

HAMILTON-CONNECTED PROPERTIES IN CARTESIAN PRODUCT

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ABSTRACT. In this paper, we investigate a problem of finding natural condition to assure the product of two graphs to be hamilton-connected. We present some sufficient and necessary conditions for $G \square H$ being hamilton-connected when G is a hamilton-connected graph and H is a tree or G is a hamiltonian graph and H is K_2 .

1. Introduction

In this paper, we consider finite simple graphs, and refer to [1] for terms and notations not defined here. Let $G = (V, E)$ be a graph. For any vertex $v \in V$, let $d_G(v)$ denote the degree of v in G , and $\Delta(G)$ denote the maximum degree of G . Let $c(G)$ be the number of components in G . Denote P_m , C_n and $K_{1,j-1}$ to be a path with m vertices ($m \geq 2$), a cycle with n vertices ($n \geq 3$) and a star with j vertices ($j \geq 1$), respectively.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian product of G_1 and G_2 , denoted by $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ such that the vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2 \in V_1$ with $y_1 y_2 \in E_2$, or $y_1 = y_2 \in V_2$ with $x_1 x_2 \in E_1$. It follows the definition that for any $(x, y) \in V(G)$,

$$d_{G_1 \square G_2}(x, y) = d_{G_1}(x) + d_{G_2}(y).$$

For any $y \in V_2$, define G_{1y} to be the graph with vertex set $V_{1y} = \{(x, y) \mid x \in V_1\}$ and edge set $E_{1y} = \{(x_1, y)(x_2, y) \mid x_1 x_2 \in E_1\}$. Similarly, For any $x \in V_1$, define G_{2x} to be the graph with vertex set $V_{2x} = \{(x, y) \mid y \in V_2\}$ and edge set $E_{2x} = \{(x, y_1)(x, y_2) \mid y_1 y_2 \in E_2\}$.

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Note that G_{1y} is isomorphic to graph G_1 , for any $y \in V_2$; and that G_{2x} is isomorphic to graph G_2 for any $x \in V_1$. It is clear that

$$V_{1y} \cap V_{1y'} = \emptyset, E_{1y} \cap E_{1y'} = \emptyset \text{ for } y \neq y';$$

$$V_{2x} \cap V_{2x'} = \emptyset, E_{2x} \cap E_{2x'} = \emptyset \text{ for } x \neq x';$$

$$V_{1y} \cap V_{2x} = \{(x, y)\} \text{ for } x \in V_1; y \in V_2;$$

$$E(G_1 \square G_2) = (\cup_{y \in V_2} E_{1y}) \cup (\cup_{x \in V_1} E_{2x});$$

$$V(G_1 \square G_2) = (\cup_{y \in V_2} V_{1y}) = (\cup_{x \in V_1} V_{2x}).$$

A spanning path(cycle) is called a *Hamilton path(cycle)*. A graph G is *traceable* if it contains a Hamilton path, and *hamiltonian* if it contains a Hamilton cycle. A graph G is *hamilton – connected* if there exists a Hamilton path joining any two different vertices of G .

Let F be a subgraph of a graph G . An *ear* of F in G is a nontrivial path in G whose ends lie in F but whose internal vertices do not.

A graph G is called a *cactus* if it has at least 3 vertices, all cycles of G are vertex-disjoint, maximum degree of G is 3 and all vertices of degree 3 are on a cycle of G .

We denote by \mathcal{E} the class of graphs with following properties:

- (i) any graph $H \in \mathcal{E}$ can be edge-covered by two subgraphs H_C and H_F , such that $H = H_C \cup H_F$, H_C and H_F are edge-disjoint, H_C is an edge disjoint union of cycles C_1, \dots, C_p , and H_F is a forest.
- (ii) there is no vertex in H_C common to more than two cycles among the cycles C_1, \dots, C_p ,
- (iii) $H \in \mathcal{E}$ has at least two vertices.

We call the pair (H_C, H_F) a *cycle – tree covering* of H .

A graph $H \in \mathcal{E}$ satisfying the following:

- (i) for every vertex on exactly one cycle of H_C in the cycle-tree covering of H all its neighbors are either all pendent (vertices of degree one) or all nonpendant, i.e. for such vertex u we have either $d_G^p(u) \geq 0$ and $d_G^{np}(u) = 2$ or $d_G^p(u) = 0$ and $d_G^{np}(u) \geq 2$,
- (ii) a vertex common to exactly two cycles in the cycle-tree covering of H has neighbors on these cycles only,

is called *generalized cactus*. In particular, such cactus is *even* if all its cycles are of even length.

The *generalized b – cactus* is a generalized cactus with every branch vertex (vertices of degree more than 3) on a cycle.

We denote by \mathcal{F} the class of graphs with following properties:

- (i) any graph $G \in \mathcal{F}$ is union of cycles C_1, \dots, C_p ,

- (ii) for two cycles for $C_i, C_{i+1}, |V(C_i \cap C_{i+1})| = l_i, 1 \leq i \leq p - 1, 1 \leq l_i \leq \max[\frac{|V(C_i)|}{2}, \frac{|V(C_{i+1})|}{2}]$
- (iii) at least one of these cycles is odd cycle.

When $p = 2, l_i = l$, graph G is denoted by $\Theta(l, m, n)$, where $|V(C_1)| = n, |V(C_2)| = m$.

Here we mention some related results. Gould in [4] raised a research problem to find natural conditions to assure the product of two graphs to be hamiltonian. Paulraja in [11] gave the sufficient and necessary conditions for the prism over graphs to be hamiltonian. And Lu et.al in [9] present some sufficient and necessary conditions for $G \square H$ being hamiltonian when G is a hamiltonian graph and H is tree.

The followings are some results related with our main Theorem.

Theorem 1.1. [1] *Let S be a set of vertices of a hamiltonian graph G . Then $c(G - S) \leq |S|$.*

Theorem 1.2. [11] *Let G be a graph. The Cartesian product $G \square K_2$ is hamiltonian if and only if G has an even generalized b -cactus as a subgraph.*

Lemma 1.3. [9] *Let C_n be a cycle ($n \geq 3$). For any tree T , if T contains a subdivision of $K_{1,3}^{(n)}$ as a subgraph. Then $G = C_n \square T$ is not traceable, where $K_{1,3}^{(n)}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$.*

Lemma 1.4. [7] *Suppose that m is an odd integer. Then $C_m \square K_2$ is hamilton-connected.*

Note that when m is an even integer, $C_m \square K_2$ is not hamilton-connected. In this paper, we shall investigate the sufficient and necessary conditions for $G \square H$ being hamilton-connected when G is a hamilton-connected graph and H is a tree or G is hamiltonian graph and H is K_2 . Our main theorems are as follows:

Theorem 1.5. *Let G be a hamilton-connected graph, and let T be a tree with maximum degree Δ . Then graph $G \square T$ is hamilton-connected if and only if $\Delta(T) \leq |V(G)| - 1$ and T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph, where $K_{1,3}^{(n)}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$.*

Theorem 1.6. *Let G be a hamiltonian graph. The Cartesian product $G \square K_2$ is hamilton-connected if and only if:*

- (i) G is of odd order, or
- (ii) G is of even order and G contains $\Theta(1, 2k + 1, 2l + 1)$ as a spanning subgraph, where $\Theta(1, 2k + 1, 2l + 1)$ is the graph union of two odd cycles with a common edge.

2. Proof of Theorem 1.5

In this section we will give a proof of Theorem 1.5.

Lemma 2.1. *Let $T = K_{1,m}$ be a star, and let G be a hamilton-connected graph with n vertices ($n \geq 3$). If $m \leq n - 1$, then the graph $H = G \square T$ is hamilton-connected.*

Proof. Let $V(K_{1,m}) = \{y_0, y_1, \dots, y_m\}$, where $d(y_0) = m$ and $d(y_i) = 1$ for $1 \leq i \leq m$. Then $G_{1y_i} \cong G$ for $i = 0, 1, \dots, m$. Particularly, $V(G_{1y_i}) = \{(v_1, y_i), (v_2, y_i), \dots, (v_n, y_i)\}$ for $i = 0, 1, \dots, m$. We shall determine a Hamilton path between any two given vertices in $G \square T$. We distinguish the following cases.

Case 1. Any $(v_l, y_0), (v_f, y_0) \in G_{1y_0}$, $1 \leq l, f \leq n$.

Because graph G is hamilton-connected, there exists a Hamilton path joining any two distinct vertices of the graph G . Let P_0 be a Hamilton path of G_{1y_0} between (v_l, y_0) and (v_f, y_0) . Let $x_1 = v_l$, $x_n = v_f$ and $P_0 = \langle (x_1, y_0), (x_2, y_0), \dots, (x_n, y_0) \rangle$. Let P_i be a Hamilton path of G_{1y_i} between (x_i, y_i) and (x_{i+1}, y_i) . Then

$$\begin{aligned} P = & (P_0 - \{(x_1, y_0)(x_2, y_0), (x_2, y_0)(x_3, y_0), \dots, (x_{m-1}, y_0)(x_m, y_0)\}) \\ & \cup P_1 \cup P_2 \cup \dots \cup P_m \cup \{(x_1, y_0)(x_1, y_1), (x_2, y_0)(x_2, y_1), (x_2, y_0)(x_2, y_2), \\ & (x_3, y_0)(x_3, y_2), (x_3, y_0)(x_3, y_3), \dots, (x_m, y_0)(x_m, y_m)\} \end{aligned}$$

is a Hamilton path between (v_l, y_0) and (v_f, y_0) in H .

Case 2. Any $(v_l, y_0) \in G_{1y_0}$, $(v_f, y_i) \in G_{1y_i}$, $1 \leq l, f \leq n$, $1 \leq i \leq m$.

Let P_1 be a Hamilton path between $(v_{l+1}, y_i), (v_f, y_i)$ in G_{1y_i} . From *Case 1*, we can find a Hamilton path P_2 between (v_l, y_0) and (v_{l+1}, y_0) in $G \square (T - y_i)$. Then

$$P = P_1 \cup P_2 \cup \{(v_{l+1}, y_0)(v_{l+1}, y_i)\}$$

is a Hamilton path between (v_l, y_0) and (v_f, y_i) in H .

Case 3. Any $(v_l, y_i) \in G_{1y_i}$, $(v_f, y_j) \in G_{1y_j}$, $1 \leq l, f \leq n$, $1 \leq i, j \leq m$, $i \neq j$.

Let P_1 be a Hamilton path between (v_l, y_i) and (v_p, y_i) in G_{1y_i} for $1 \leq p \leq m$. Let P_2 be a Hamilton path between (v_f, y_j) and (v_q, y_j) in G_{1y_j} for $1 \leq q \leq m$. From *Case 1*, we can find a Hamilton path P_3 between (v_p, y_0) and (v_q, y_0) in $G \square (T - \{y_i, y_j\})$. Then

$$P = P_1 \cup P_2 \cup \{(v_p, y_0)(v_p, y_i), (v_q, y_0)(v_q, y_j)\}$$

is a Hamilton path between (v_l, y_i) and (v_f, y_j) in H .

Case 4. Any $(v_l, y_i), (v_f, y_i) \in G_{1y_i}$, $1 \leq l, f \leq n$, $1 \leq i \leq m$.

Let Let P_1 be a Hamilton path between (v_l, y_i) and (v_f, y_i) in G_{1y_i} . From *Case 1*, we can find a Hamilton path P_2 between (v_r, y_0) and (v_s, y_0) in $G \square (T - y_i)$ for $\{(v_s, y_i)(v_r, y_i)\} \in P_1$ for $1 \leq r, s \leq n$. Then

$$P = (P_1 - \{(v_r, y_i)(v_s, y_i)\}) \cup \{(v_r, y_i)(v_r, y_0), (v_s, y_i)(v_s, y_0)\} \cup P_2$$

is a Hamilton path between (v_l, y_i) and (v_f, y_i) in H . □

Remark 2.2. By the argument used in Case 1 in the proof of Lemma 2.1, $|E_{1y_0} \cap E(P)| = \emptyset$ if $m = n - 1$. If $n - 1 > m$, then $(x_i, y_0)(x_{i+1}, y_0) \in E(P)$ for $m \leq i \leq (n - 1)$, that is, $|E_{1y_0} \cap E(P)| = n - m - 1$. If $n - 1 < m$, there exists no Hamilton path between $(v_l, y_0), (v_f, y_0) \in G_{1y_0}$.

Corollary 2.3. Let G be a hamilton-connected graph with n vertices ($n \geq 3$). Then the graph $H = G \square K_{1,n}$ is not hamilton-connected.

Recall that $K_{1,3}^{(n)}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$. Note that $\Delta(K_{1,3}^{(n)}) = n + 1$. Since a hamilton-connected graph is also hamiltonian, by Lemma 1.3 in [9], we have the following corollary.

Corollary 2.4. Let G be a hamilton-connected graph with n vertices ($n \geq 3$). If T contains a subdivision of $K_{1,3}^{(n)}$ as a subgraph, then the graph $H = G \square T$ is not hamilton-connected.

Proof of Theorem 1.5.

Let $H = G \square T$ be a hamilton-connected graph and $\Delta(T) \geq n + 1$, where $|V(G)| = n$. If there exists $y \in V(T)$ such that $d_T(y) \geq n + 1$, then $c(H - G_{1y}) = c(T - y) = d_T(y) \geq n + 1$. By Theorem 1.1, $H = G \square T$ is not hamiltonian and hence is not hamilton-connected, a contradiction. If $\Delta(T) = V(G) = n$, by Corollary 2.3, H is not hamilton-connected. Therefore $\Delta(T) \leq n - 1$. By Corollary 2.4, T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph.

So it suffices to show that if $\Delta(T) \leq n - 1$ and T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph, then $H = G \square T$ is a hamilton-connected graph. If T is a star, then it follows from Lemma 2.1. Therefore we may assume that T is not a star. By way of contradiction, let T be a tree with minimal number of vertices such that $\Delta(T) \leq n - 1$, T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph and $H = G \square T$ is not hamilton-connected.

Note that T can be viewed as a graph obtained from finite stars T_1, T_2, \dots, T_k by connecting their centers with edges and there exists such a star T_i that is connected to the other stars with only one edge. Without lose of generality, we may assume that T_1 is only connected to T_2 . Let y_i the center of T_i ($i = 1, 2$). Since $T - T_1$ is also a tree and $|V(T - T_1)| \leq |V(T)|$, $G \square (T - T_1)$ is hamilton-connected. Since $\Delta(T_1) \leq n - 1$, by Lemma 2.1, $G \square T_1$ is hamilton-connected.

Now we shall construct a Hamiltonian path between any two distinct vertices of $H = G \square T$, and then obtain a contradiction.

Case 1. Any $(v_l, y_i), (v_f, y_j) \in G \square T_1, 1 \leq l, f \leq n$.

Let P_1 be a Hamilton path between (v_l, y_i) and (v_f, y_j) in $G \square T_1$. Since $d_{T_1}(y_1) \leq n - 2$, at least one edge of G_{1y_1} lies in P_1 . By Remark 2.2, we may assume $(v_1, y_1)(v_2, y_1) \in E_{1y_1} \cap E(P_1)$. Then there exists a Hamilton path P_2 between $(v_1, y_2)(v_2, y_2)$ in $G \square (T - T_1)$. Hence

$$P = P_2 \cup (P_1 - \{(v_1, y_1)(v_2, y_1)\}) \cup \{(v_1, y_1)(v_1, y_2), (v_2, y_1)(v_2, y_2)\}$$

is a Hamilton path between (v_l, y_i) and (v_f, y_j) in $G \square T$.

Case 2. Any $(v_l, y_i), (v_f, y_j) \in G \square (T - T_1)$, $1 \leq l, f \leq n$.

The proof of this case is similar to the proof of Case 1. So it is omitted.

Case 3. Any $(v_l, y_i) \in G \square T_1$, $(v_f, y_j) \in G \square (T - T_1)$, $1 \leq l, f \leq n$.

Let P_1 be a Hamilton path between (v_l, y_i) and (v_t, y_1) in $G \square T_1$. Let P_2 be a Hamilton path between (v_t, y_2) and (v_f, y_j) in $G \square (T - T_1)$. Then

$$P = P_1 \cup P_2 \cup \{(v_t, y_1)(v_t, y_2)\}$$

is a Hamilton path between (v_l, y_i) and (v_f, y_j) in $G \square T$.

3. Proof of Theorem 1.6

In this section we will give the sufficient and necessary condition for $G \square K_2$ being hamilton-connected. We know that prism over odd cycle is hamilton-connected[7], but the prism over even cycle is not hamilton-connected. We now can consider the case when G contains $\Theta(1, 2k + 1, 2l + 1)$ or $\Theta(1, 2k + 1, 2l + 1)$ as a spanning subgraph. Note that G is a hamiltonian graph with even order.

Lemma 3.1. Let $V(P_m) = \{x_1, x_2, \dots, x_m\}$, and let X, Y be the bipartite partition of bipartite graph $G = P_m \square K_2$. Then there exists Hamilton path joining any two vertices $(x_e, y_i) \in X$ and $(x_f, y_j) \in Y$ for $1 \leq e, f \leq m$, $1 \leq i, j \leq 2$, but no Hamilton path joining $(x_l, y_1) \in X$ and $(x_l, y_2) \in Y$ for $1 < l < m$.

Proof. Let $V(P_m) = \{x_1, x_2, \dots, x_m\}$, $V(K_2) = \{y_1, y_2\}$, and $V(P_m \square K_2) = \{(x_1, y_1), (x_2, y_1), \dots, (x_m, y_1), (x_1, y_2), (x_2, y_2), \dots, (x_m, y_2)\}$. We can see that $P_m \square K_2$ is a bipartite graph. Let X and Y be the bipartite partition of the G . We will show that for any $(x_e, y_i) \in X$ and $(x_f, y_j) \in Y$ for $1 \leq e, f \leq m$, $1 \leq i, j \leq 2$ there exists a Hamilton path joining them, but no Hamilton path joining $(x_l, y_1) \in X$ and $(x_l, y_2) \in Y$ for $1 < l < m$.

Obviously, there exists Hamilton path between (x_1, y_1) and (x_1, y_2) or (x_m, y_1) and (x_m, y_2) in G . But no Hamilton path between $(x_l, y_1) \in X$ and $(x_l, y_2) \in Y$ for $1 < l < m$.

Now consider any $(x_e, y_i) \in X$ and $(x_f, y_j) \in Y$, where $e \neq f$.

By induction on m . suppose that it is true for $P_k \square K_2$ with $k < m$. Let $P_1 = \langle x_1, x_2, \dots, x_{f-1} \rangle$, $P_2 = \langle x_f, x_{f+1}, \dots, x_m \rangle$. We may assume $x_e \in V(P_1)$. By the induction hypothesis, there is a Hamilton path P' between (x_e, y_i) and (x_{f-1}, y_{j+1}) in $P_1 \square K_2$ and there exists Hamilton a path P'' between (x_f, y_j) and (x_f, y_{j+1}) , where $(x_f, y_j)(x_f, y_{j+1}) \in E(P_m \square K_2)$. Then

$$P = P' \cup P'' \cup \{(x_f, y_{j+1})(x_{f-1}, y_{j+1})\}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_j) in G . □

Proposition 3.2. Let $G = \Theta(1, 2k, 2l)$. Then $H = G \square K_2$ is not hamilton-connected.

Proof. Let C_1 and C_2 be two even cycles, and let $V(C_1) = \{x_1, x_2, \dots, x_{2k}\}$, $V(C_2) = \{x_1, x_2, x'_3, \dots, x'_{2l}\}$, $E(C_1 \cap C_2) = \{x_1x_2\}$, $V(K_2) = \{y_1, y_2\}$.

We suppose that $G \square K_2$ is hamilton-connected. There exists a Hamilton path P between (x_1, y_1) and (x_2, y_2) . Let $C = G \setminus \{x_1x_2\}$, and note that C is an cycle with even order. Then $C \square K_2$ is not hamilton-connected, and there exists no Hamilton path between (x_1, y_1) and (x_2, y_2) in $C \square K_2$. So edges $(x_1, y_1)(x_2, y_1)$ or $(x_1, y_2)(x_2, y_2)$ must be contained in P , and one of them must be the first or last edge of P .

Now let $P_1 = \langle x_3, x_4, \dots, x_{2k} \rangle$, and let $P_2 = \langle x'_3, x'_4, \dots, x'_{2l} \rangle$. Say edge $(x_1, y_1)(x_2, y_1)$ is first edge of P . From the argument earlier, there must exist Hamilton path between (x_3, y_1) and (x_{2k}, y_2) in $P_1 \square K_2$. Because (x_3, y_1) and (x_{2k}, y_2) is in same partite in $P_1 \square K_2$, by the Lemma 3.1, there exists no Hamilton path joining (x_3, y_1) and (x_{2k}, y_2) , contradiction. \square

Lemma 3.3. *Let $G = \Theta(1, 2k + 1, 2l + 1)$. Then $H = G \square K_2$ is hamilton-connected.*

Proof. Let C_1 and C_2 be two odd cycles, and $V(C_1) = \{x_1, x_2, \dots, x_{2k+1}\}$, $V(C_2) = \{x_1, x_2, x'_3, \dots, x'_{2l+1}\}$, $E(C_1 \cap C_2) = \{x_1x_2\}$, $V(K_2) = \{y_1, y_2\}$.

Case 1. *Any $(x_e, y_i), (x_f, y_j) \in V(C_1 \square K_2)$, $1 \leq e, f \leq 2k + 1$, $1 \leq i, j \leq 2$.*

Let $P_1 = \langle x_3, \dots, x_e \rangle$, $P_2 = \langle x_{e+1}, \dots, x_f, \dots, x_{2k+1} \rangle$.

Let X_1, Y_1 be bipartite partition of $P_1 \square K_2$; X_2, Y_2 be bipartite partition of $P_2 \square K_2$. Without loss of generality, we may assume that $(x_3, y_1) \in X_1$, $(x_3, y_2) \in Y_1$ and $(x_{2k+1}, y_1) \in X_2$, $(x_{2k+1}, y_2) \in Y_2$.

If $(x_e, y_i) \in X_1$ (if $(x_e, y_i) \in Y_1$ we can choose the (x_3, y_1)), by Lemma 3.1, we can find a Hamilton path P_3 between (x_e, y_i) and (x_3, y_2) in $P_1 \square K_2$. Similarly if $(x_f, y_j) \in X_2$ (if $(x_f, y_j) \in Y_2$ we can choose the (x_{2k+1}, y_1)), we can find a Hamilton path P_4 between (x_f, y_j) and (x_{2k+1}, y_2) in $P_2 \square K_2$.

By Lemma 1.4, there is a Hamilton path P_5 between (x_1, y_2) and (x_2, y_2) in $C_2 \square K_2$. Then

$$P = P_3 \cup P_4 \cup P_5 \cup \{(x_1, y_2)(x_{2k+1}, y_2), (x_2, y_2)(x_3, y_2)\}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_j) in $G \square K_2$.

Case 2. *Any $(x'_e, y_i)(x'_f, y_j) \in V(C_2 \square K_2)$, $3 \leq e, f \leq 2l + 1$, $1 \leq i, j \leq 2$.*

The proof of this case is similar to that of Case 1. So it is omitted.

Case 3. *Any $(x_e, y_i) \in V(C_1 \square K_2)$, $(x'_f, y_j) \in V(C_2 \square K_2)$, $1 \leq e \leq 2k+1$, $3 \leq f \leq 2l+1$, $1 \leq i, j \leq 2$.*

Let $P_{2l-2} = \langle x'_3, x'_4, \dots, x'_{2l+1} \rangle$ and let X, Y be bipartite partition of $P_{2l-2} \square K_2$. Consider the vertices $(x'_{2l+1}, y_1), (x'_{2l+1}, y_2)$ or $(x'_3, y_1), (x'_3, y_2)$ that are in different partite sets. Say the former, we can assume $(x'_{2l+1}, y_1) \in X, (x'_{2l+1}, y_2) \in Y$.

Without loss of generality, we may assume $(x'_f, y_j) \in X$, by Lemma 3.1, there is a Hamilton path P_1 between (x'_f, y_j) and (x'_{2l+1}, y_2) in $P_{2l-2} \square K_2$. Also there is a Hamilton path P_2 between (x_1, y_2) and (x_e, y_i) in $C_1 \square K_2$. Then

$$P = P_1 \cup P_2 \cup \{(x'_{2l+1}, y_2)(x_1, y_2)\}$$

is a Hamilton path between (x_e, y_i) and (x'_f, y_j) in $G \square K_2$. □

Proposition 3.4. *Suppose that $H = C_n \square K_2$ is a hamilton-connected graph. Then n is an odd integer.*

Proof. Let $V(C_n) = \{x_1, x_2, \dots, x_n\}$, $V(K_2) = \{y_1, y_2\}$. Let P_1 be a Hamilton path between (x_1, y_1) and (x_2, y_2) , then $P_1 = \langle (x_1, y_1), (x_1, y_2), (x_n, y_1), (x_n, y_2), (x_{n-1}, y_1), \dots, (x_3, y_1), (x_2, y_1), (x_2, y_2) \rangle$ and P_1 contains every pillar of the prism. Let $X = V(G_{1y_1})$, then $|\partial(X)| = 2t + 1$, where $\partial(X)$ is set of edges with one end in X .

Since $H = C_n \square K_2$ is an odd graph, and by $|\partial(X)| + 2e(X) = \sum_{v \in X} d(v)$, we have $|\partial(X)| = |X| \pmod{2}$, so $|X| = |V(G_{1y_1})| = n$ is an odd integer. □

Conclude all the above, we can obtain the sufficient and necessary condition for prism over hamiltonian graph being hamilton-connected, that is Theorem 1.6. Therefore we can give a sufficient condition for $G \square P_m$ being hamilton-connected.

Theorem 3.5. *Let m be an integer, where $m \geq 2$. Then $H = G \square P_m$ is hamiltonian-connected, if:*

- (i) G is a hamiltonian graph with odd order, or
- (ii) G is a hamiltonian graph with even order, and G contains $\Theta(1, 2k + 1, 2l + 1)$ as a spanning subgraph, where $\Theta(1, 2k + 1, 2l + 1)$ is the graph union of two odd cycles with a common edge.

Proof. Let C_n be a Hamilton cycle of G , $V(C_n) = \{x_1, x_2, \dots, x_n\}$, $V(P_m) = \{y_1, y_2, \dots, y_m\}$.

We prove it by induction on m .

When $m = 2$, by Theorem 1.6 it is right. Now we suppose that $G \square P_{m-1}$ is hamilton-connected when G satisfies those conditions. We shall show that $H = G \square P_m$ is hamilton-connected.

Case 1. *Any $(x_e, y_i), (x_f, y_j) \in V(C_n \square P_{m-1})$, $1 \leq e, f \leq n$, $1 \leq i, j \leq m - 1$.*

By induction hypothesis there is a Hamilton path P_1 between (x_e, y_i) and (x_f, y_j) in $G \square P_{m-1}$. Without loss of generality, assume $(x_i, y_{m-1})(x_{i+1}, y_{m-1}) \in E(C_n \square P_{m-1} \cap P_1)$. Then

$$P = (P_1 \setminus \{(x_i, y_{m-1})(x_{i+1}, y_{m-1})\}) \cup \{(x_i, y_{m-1})(x_i, y_m), (x_{i+1}, y_{m-1})(x_{i+1}, y_m)\} \cup \{(x_i, y_m)(x_{i-1}, y_m), (x_{i-1}, y_m)(x_{i-2}, y_m), \dots, (x_{i+2}, y_m)(x_{i+1}, y_m)\}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_j) in H .

Case 2. *Any $(x_e, y_i) \in V(C_n \square P_{m-1})$, $(x_f, y_m) \in V(G_{1y_m})$, $1 \leq e, f \leq n$, $1 \leq i \leq m - 1$,*

By induction hypothesis there is a Hamilton path P_1 between (x_e, y_i) and (x_{f+1}, y_{m-1}) in $G \square P_{m-1}$. Then

$$P = P_1 \cup \{(x_{f+1}, y_{m-1})(x_{f+1}, y_m)\} \cup \{(x_{f+1}, y_m)(x_{f+2}, y_m), (x_{f+2}, y_m)(x_{f+3}, y_m), \dots, (x_{f-1}, y_m)(x_f, y_m)\}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_m) in H .

Case 3. Any $(x_e, y_m), (x_f, y_m) \in V(G_{1y_m}), 1 \leq e, f \leq n$.

Let $P_1 = \langle (x_{e+1}, y_m), (x_{e+2}, y_m), \dots, (x_f, y_m) \rangle$, $P_2 = \langle (x_{f+1}, y_m), (x_{f+2}, y_m), \dots, (x_e, y_m) \rangle$. By induction hypothesis there is a Hamilton path P_3 between (x_{e+1}, y_{m-1}) and (x_{f+1}, y_{m-1}) in $G \square P_{m-1}$. Then

$$P = P_3 \cup \{(x_{e+1}, y_{m-1})(x_{e+1}, y_m), (x_{f+1}, y_{m-1})(x_{f+1}, y_m)\} \cup P_1 \cup P_2$$

is a Hamilton path between (x_e, y_m) and (x_f, y_m) in H . \square

By the proof of the Lemma 3.3 we can obtain the following corollary.

Corollary 3.6. Let G be a graph such that $G \square K_2$ is a hamilton-connected graph. If $H = G \cup P$ with $|E(G \cap P)| = 1$, where P is an ear of G , then $H \square K_2$ is hamilton-connected.

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