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ON THE TOTAL DOMATIC NUMBER OF REGULAR GRAPHS

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ABSTRACT. A set S of vertices of a graph G = (V, E) without isolated vertex is a total dominating set if every vertex of V(G) is adjacent to some vertex in S. The total domatic number of a graph G is the maximum number of total dominating sets into which the vertex set of G can be partitioned. We show that the total domatic number of a random r-regular graph is almost surely at most r - 1, and that for 3-regular random graphs, the total domatic number is almost surely equal to 2. We also give a lower bound on the total domatic number of a graph in terms of order, minimum degree and maximum degree. As a corollary, we obtain the result that the total domatic number of an r-regular graph is at least $r/(3\ln(r))$.

1. Introduction

Let G = (V(G), E(G)) = (V, E) be a simple graph of order n with minimum degree $\delta(G) \ge 1$. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $d_G(u)$ (briefly N(u) and d(u) when no ambiguity on the graph is possible). The minimum and maximum degree of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. A matching is a set of edges with no shared endvertices. A perfect matching M of G is a matching with V(M) = V(G). The maximum number of edges of a matching in G is denoted by $\alpha'(G)$ (α' for short). If $C = (v_1, v_2, \ldots, v_n)$ is a cycle and v_i, v_k are distinct vertices of C, then the segment $[v_i, v_k]$ of C is defined as the set $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_k\}$, where the subscripts are taken modulo n. If f(n) and g(n)are real valued functions of an integer variable n, then we write f(n) = O(g(n)) (or $f(n) = \Omega(g(n))$)

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if there exist constants C > 0 and n_0 such that $f(n) \leq Cg(n)$ (or $f(n) \geq Cg(n)$) for $n \geq n_0$. We also write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. We use [9] for terminology and notation which are not defined here.

A set S of vertices of a graph G with minimum degree $\delta(G) > 0$ is a total dominating set if N(S) = V(G). The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the total domination number of G. A $\gamma_t(G)$ -set is a total dominating set of G of cardinality $\gamma_t(G)$.

A partition of V(G), all of whose classes are total dominating sets in G, is called a *total domatic partition* of G. The maximum number of classes of a total domatic partition of G is called the *total domatic number* of G and is denoted by $d_t(G)$. The total domatic number was introduced by Cockayne, Dawes and Hedetniemi in [5] and has been studied by several authors (see for example, [2, 4, 11, 12]). More information on the total domination number and the total domatic number can be found in the monographs [7, 8] by Haynes, Hedetniemi and Slater.

We use the following standard model $\mathcal{G}_{n,r}$ to generate *r*-regular graphs on *n* vertices uniformly: to construct a random *r*-regular graph on the vertex set $\{v_1, v_2, \ldots, v_n\}$, take a random matching on the vertex set $\{v_{1,1}, v_{1,2}, \ldots, v_{1,r}, v_{2,1}, \ldots, v_{2,r}, \ldots, v_{n,r}\}$ and collapse each set $\{v_{i,1}, v_{i,2}, \ldots, v_{i,r}\}$ into a single vertex v_i . If the resulting graph contains any loops or multiple edges, discard it. All *r*-regular graphs are generated uniformly with this method. Wormald [10] has shown that 3-regular graphs are almost surely Hamiltonian, and that the model $\mathcal{G}_{n,r}$ and $\mathcal{H}_n \oplus \mathcal{G}_{n,r-2}$ are contiguous, meaning roughly that events that are almost sure in one model are almost sure in the other. Thus if an event is almost surely true in a random graph constructed from a random Hamilton cycle plus a random matching, then it is almost surely true in a random 3-regular graph. For more details the reader is referred to [10].

We make use of the following results.

Theorem A. ([10]) If G is a 3-regular random graph, then a.a. G consists of a Hamilton cycle plus a random matching.

Theorem B. ([5]) For every graph G of order n without isolated vertices,

$$d_t(G) \le \min\left\{\delta(G), \frac{n}{\gamma_t(G)}\right\}.$$

2. A lower bound on the total domatic number

In this section we will show that the total domatic number of a random 3-regular graph is at least 2.

Definition 2.1. Let G be a 3-regular graph obtained from a cycle $C = (v_1, v_2, \dots, v_n)$ by adding a perfect matching M. An edge $v_i v_{i+1}$ of C (the indices are taken modulo n) is a 4-edge if v_i and v_{i+1} have matching partners v_j and v_k respectively, such that the cycle segments $[v_j, v_i]$ and $[v_{i+1}, v_k]$ are disjoint and have cardinality 0 (mod 4).

Lemma 2.2. Let $G = C \cup M$ as above. If C has a 4-edge then $d_t(G) \ge 2$.

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Proof. Let $v_i v_{i+1}$ be a 4-edge of C, and let v_j, v_k be their matching partners, respectively. Without loss of generality, we may assume that i = 1. If $n \equiv 0 \pmod{4}$, then obviously $S_1 = \{v_{4i+1}, v_{4i+2} \mid 0 \leq i \leq \frac{n}{4} - 1\}$ and $S_2 = V(G) - S_1$ are two disjoint total dominating sets and hence $d_t(G) \geq 2$.

Now let $n \equiv 2 \pmod{4}$. Then n = 4s + 2 for some positive integer s. If n = 6, then the result is immediate. Assume that $n \geq 10$. Then $k \equiv 1 \pmod{4}$ and $j \equiv 0 \pmod{4}$. Let $k = 4\ell + 1$, j = 4r and define $S_1 = \{v_{4i+1}, v_{4i+2} \mid 0 \leq i \leq \frac{k-1}{4} - 1\} \cup \{v_{4i+3}, v_{4i} \mid r \leq i \leq s\}$ if j = k + 3, or $S_1 = \{v_{4i+1}, v_{4i+2} \mid 0 \leq i \leq \frac{k-1}{4} - 1\} \cup \{v_{4i+3}, v_{4i} \mid r \leq i \leq s\} \cup \{v_{4i}, v_{4i+1} \mid \ell + 1 \leq i \leq r - 1\}$ when j > k + 3 and $S_2 = V(G) - S_1$. Obviously, S_1 and S_2 are two disjoint total dominating sets. Thus $d_t(G) \geq 2$ and the proof is complete. \Box

The proof of the following lemma is essentially similar to the proof of Lemma 2 of [6].

Lemma 2.3. Let G be a graph obtained from a cycle $C = (v_1, v_2, \dots, v_n)$ of even order by adding a random matching M. Then G has a 4-edge a.a.

Proof. Define random variables X_i for $i = 1, 2, \dots, n$ by

$$X_{i} = \begin{cases} 1 & \text{if } v_{i}v_{i+1} \in E(C) \text{ is a } 4 - \text{edge} \\ 0 & \text{otherwise.} \end{cases}$$

and let $X = \sum_{i=1}^{n} X_i$. Then each X_i has expectation

$$E(X_i) = P(X_i = 1) = \frac{1}{2}(\frac{1}{4} + \frac{1}{n})^2 = \frac{1}{32} + O(1/n)$$

and variance

$$\operatorname{var}(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i) - E(X_i)^2 = \frac{31}{1024} + O(1/n).$$

The covariance of X_i and X_j for i < j equals

$$\begin{aligned} \operatorname{cov}(X_i, X_j) &= E(X_i X_j) - E(X_i) E(X_j) \\ &= \begin{cases} 1/1024 - (1/32)^2 + O(1/n) & \text{if } i < j - 1 \\ \frac{8}{3}(\frac{1}{1024}) - (1/32)^2 + O(1/n) & \text{if } i = j - 1 \text{ and } n \equiv 2 \pmod{4} \\ 0 - (1/32)^2 + O(1/n) & \text{if } i = j - 1 \text{ and } n \equiv 0, 1, 3 \pmod{4} \\ &= \begin{cases} O(1/n) & \text{if } i < j - 1 \\ O(1) & \text{if } i = j - 1. \end{cases} \end{aligned}$$

Note that $X_i X_{i+1} = 1$ implies that $n \equiv 2 \pmod{4}$. To see this, let v_k be the matching partner of v_{i+1} . If $X_i X_{i+1} = 1$, then $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ are 4-edges and thus $n+2 \equiv |[v_{i+1}, v_k]| + |[v_k, v_{i+1}]| \equiv 0 + 0 \equiv 0 \pmod{4}$, i.e., $n \equiv 2 \pmod{4}$.

Hence the random variable X has expectation

$$E(X) = \sum_{i=1}^{n} E(X_i) = n/32 + O(1) = O(n)$$

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and variance

$$\operatorname{var}(X) = \sum_{i=1}^{n} \operatorname{var}(X_i) + 2 \sum_{i < j-1} \operatorname{cov}(X_i, X_j) + 2 \sum_{i=1}^{n} \operatorname{cov}(X_i, X_{i+1})$$
$$= \frac{31}{1024} n + 2 \sum_{i < j-1} O(1/n) + 2 \sum_{i=1}^{n} O(1)$$

= O(n)

By Chebyschev's inequality, we have

$$\operatorname{prob}(X=0) \le \frac{\operatorname{var}(X)}{E(X)^2} = \frac{O(n)}{(O(n))^2} = O(1/n).$$

Hence X > 0 a.a., i.e., G has a 4-edge.

An immediate consequence of Theorem A and Lemmas 2.2 and 2.3 now follows.

Theorem 2.4. If G is a random 3-regular graph, then $d_t(G) \ge 2$ a.a.

3. An upper bound for the total domatic number

The proof of the following theorem is essentially similar to the proof of Theorem 2 of [6].

Theorem 3.1. Let $r \ge 3$ and let G be a random r-regular graph. Then $d_t(G) \le r - 1$.

Proof. Suppose to the contrary that G is an r-regular graph with $d_t(G) > r - 1$. It follows from Theorem B that $d_t(G) = r$. Let V_1, V_2, \dots, V_r be a total domatic partition for G. Then each vertex has a neighbor in every V_i . Since every vertex has precisely r neighbors, we deduce that every vertex in V_i has precisely one neighbor in V_j for each j. Hence,

$$|N(v) \cap V_i| = 1$$
 for all $v \in V(G)$ and $i \in \{1, 2, \dots, r\}$.

For $i \neq j$ we deduce that

(3.1)
$$E_{ij} := \{ uv \in E(G) | u \in V_i, v \in V_j \} \text{ is a perfect } V_i - V_j \text{ matching}$$

and so

(3.2)
$$|V_1| = |V_2| = \dots = |V_r| = \frac{n}{r}$$

It follows from the above argument that every r-regular graph with $d_t = r$ on the vertex set V(G) can be obtained by first partitioning V(G) into r sets, all of equal cardinality, and then adding a perfect matching between the vertices of every partition, implying that n/r is even, and finally adding a perfect matching between all pairs of partition sets. Suppose n is a multiple of r. Since the sets are not distinguishable, the first step can be done in

$$\begin{pmatrix} n \\ n/r, n/r, \cdots, n/r \end{pmatrix} \frac{1}{r!}$$

ways, the second step can be done in

$$[(\frac{n}{r}-1)(\frac{n}{r}-3)\cdots 1]^r = \frac{((\frac{n}{r})!)^r}{2^{n/2}((\frac{n}{2r})!)^r}$$

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ways, and the last step can be done in

$$\left(\left(\frac{n}{r}\right)!\right)^{\binom{r}{2}}$$

ways, since there are $\binom{r}{2}$ different pairs of sets V_i, V_j , and between each pair a matching can be added in $(\frac{n}{r})!$ ways. Hence, an upper bound on the number of labeled *r*-regular graphs of order *n* with $d_t = r$, is

$$\begin{array}{c} n \\ n/r, n/r, \cdots, n/r \end{array} \right) \frac{1}{r!} \left(\left(\frac{n}{r}\right)! \right)^{\binom{r}{2}} \frac{\left(\left(\frac{n}{r}\right)!\right)^r}{2^{n/2} \left(\left(\frac{n}{2r}\right)!\right)^r} \\ = \frac{n!}{\left(\left(\frac{n}{r}\right)!\right)^r r!} \left(\left(\frac{n}{r}\right)!\right)^{\frac{r(r-1)}{2}} \frac{\left(\left(\frac{n}{r}\right)!\right)^r}{2^{n/2} \left(\left(\frac{n}{2r}\right)!\right)^r}, \end{array}$$

and hence, by Stirling's formula $(n! \sim (\frac{n}{e})^n \sqrt{2\pi n} (1 + \frac{1}{12n} + O(\frac{1}{n^2})))$ the upper bound is, for large n and constant r,

$$(\frac{n}{e})^n \sqrt{2\pi n} (1 + \frac{1}{12n} + O(\frac{1}{n^2})) \cdot \frac{1}{r! 2^{n/2}} \cdot (\frac{n}{re})^{\frac{r(r-1)}{2}}$$

$$\cdot \left[\sqrt{2\pi n/r} \left(1 + \frac{1}{12n} + O(\frac{1}{n^2})\right)\right]^{\frac{r(r-1)}{2}} \cdot \frac{1}{\left(\frac{n}{2re}\right)^{rn} \left(\sqrt{\pi n/r} \left(1 + \frac{1}{12n} + O(\frac{1}{n^2})\right)\right)^r}$$

Denote this last expression by DOMT(r, n). The total number of r-regular graphs, as given in [3] is asymptotic to

$$e^{-(r^2-1)/4}\frac{(rn)!}{(rn/2)!2^{rn/2}(r!)^n} = \frac{e^{-(r^2-1)/4}}{2^{rn/2}(r!)^n} \cdot \frac{(\frac{rn}{e})^n \sqrt{2\pi rn}(1+\frac{1}{12n}+O(\frac{1}{n^2}))}{(\frac{rn}{2e})^n \sqrt{\pi rn}(1+\frac{1}{12n}+O(\frac{1}{n^2}))}$$

Denote this last expression by TOTAL(r, n). Then the proportion of r-regular graphs with $d_t = r$, DOMT(r, n)/TOTAL(r, n), is at most

$$(\frac{r!}{r^{r-1}})^n O(n^{\frac{(r-1)(r-2)}{4}}).$$

Since $\frac{r!}{r^{r-1}}$ is less than 1, so the limit DOMT(r, n)/TOTAL(r, n) tends to 0, as desired. This completes the proof.

4. Total domatic number and minimum degree

If G is a graph of order n, then Zelinka [12] gave the following lower bound on the total domatic number

$$d_t(G) \ge \lfloor \frac{n}{n-\delta(G)+1} \rfloor.$$

The proof of the following theorem is essentially similar to the proof of Theorem 3 of [6] and we leave it to the reader.

Theorem 4.1. Let G be a graph of order n with minimum degree δ and maximum degree Δ , and let k be a nonnegative integer. If

$$e(\Delta^2 + 1)k(1 - \frac{1}{k})^{\delta} < 1,$$

then $d_t(G) \ge k$.

For the special case of a regular graph, we obtain a significant improvement of Zelinka's bound.

Corollary 4.2. Let G be an r-regular graph with $r \ge 3$. Then

$$d_t(G) \ge \frac{r}{3\ln r}.$$

Proof. With $\Delta = \delta = r$ and $k = \frac{r}{3 \ln r}$ we have

$$e(\Delta^{2}+1)k(1-\frac{1}{k})^{\delta} = e(r^{2}+1)k(1-\frac{1}{k})^{r}$$

$$\leq e(r^{2}+1)\frac{r}{3\ln r}e^{-\frac{3\ln r}{r}r}$$

$$= \frac{e(r^{2}+1)}{3r^{2}\ln r}$$

$$< 1.$$

Now it follows from Theorem 4.1 that $d_t(G) \geq \frac{r}{3\ln(r)}$.

A question that arises naturally is whether the bound in Corollary 4.2 is best possible. For a positive integer r, let f(r) be the minimum total domatic number of all r-regular graphs. By Corollary 4.2 we have $f(r) \geq \frac{r}{3\ln(r)}$. On the other hand, it follows from [1] that there exist r-regular graphs of order n with total domination number $(1 + o(1))\frac{n\ln(r)}{r}$. According to Theorem B, the total domatic number of those graphs is at most $n/\gamma_t(G) = (1 + o(1))\frac{r}{\ln(r)}$. This proves $f(r) = \Omega(\frac{r}{\ln(r)})$, and the order of magnitude of the bound in Corollary 4.2 is best possible.

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