mathbf{MCN}(G)$ there exist $2(p-1)$ minimal CN-dominating has one common elements with u. Therefore $\mathbf{MCN}(G)$ is $(2(p-1))$-regular graph. Now suppose u and v be any two adjacent vertices in $\mathbf{MCN}(G)$ (two minimal CN-dominating set of G which has common element). Then it is clear there is $p-2$ minimal CN-dominating set of G have common elements with both minimal CN-dominating sets which They are corresponding to u and v,and similarly if u and v be any two nonadjacent vertices in $\mathbf{MCN}(G)$ (two minimal CN-dominating set of G which has no common element) then there are only two minimal CN-dominating set of G have common elements with both minimal CN-dominating sets which they are corresponding to u and v. Hence $\mathbf{MCN}(G)$ is strongly regular graph with the parameters $(p^2, 2(p-1), p-2, 2)$. □

Corollary 2.4. *If G is the two copy of the complete graph K_m, then $\mathbf{MCN}(G)$ is isomorphic to the line graph of K_m.*

Example. Let G be the two copy of the complete graph K_3. Then $\mathbf{MCN}(G)$ is shown in Figure 1.

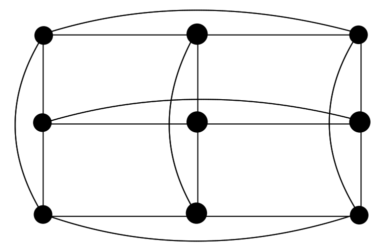

Figure 1

Theorem 2.A [1]
Let S be a maximal CN-independent set. Then S is minimal CN-dominating set.

Theorem 2.5. *For any graph G with p vertices ($p \geq 2$), $\mathbf{MCN}(G)$ is connected if and only if $\Delta_{cn}(G) < p - 1$.*

Proof. Let $\Delta_{cn}(G) < p - 1$ and D_1, D_2 be any two minimal CN-dominating sets of G. We have two cases:
Case 1: Every vertex in D_1 is adjacent and has common neighbourhood to every vertex in D_2 and according to that we have two cases:
(1) Suppose for any two vertices $u \in D_1$ and $v \in D_2$ there exist a vertex $w \notin (D_1 \cup D_2)$ such that u not adjacent to both u and v, then we have two maximal CN-independent D_3 and D_4 containing u, w and v, w respectively, and since by Theorem 2.A every maximal CN-independent set is minimal CN-dominating set, D_3 and D_4 are minimal CN-dominating sets. Then D_1 and D_2 are connected in $\mathbf{MCN}(G)$ through D_3 and D_4.
(2) There exist two vertices $u \in D_1$ and $v \in D_2$ such that every vertex not in $D_1 \cup D_2$ is adjacent and has common neighbourhood to either u or v that is $\{u, v\}$ is minimal CN-dominating set of G. Then D_1 and D_2 are connected through$\{u, v\}$.
Case 2: Suppose there exist two vertices $u \in D_1$ and $v \in D_2$ such that u and v are not adjacent. Then there exist maximal cn-independent D_3 containing u and v then D_3 is minimal CN-dominating set of G. Hence D_1 and D_2 are connected through D_3.
Conversely, Suppose that $\mathbf{MCN}(G)$ is connected and $\Delta_{cn}(G) = p - 1$. Then $\{u\}$ is minimal CN-dominating set of G and $V - \{u\}$ contains a minimal CN-minimal dominating set, that is $\mathbf{MCN}(G)$ is not connected, a contradiction. Hence $\mathbf{MCN}(G)$ is connected. $\square$

Theorem 2.6. $\mathbf{MCN}(G)$ *either connected or it has at least one component which is K_1.*

Proof. If $\Delta_{cn} < p - 1$ then from Theorem 2.5, $\mathbf{MCN}(G)$ is connected, then we have only two cases:
Case 1: $\delta_{cn} = \Delta_{cn} = p - 1$. Then $\mathbf{MCN}(G)$ is complete graph and all the singleton $\{u\}$, where $u \in G$

are minimal CN-dominating sets. then all the components of $\mathbf{MCN}(G)$ are K_1.

Case 2: Let $\delta_{cn}(G) < \Delta_{cn}(G) = p-1$. Let $\{u_1, \ldots, u_s\}$ be the set of vertices in G such that $d_{cn}(u_i) = p-1$, where $i = 1, \ldots, t$, then it is clear u_i is minimal CN-dominating set. Then the minimal CN-dominating sets $\{u_i\}$, where $i = 1, \ldots, s$ form component isomorphic to K_1. Hence has at least one component which is K_1. □

Theorem 2.7. *For any graph G, $\beta(\mathbf{MCN}(G)) = d_{cn}(G)$, where $d_{cn}(G)$ is the CN-domatic number of G.*

Proof. let F be the maximum order CN-domatic partition of $V(G)$. If each dominating set in F is minimal. Then F is maximum independent set in $\mathbf{MCN}(G)$ and hence $\beta(\mathbf{MCN}(G)) = d_{cn}(G)$. Otherwise, let $D \subseteq F$ be CN-dominating set in F which is not minimal. Then there is minimal CN-dominating set $D' \subset D$ by replacing each D in F by its subset D' we see that F is maximum independent set in $\mathbf{MCN}(G)$. Hence $\beta(\mathbf{MCN}(G)) = d_{cn}(G)$. □

3. Commonality minimal CN-Dominating Graphs

Definition 3.1. *The commonality minimal CN-dominating graph is denoted by $\mathbf{CMCN}(G)$ is the graph which has the same vertex set as G with two vertices adjacent if and only if there exist minimal CN-dominating in G containing them.*

Example Let G be a graph as in Figure 2a. Then the minimal CN-dominating sets are $\{1,2\}, \{1,3\}$ and $\{1,4\}$.The $\mathbf{CMCN}(G)$ is shown in Figure 2b.

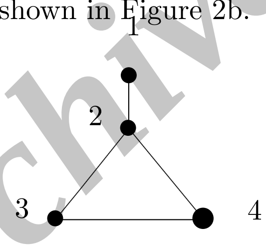

Figure 2a

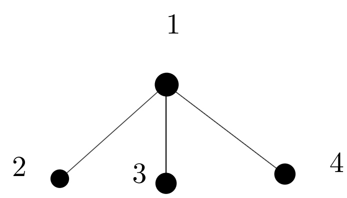

Figure 2b

Proposition 3.2. *For any graph G,*
(i) If G is complete graph, then $\mathbf{CMCN}(G)$ is totally disconnected.
(ii) If G is totally disconnected, then $\mathbf{CMCN}(G)$ is complete graph.

Theorem 3.3. *For any graph G*
(i) $\overline{G} \subseteq \mathbf{CMCN}(G)$.
(ii) $\overline{G} \cong \mathbf{CMCN}(G)$ if and only if every minimal CN-dominating set of G is independent.

Proof. **(i)** let u and v be any two adjacent vertices in $\overline{G}$, then we can extend the set $\{u, v\}$ into maximal CN-independent set S in G which is also minimal CN-dominating set that is u and v also

adjacent vertices in $\mathbf{CMCN}(G)$. Hence $\overline{G} \subseteq \mathbf{CMCN}(G)$.
(ii) Let every minimal CN-dominating of G is CN-independent. Then any two adjacent vertices in G can not adjacent in $\mathbf{CMCN}(G)$, that is $\mathbf{CMCN}(G) \subseteq \overline{G}$ and by (i), we get $\overline{G} \cong \mathbf{CMCN}(G)$.
Conversely, if $\mathbf{CMCN}(G) \subseteq \overline{G}$, then any two vertices in the same minimal CN-dominating set S of G are not adjacent in G. Hence S is independent set. □

Let $u \in V(G)$, the CN-neighbourhood of u denoted by $N_{cn}(u) = \{v \in N(u : |\Gamma(u,v)| \geq 1\}$, where $|\Gamma(u,v)|$ is the number of common neighbours between u and v, the cardinality of $N_{cn}(u)$ is denoted by $d_{cn}(u)$, the CN-maximum degree $\Delta_{cn}(G)$ and the CN-minimum degree are defined respectively $\Delta_{cn}(G) = max_{u \in V(G)}|d_{cn}(u)|$, $\delta_{cn}(G) = min_{u \in V(G)}|d_{cn}(u)|$.

Theorem 3.4. *For any graph G with p vertices, where $p \geq 2$,* $\mathbf{CMCN}(G)$ *is connected graph if and only if $\Delta_{cn} < p - 1$.*

Proof. Let $\Delta_{cn} < p - 1$ and u, v be any two vertices of G. Then we have four cases:
Case 1: If u and v are not adjacent in G then by Theorem 3.3, u is adjacent to v in $\mathbf{CMCN}(G)$.
Case 2: If u and v are adjacent in G and there is a vertex w not adjacent to both u and v, then in $\mathbf{CMCN}(G)$, u and v are joining by the path uwv.
Case 3: If u and v are adjacent in G and every other vertex w is adjacent and has common neighbours to at least one of u and v. Then $\{u, v\}$ is minimal CN-dominating set of G. Hence u is adjacent to v in $\mathbf{CMCN}(G)$.
Case 4: If u and v are adjacent in G and there exist a vertex w adjacent to u or v but has not common neighbours, then there exist two maximum Cn-independent sets D_1 and D_2 contains u, w and v, w respectively and by Theorem 2.A, D_1 and D_2 are minimal CN-dominating set in G. Hence u and v are connected in $\mathbf{CMCN}(G)$ through w.
From the four cases we get that $\mathbf{CMCN}(G)$ is connected graph.
Conversely, suppose that $\mathbf{CMCN}(G)$ is connected graph. If possible suppose $\Delta_{cn} = p - 1$, then there exist at least one vertex u in G such that $d_{cn}(u) = p - 1$, then u is isolated vertex in $\mathbf{CMCN}(G)$, and since G has at least two vertices implies that $\mathbf{CMCN}(G)$ has at least two component, a contradiction. Hence $\Delta_{cn} < p - 1$. □

Definition A triangle-free graph is a graph containing no graph cycles of length three.
Theorem 3.A.[1] Let G be a graph, $\gamma_{cn}(G) = p$ if and only if G is a triangle free.

Proposition 3.5. *If G is triangle-free graph, then* $\mathbf{CMCN}(G)$ *is complete graph.*

Proof. We know by Theorem 3.A, for any graph G, $\gamma_{cn}(G) = p$ if and only if G is triangle-free, that means there is only one minimal CN-dominating set which contains all the vertices and by using the definition of $\mathbf{CMCN}(G)$ it is clear any two vertices are adjacent that means $\mathbf{CMCN}(G)$ is complete graph.

Lemma 3.6. *If G be a triangle-free graph, then G is totally disconnected if and only if every CN-independent set in G is independent set.*

Proof. Let G be a triangle-free graph, and every CN-independent set in G is independent set, that means the set $V(G)$ is CN-independent set since G is triangle-free graph, and since every CN-independent set in G is independent set, then $V(G)$ is also independent set that is G is totally disconnected.
Conversely, clearly if G is totally disconnected then every CN-independent set in G is also independent set. □

Theorem 3.7. *For any graph G with the property every CN-independent set in G is independent set, $\gamma_{cn}(\mathbf{CMCN}(G)) = p$ if and only if G is K_p.*

Proof. If G is K_p, then it is clear that the $\mathbf{CMCN}(G)$ is totally disconnected graph. Then $\gamma_{cn}(\mathbf{CMCN}(G)) = \gamma(\mathbf{CMCN}(G)) = p$.
Conversely, suppose $\gamma_{cn}(\mathbf{CMCN}(G)) = p$, then $\mathbf{CMCN}(G)$ is triangle-free graph by Theorem 3.A, and by Lemma 3.6, $\mathbf{CMCN}(G) = \overline{K_p}$, since all the minimal CN-dominating sets in G are CN-independent and by Theorem 3.3, $\overline{G} = \overline{K_p}$. Hence G is K_p.
Conversely, if G is K_p, then every minimal CN-dominating set of G is independent, and by Theorem 3.3 $\mathbf{CMCN}(G) = \overline{G} = \overline{K_p}$. Hence $\gamma_{cn}(\mathbf{CMCN}(G)) = p$. □

It is not true in general that if $\gamma_{cn}(\mathbf{CMCN}(G)) = p$, then $\mathbf{CMCN}(G)$ is totally disconnected graph we show that by the following example
Example. Let G be a graph as in Figure 3a, then the CN-minimal independent sets are $\{2\}, \{4\} and\ \{1, 3\}$ and it is clear from Figure 3b that $\gamma_{cn}(\mathbf{CMCN}(G)) = p$ but $\mathbf{CMCN}(G)$ is not totally disconnected.

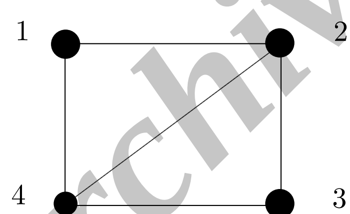

Figure 3a

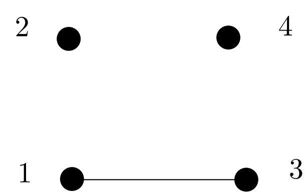

Figure 3b

Proposition 3.8. *The graph $\mathbf{CMCN}(G)$ is complete bipartite graph $K_{r,m}$, if and only if G is the disjoint union of K_r and K_m.*

4. Vertex Minimal CN-Dominating Graphs

Definition 4.1. *The vertex minimal CN-dominating graph $\mathbf{M_v CN}(G)$ of a graph G is a graph with $V \cup S$ as vertex set, where S is the collection of all minimal CN-dominating set of G with two vertices*

$u, v \in V \cup S$ are adjacent if they are adjacent in G or $v = D$ is a minimal CN-dominating set of G containing u.

Example Let G be a graph as in Figure 2a, then the vertex minimal CN-dominating graph is shown in Figure 4.

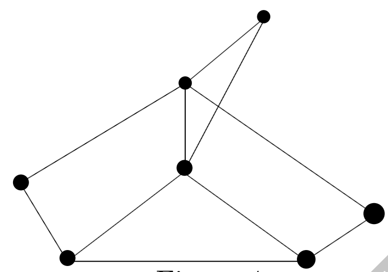

Figure 4

Theorem 4.2. *For any graph G, $\mathbf{M_vCN}(G)$ is connected.*

Proof. Since for each vertex $v \in V(G)$ there exist a minimal CN-dominating set containing v, every vertex in $\mathbf{M_vCN}(G)$ is not isolated vertex.
Now suppose $\mathbf{M_vCN}(G)$ is disconnected, then there exist at least two component say G_1 and G_2 and there exist two nonadjacent vertices u, v such that $u \in G_1$ and $v \in G_2$ that means there is no minimal CN-dominating set in G containing u and v, a contradiction. Hence $\mathbf{M_vCN}(G)$ is connected. □

Theorem 4.3. *For any graph G, $diam(\mathbf{M_vCN}(G)) \leq 3$.*

Proof. Suppose G has at least two vertices. Then $\mathbf{M_vCN}(G)$ has at least three vertices , let $u, v \in V(\mathbf{M_vCN}(G))$, we consider the following cases:
Case 1: Suppose $u, v \in V(G)$. Then in $\mathbf{M_vCN}(G)$, $d(u, v) \leq 2$.
Case 2: Suppose that $u \in V(G)$ and $v \notin V(G)$. Then $v = D$ is minimal CN-dominating set of G, if $u \in D$, then in $\mathbf{M_vCN}(G)$, $d(u, v) = 1$, if $u \notin D$, then there exist vertex $w \in D$ adjacent to u and has common neighbours with u. Hence in $\mathbf{M_vCN}(G)$ $d(u, v) = d(u, w) + d(w, v) = 2$.
Case 3: Suppose $u, v \notin V(G)$. Then $u = D$ and $v = D'$ are two minimal CN-dominating set in G, if D and D' are disjoint, then every vertex in D is adjacent to some vertex $x \in D'$ and vice versa this implies that in $\mathbf{M_vCN}(G)$ $d(u, v) = d(u, w) + d(w, x) + d(x, v) = 3$, and if D and D' are not disjoint then in $\mathbf{M_vCN}(G)$, $d(u, v) = d(u, w) + d(w, v) = 2$, where w is common vertex between D and D'.
Hence $diam(\mathbf{M_vCN}(G)) \leq 3$. □

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