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A NOTE ON THE TOTAL DOMINATION SUPERCRITICAL GRAPHS

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Abstract. Let G be a connected spanning subgraph of $K_{s,s}$ and let H be the complement of G relative to $K_{s,s}$. The graph G is *k-supercritical* relative to $K_{s,s}$ if $\gamma_t(G) = k$ and $\gamma_t(G+e) = k-2$ for all $e \in E(H)$. The 2002 paper by T.W. Haynes, M.A. Henning and L.C. van der Merwe, "Total domination supercritical graphs with respect to relative complements" that appeared in Discrete Mathematics, 258 (2002), 361-371, presents a theorem (Theorem 11) to produce $(2k+2)$-supercritical graphs relative to $K_{2k+1,2k+1}$ of diameter 5, for each $k \geq 2$. However, the families of graphs in their proof are not the case. We present a correction of this theorem.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. A set of vertices S in G is a *total dominating set*, (or just TDS), if $N(S) = V(G)$. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. For Graph Theory notation and terminology in general we follow [2].

Haynes, Henning and Van der Merwe in [3] studied *total domination supercritical graphs with respect to relative complements*. Let G be a connected spanning subgraph of $K_{s,s}$ and let H be the complement of G relative to $K_{s,s}$. The graph G is *k-supercritical* relative to $K_{s,s}$ if $\gamma_t(G) = k$ and $\gamma_t(G+e) = k-2$ for all $e \in E(H)$. They presented a construction to produce 6-supercritical graphs of diameter 5.

For $k \geq 2$, let $\mathcal{G}_k$ be the class of all graphs G such that $G \in \mathcal{G}_k$ if and only if G is formed as follows. Form G from k copies of the cycle C_6 by identifying an edge, say ab, common to every cycle. Let

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$A = N(a) - \{b\}$ and $B = N(b) - \{a\}$, and label the vertices of A and B as $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ such that a_i and b_i are in the ith copy of C_6. Finally, for each $i \neq j$, add exactly one of the edges a_ib_j and a_jb_i. Clearly, G is a bipartite spanning subgraph of $K_{2k+1,2k+1}$ and $diam(G) = 5$.

Theorem 1.1. *(Theorem 11 of [3]) For each $k \geq 2$, $G \in \mathcal{G}_k$ is a $(2k+2)$-supercritical graph relative to $K_{2k+1,2k+1}$ of diameter 5.*

We show that Theorem 1.1 is not correct, and the construction presented for $2k+2$-supercritical graphs of diameter 5 is incorrect, and then we give a corrected construction which produces $2k+2$-supercritical graphs of diameter 5. We remark that in [1] the authors corrected the construction presented in the first paragraph of page 370 of [3].

2. Main results

Consider the construction presented before Theorem 1.1. For $i \in \{1, \ldots, k-1\}$, let $b_i \in N(a_{i+1})$, and $b_k \in N(a_1)$. Then clearly $S = \{a_i, b_i : i = 1, 2, \ldots, k\}$ is a TDS for G, implying that $\gamma_t(G) = 2k$. Thus the above construction does not produce $(2k+2)$-supercritical graphs.

We will now give a corrected construction.

- For $k \geq 2$, let $\mathcal{H}_k$ be the class of all graphs G such that $G \in \mathcal{H}_k$ if and only if G is formed as follows. Form G from k copies of the cycle C_6 by identifying an edge, say ab, common to every cycle. Let $A = N(a) - \{b\}$ and $B = N(b) - \{a\}$, and label the vertices of A and B as $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ such that a_i and b_i are in the ith copy of C_6. Finally, for $i = 1, 2, \ldots, k-1$, join a_i to every vertex in $\{b_{i+1}, b_{i+2}, \ldots, b_k\}$. Clearly, G is a bipartite spanning subgraph of $K_{2k+1,2k+1}$ and $diam(G) = 5$.

Theorem 2.1. *For each $k \geq 2$, $G \in \mathcal{H}_k$ is a $(2k+2)$-supercritical graph relative to $K_{2k+1,2k+1}$ of diameter 5.*

Proof. Let $k \geq 2$ and $G \in \mathcal{H}_k$. Clearly $diam(G) = 5$. Let $C_6^1, C_6^2, \ldots, C_6^k$ be the k copies of C_6, and $V(C_6^i) = \{a, b, a_i, b_i, c_i, d_i\}$, and $E(C_6^i) = \{ab, aa_i, a_id_i, d_ic_i, c_ib_i, b_ib\}$ for $i = 1, 2, ..., k$.

We first show that $\gamma_t(G) = 2k+2$. Let S be a $\gamma_t(G)$-set. For each i, since d_i is totally dominated by S, we find that $S \cap \{a_i, c_i\} \neq \emptyset$, and since c_i is totally dominated by S, we find that $S \cap \{d_i, b_i\} \neq \emptyset$, and so $|S \cap (V(C_6^i) - \{a, b\})| \geq 2$. Thus $|S| \geq 2k$. We show that $|S| = 2k+2$.

Suppose that $|S| = 2k+1$. Then $|S \cap \{a, b\}| \leq 1$. We consider the following cases.

Case 1. $|S \cap \{a, b\}| = 1$. Then $|S \cap (V(C_6^i) - \{a, b\})| = 2$ for any $i \in \{1, 2, \ldots, k\}$. Without loss of generality assume that $a \in S$ and $b \notin S$. Since a is totally dominated by S, we find that there is an integer $j \in \{1, 2, \ldots, k\}$ such that $a_j \in S$. If $c_j \in S$, then $S \cap \{b_j, d_j\} \neq \emptyset$, since c_j is totally dominated by S. Then $|S \cap (V(C_6^j) - \{a, b\})| = 3 > 2$, a contradiction. Thus $c_j \notin S$. Since b_j is totally dominated by S, there is an integer $t < j$ such that $a_t \in S$. As before we find that $c_t \notin S$, and there is an integer $l < t$ such that $a_l \in S$. By continuing this process, we obtain that $a_1 \in S$. Since b_1 is totally dominated by S, we find that $c_1 \in S$, and so $S \cap \{b_1, d_1\} \neq \emptyset$. Thus, $|S \cap (V(C_6^1) - \{a, b\})| = 3 > 2$, a contradiction.

Case 2. $|S \cap \{a, b\}| = 0$. Then there is an integer $m \in \{1, 2, \ldots, k\}$ such that $|S \cap (V(C_6^m) - \{a, b\})| = 3$, and

$$\text{(2.1)} \qquad for \;\; any \;\; i \neq m, \;\; |S \cap (V(C_6^i) - \{a, b\})| = 2.$$

Claim 1. $m \notin \{1, k\}$.

Proof of Claim 1. Assume that $m = 1$. Then $|S \cap (V(C_6^1) - \{a, b\})| = 3$ and

$$\text{(2.2)} \qquad for \;\; any \;\; i \in \{2, 3, \ldots, k\}, \;\; |S \cap (V(C_6^i) - \{a, b\})| = 2.$$

Since b_1 is totally dominated by S, $c_1 \in S$, and since c_1 is totally dominated by S, $S \cap \{b_1, d_1\} \neq \emptyset$. Since a is totally dominated by S, there is an integer i such that $a_i \in S$. If $i \neq 1$, then $a_1 \notin S$. Since c_i is totally dominated by S, $S \cap \{b_i, d_i\} \neq \emptyset$, and since b_i is totally dominated by S, $a_t \in S$ for some integer $t < i$. By continuing this process as seen in Case 1, we obtain that $a_1 \in S$, a contradiction. Thus $i = 1$, and $a_1 \in S$.

Since b is dominated by S, there is an integer j such that $b_j \in S$. Assume that $j \neq 1$. Then $b_1 \notin S$. Since d_j is totally dominated by S, $S \cap \{a_j, c_j\} \neq \emptyset$, and since a_j is totally dominated by S, $b_n \in S$, for some $n > j$. By continuing this process we obtain that $b_k \in S$.

Since a_k is totally dominated by S, $d_k \in S$, and since d_k is totally dominated by S, $S \cap \{a_k, c_k\} \neq \emptyset$. Then $|S \cap (V(C_6^k) - \{a, b\})| = 3 > 2$ contradicting (2.2). Thus $j = 1$ and $b_1 \in S$.

Now $a_1, b_1, c_1 \in S$, and $d_1 \notin S$. Since a_1 is totally dominated by S, there is an integer l such that $b_l \in S$, and since d_l is totally dominated by S, $S \cap \{a_l, c_l\} \neq \emptyset$. Since a_l is totally dominated by S, $b_p \in S$ for some integer p, and by continuing this process we obtain that $b_k \in S$. Since a_k is totally dominated by S, $d_k \in S$, and since d_k is totally dominated by S, $S \cap \{a_k, c_k\} \neq \emptyset$. Thus $|S \cap (V(C_6^k) - \{a, b\})| = 3 > 2$, contradicting (2.2). Thus $m \neq 1$. The proof for $m \neq k$ is similar. $\square$

Since a is totally dominated by S, there is an integer $j \in \{1, 2, \ldots, k\}$ such that $a_j \in S$, and since b is totally dominated by S, there is an integer $l \in \{1, 2, \ldots, k\}$ such that $b_l \in S$. Since c_j is totally dominated by S, $S \cap \{d_j, b_j\} \neq \emptyset$.

We show that $j = m$. If $j \neq m$, then by (2.1), for b_j to be totally dominated by S, there is an integer t such that $t < j$ and $a_t \in S$. If $t > 1$, we do with b_t similarly to b_j, and thus we may assume that $t = 1$, and so $a_1 \in S$. Since b_1 is totally dominated by S, we find that $c_1 \in S$, and since c_1 is dominated by S, $S \cap \{b_1, d_1\} \neq \emptyset$. Then $|S \cap (V(C_6^1) - \{a, b\})| = 3 > 2$, and so $m = 1$. But by Claim 1, $m \notin \{1, m\}$, a contradiction. Thus $j = m$.

Similarly, $l = m$. Thus $j = l = m$. Since b_m is totally dominated by S, $S \cap \{c_m, a_t\} \neq \emptyset$ for some $t < m$, and since a_m is totally dominated by S, $S \cap \{d_m, b_n\} \neq \emptyset$ for some $n > m$. Without loss of generality assume that $c_m \in S$. As before, we can see that $a_1 \in S$. Since b_1 is totally dominated by

S we find that $c_1 \in S$, and since c_1 is totally dominated by S we find that $S \cap \{b_1, d_1\} \neq \emptyset$. Thus $|S \cap (V(C_6^1) - \{a, b\})| = 3 > 2$, a contradiction.

We conclude that $|S| \geq 2k + 2$. On the other hand $\{a, b, a_i, b_i : i = 1, 2, \ldots, k\}$ is a TDS for G, implying that $\gamma_t(G) = 2k + 2$.

Now we show that $\gamma_t(G + e) = 2k$ for all $e \in E(H)$, where H is the complement of G relative to $K_{2k+1,2k+1}$. Since $\gamma_t(G) = 2k + 2$, for any $e \in E(H)$, it is obvious that $\gamma_t(G + e) \geq 2k$. Thus for any $e \in E(H)$, it is sufficient to present a TDS for $G + e$ of cardinality $2k$.

If $e = ac_i$ for some i, then $\{a, c_i, d_j, c_j : j = 1, 2, \ldots, k, j \neq i\}$ is a TDS for $G + e$. If $e = a_ib_i$ for some i, then $\{a_i, b_i, d_j, c_j : j = 1, 2, \ldots, k, j \neq i\}$ is a TDS for $G + e$. If $e = b_ia_j$ for some i, j with $i < j$, then $\{a_i, b_i, a_j, b_j, c_l, d_l : l = 1, 2, \ldots, k, l \neq i, j\}$ is a TDS for $G + e$. If $e = b_ic_l$ for some i, l with $i \neq l$, then $\{b_i, c_l, a, a_i, c_j, d_j : j = 1, 2, \ldots, k, j \neq i, l\}$ is a TDS for $G + e$. If $e = bd_i$ for some i, then $\{b, d_i, c_j, d_j : j = 1, 2, \ldots, k, j \neq i\}$ is a TDS for $G + e$. If $e = a_jd_i$ for some i, j with $i \neq j$, then $\{a_j, d_i, b, b_j, c_l, d_l : l = 1, 2, \ldots, k, l \neq i, j\}$ is a TDS for $G + e$. Finally if $e = d_ic_l$ for some i, l with $i \neq l$, then $\{a, b, d_i, c_l, c_j, d_j : j = 1, 2, \ldots, k, j \neq i, l\}$ is a TDS for $G + e$. Note that the other possibilities for e are similarly verified.

□

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