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HAMILTON-CONNECTED PROPERTIES IN CARTESIAN PRODUCT

R. HOSHUR AND E. VUMAR*

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Abstract. In this paper, we investigate a problem of finding natural condition to assure the product of two graphs to be hamilton-connected. We present some sufficient and necessary conditions for $G\square H$ being hamilton-connected when G is a hamilton-connected graph and H is a tree or G is a hamiltonian graph and H is K_2.

1. Introduction

In this paper, we consider finite simple graphs, and refer to [1] for terms and notations not defined here. Let $G = (V, E)$ be a graph. For any vertex $v \in V$, let $d_G(v)$ denote the degree of v in G, and $\triangle(G)$ denote the maximum degree of G. Let $c(G)$ be the number of components in G. Denote P_m, C_n and $K_{1,j-1}$ to be a path with m vertices ($m \geq 2$), a cycle with n vertices ($n \geq 3$) and a star with j vertices ($j \geq 1$), respectively.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian product of G_1 and G_2, denoted by $G_1\square G_2$, is the graph with vertex set $V_1 \times V_2$ such that the vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2 \in V_1$ with $y_1y_2 \in E_2$, or $y_1 = y_2 \in V_2$ with $x_1x_2 \in E_1$. It follows the definition that for any $(x, y) \in V(G)$,

$$d_{G_1\square G_2}(x, y) = d_{G_1}(x) + d_{G_1}(y).$$

For any $y \in V_2$, define G_{1y} to be the graph with vertex set $V_{1y} = \{(x, y) \mid x \in V_1\}$ and edge set $E_{1y} = \{(x_1, y)(x_2, y) \mid x_1x_2 \in E_1\}$. Similarly, For any $x \in V_1$, define G_{2x} to be the graph with vertex set $V_{2x} = \{(x, y) \mid x \in V_2\}$ and edge set $E_{2x} = \{(x, y_1)(x, y_2) \mid y_1y_2 \in E_2\}$.

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*Corresponding author.

Note that G_{1y} is isomorphic to graph G_1, for any $y \in V_2$; and that G_{2x} is isomorphic to graph G_2 for any $x \in V_1$. It is clear that

$$V_{1y} \cap V_{1y'} = \emptyset,\ E_{1y} \cap E_{1y'} = \emptyset\ for\ y \neq y';$$

$$V_{2x} \cap V_{2x'} = \emptyset,\ E_{2x} \cap E_{2x'} = \emptyset\ for\ x \neq x';$$

$$V_{1y} \cap V_{2x} = \{(x, y)\}\ for\ x \in V_1;\ y \in V_2;$$

$$E(G_1 \Box G_2) = (\cup_{y \in V_2} E_{1y}) \cup (\cup_{x \in V_1} E_{2x});$$

$$V(G_1 \Box G_2) = (\cup_{y \in V_2} V_{1y}) = (\cup_{x \in V_1} V_{2x}).$$

A spanning path(cycle) is called a *Hamilton path*(*cycle*). A graph G is *traceable* if it contains a Hamilton path, and *hamiltonian* if it contains a Hamilton cycle. A graph G is $hamilton-connected$ if there exists a Hamilton path joining any two different vertices of G.

Let F be a subgraph of a graph G. An *ear* of F in G is a nontrivial path in G whose ends lie in F but whose internal vertices do not.

A graph G is called a *cactus* if it has at least 3 vertices, all cycles of G are vertex-disjoint, maximum degree of G is 3 and all vertices of degree 3 are on a cycle of G.

We denote by $\mathscr{E}$ the class of graphs with following properties:

(i) any graph $H \in \mathscr{E}$ can be edge-covered by two subgraphs H_C and H_F, such that $H = H_C \cup H_F$, H_C and H_F are edge-disjoint, H_C is an edge disjoint union of cycles $C_1, \cdots, C_p$, and H_F is a forest.

(ii) there is no vertex in H_C common to more than two cycles among the cycles $C_1, \cdots, C_p$,

(iii) $H \in \mathscr{E}$ has at least two vertices.

We call the pair (H_c, H_F) a $cycle-tree\ covering$ of H.

A graph $H \in \varepsilon$ satisfying the following:

(i) for every vertex on exactly one cycle of H_C in the cycle-tree covering of H all its neighbors are either all pendent (vertices of degree one) or all nonpendant, i.e. for such vertex u we have either $d_G^p(u) \geq 0$ and $d_G^{np}(u) = 2$ or $d_G^p(u) = 0$ and $d_G^{np}(u) \geq 2$,

(ii) a vertex common to exactly two cycles in the cycle-tree covering of H has neighbors on these cycles only,

is called *generalized cactus*. In particular, such cactus is *even* if all its cycles are of even length.

The *generalized* $b-cactus$ is a generalized cactus with every branch vertex (vertices of degree more than 3) on a cycle.

We denote by $\mathscr{F}$ the class of graphs with following properties:

(i) any graph $G \in \mathscr{F}$ is union of cycles $C_1, \cdots, C_p$,

(ii) for two cycles for $C_i,\ C_{i+1}$, $|V(C_i \cap C_{i+1})| = l_i$, $1 \le i \le p-1$, $1 \le l_i \le max[\frac{|V(C_i)|}{2}, \frac{|V(C_{i+1})|}{2}]$

(iii) at least one of these cycles is odd cycle.

When $p = 2,\ l_i = l$, graph G is denoted by $\Theta(l, m, n)$, where $|V(C_1)| = n,\ |V(C_2)| = m$.

Here we mention some related results. Gould in [4] raised a research problem to find natural conditions to assure the product of two graphs to be hamiltonian. Paulraja in [11] gave the sufficient and necessary conditions for the prism over graphs to be hamiltonian. And Lu et.al in [9] present some sufficient and necessary conditions for $G\Box H$ being hamiltonian when G is a hamiltonian graph and H is tree.

The followings are some results related with our main Theorem.

Theorem 1.1. [1] *Let S be a set of vertices of a hamiltonian graph G. Then $c(G - S) \le |S|$.*

Theorem 1.2. [11] *Let G be a graph. The Cartesian product $G\Box K_2$ is hamiltonian if and only if G has an even generalized b-cactus as a subgraph.*

Lemma 1.3. [9] *Let C_n be a cycle ($n \ge 3$). For any tree T, if T contains a subdivision of $K_{1,3}^{(n)}$ as a subgraph. Then $G = C_n\Box T$ is not traceable, where $K_{1,3}^{(n)}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$.*

Lemma 1.4. [7] *Suppose that m is an odd integer. Then $C_m\Box K_2$ is hamilton-connected.*

Note that when m is an even integer, $C_m\Box K_2$ is not hamilton-connected. In this paper, we shall investigate the sufficient and necessary conditions for $G\Box H$ being hamilton-connected when G is a hamilton-connected graph and H is a tree or G is hamiltonian graph and H is K_2. Our main theorems are as follows:

Theorem 1.5. *Let G be a hamilton-connected graph, and let T be a tree with maximum degree $\triangle$. Then graph $G\Box T$ is hamilton-connected if and only if $\triangle(T) \le |V(G)| - 1$ and T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph, where $K_{1,3}^{(n)}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$.*

Theorem 1.6. *Let G be a hamiltonian graph. The Cartesian product $G\Box K_2$ is hamilton-connected if and only if:*

(i) *G is of odd order, or*

(ii) *G is of even order and G contains $\Theta(1, 2k+1, 2l+1)$ as a spanning subgraph, where $\Theta(1, 2k+1, 2l+1)$ is the graph union of two odd cycles with a common edge.*

2. **Proof of Theorem 1.5**

In this section we will give a proof of Theorem 1.5.

Lemma 2.1. *Let $T = K_{1,m}$ be a star, and let G be a hamilton-connected graph with n vertices ($n \ge 3$). If $m \le n - 1$, then the graph $H = G\Box T$ is hamilton-connected.*

Proof. Let $V(K_{1,m}) = \{y_0, y_1, \cdots, y_m\}$, where $d(y_0) = m$ and $d(y_i) = 1$ for $1 \le i \le m$. Then $G_{1y_i} \cong G$ for $i = 0, 1, \cdots, m$. Particularly, $V(G_{1y_i}) = \{(v_1, y_i), (v_2, y_i), \cdots, (v_n, y_i)\}$ for $i = 0, 1, \cdots, m$. We shall determine a Hamilton path between any two given vertices in $G \square T$. We distinguish the following cases.

Case 1. *Any* $(v_l, y_0), (v_f, y_0) \in G_{1y_0},\ 1 \le l, f \le n.$

Because graph G is hamilton-connected, there exists a Hamilton path joining any two distinct vertices of the graph G. Let P_0 be a Hamilton path of G_{1y_0} between (v_l, y_0) and (v_f, y_0). Let $x_1 = v_l$, $x_n = v_f$ and $P_0 = \langle (x_1, y_0), (x_2, y_0), \cdots, (x_n, y_0) \rangle$. Let P_i be a Hamilton path of G_{1y_i} between (x_i, y_i) and (x_{i+1}, y_i).Then

$$\begin{aligned} P &= (P_0 - \{(x_1, y_0)(x_2, y_0),\ (x_2, y_0)(x_3, y_0),\ \cdots,\ (x_{m-1}, y_0)(x_m, y_0)\ \}) \\ &\quad \cup P_1 \cup P_2 \cup \cdots P_m \cup \{(x_1, y_0)(x_1, y_1), (x_2, y_0)(x_2, y_1), (x_2, y_0)(x_2, y_2), \\ &\quad (x_3, y_0)(x_3, y_2), (x_3, y_0)(x_3, y_3), \cdots, (x_m, y_0)(x_m, y_m)\} \end{aligned}$$

is a Hamilton path between (v_l, y_0) and (v_f, y_0) in H.

Case 2. *Any* $(v_l, y_0) \in G_{1y_0},\ (v_f, y_i) \in G_{1y_i},\ 1 \le l, f \le n,\ 1 \le i \le m.$

Let P_1 be a Hamilton path between $(v_{l+1}, y_i), (v_f, y_i)$ in G_{1y_i}. From *Case* 1, we can find a Hamilton path P_2 between (v_l, y_0) and (v_{l+1}, y_0) in $G \square (T - y_i)$. Then

$$P = P_1 \cup P_2 \cup \{(v_{l+1}, y_0)(v_{l+1}, y_i)\}$$

is a Hamilton path between (v_l, y_0) and (v_f, y_i) in H.

Case 3. *Any* $(v_l, y_i) \in G_{1y_i},\ (v_f, y_j) \in G_{1y_j},\ 1 \le l, f \le n,\ 1 \le i, j \le m,\ i \ne j.$

Let P_1 be a Hamilton path between (v_l, y_i) and (v_p, y_i) in $G_1 y_i$ for $1 \le p \le m$. Let P_2 be a Hamilton path between (v_f, y_j) and (v_q, y_j) in $G_1 y_j$ for $1 \le q \le m$. From *Case* 1, we can find a Hamilton path P_3 between (v_p, y_0) and (v_q, y_0) in $G \square (T - \{y_i, y_j\})$. Then

$$P = P_1 \cup P_2 \cup \{(v_p, y_0)(v_p, y_i), (v_q, y_o)(v_q, y_j)\}$$

is a Hamilton path between (v_l, y_i) and (v_f, y_j) in H.

Case 4. *Any* $(v_l, y_i),\ (v_f, y_i) \in G_{1y_i},\ 1 \le l, f \le n,\ 1 \le i \le m.$

Let Let P_1 be a Hamilton path between (v_l, y_i) and (v_f, y_i) in $G_1 y_i$. From *Case* 1, we can find a Hamilton path P_2 between (v_r, y_0) and (v_s, y_0) in $G \square (T - y_i)$ for $\{(v_s, y_i)(v_r, y_i)\} \in P_1$ for $1 \le r, s \le n$. Then

$$P = (P_1 - \{(v_r, y_i)(v_s, y_i)\}) \cup \{(v_r, y_i)(v_r, y_0), (v_s, y_i)(v_s, y_0)\} \cup P_2$$

is a Hamilton path between (v_l, y_i) and (v_f, y_i) in H. $\square$

Remark 2.2. *By the argument used in Case 1 in the proof of Lemma 2.1,* $|E_{1y_0} \cap E(P)| = \emptyset$ *if* $m = n-1$*. If* $n-1 > m$*, then* $(x_i, y_0)(x_{i+1}, y_0) \in E(P)$ *for* $m \le i \le (n-1)$*, that is,* $|E_{1y_0} \cap E(P)| = n-m-1$*. If* $n-1 < m$*, there exists no Hamilton path between* $(v_l, y_0), (v_f, y_0) \in G_{1y_0}$*.*

Corollary 2.3. *Let* G *be a hamilton-connected graph with* n *vertices* $(n \ge 3)$*. Then the graph* $H = G\square K_{1,n}$ *is not hamilton-connected.*

Recall that $K_{1,3}^{(n)}$ is the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$. Note that $\triangle(K_{1,3}^{(n)}) = n+1$. Since a hamilton-connected graph is also hamiltonian, by Lemma 1.3 in [9], we have the following corollary.

Corollary 2.4. *Let* G *be a hamilton-connected graph with* n *vertices* $(n \ge 3)$*. If* T *contains a subdivision of* $K_{1,3}^{(n)}$ *as a subgraph, then the graph* $H = G\square T$ *is not hamilton-connected.*

Proof of Theorem 1.5.

Let $H = G\square T$ be a hamilton-connected graph and $\triangle(T) \ge n+1$, where $|V(G)| = n$. If there exists $y \in V(T)$ such that $d_T(y) \ge n+1$, then $c(H - G_{1y}) = c(T-y) = d_T(y) \ge n+1$. By Theorem 1.1, $H = G\square T$ is not hamiltonian and hence is not hamilton-connected, a contradiction. If $\triangle(T) = V(G) = n$, by Corollary 2.3, H is not hamilton-connected. Therefore $\triangle(T) \le n-1$. By Corollary 2.4, T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph.

So it suffices to show that if $\triangle(T) \le n-1$ and T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph, then $H = G\square T$ is a hamilton-connected graph. If T is a star, then it follows from Lemma 2.1. Therefore we may assume that T is not a star. By way of contradiction, let T be a tree with minimal number of vertices such that $\triangle(T) \le n-1$, T contains no subdivision of $K_{1,3}^{(n)}$ as a subgraph and $H = G\square T$ is not hamilton-connected.

Note that T can be viewed as a graph obtained from finite stars $T_1, T_2, \cdots, T_k$ by connecting their centers with edges and there exists such a star T_i that is connected to the other stars with only one edge. Without lose of generality, we may assume that T_1 is only connected to T_2. Let y_i the center of T_i $(i = 1, 2)$. Since $T - T_1$ is also a tree and $|V(T-T_1)| \le |V(T)|$, $G\square(T-T_1)$ is hamilton-connected. Since $\triangle(T_1) \le n-1$, by Lemma 2.1, $G\square T_1$ is hamilton-connected.

Now we shall construct a Hamiltonian path between any two distinct vertices of $H = G\square T$, and then obtain a contradiction.

Case 1. *Any* $(v_l, y_i), (v_f, y_j) \in G\square T_1,\ 1 \le l, f \le n$*.*

Let P_1 be a Hamilton path between (v_l, y_i) and (v_f, y_j) in $G\square T_1$. Since $d_{T_1}(y_1) \le n-2$, at least one edge of G_{1y_1} lies in P_1. By Remark 2.2, we may assume $(v_1, y_1)(v_2, y_1) \in E_{1y_1} \cap E(P_1)$. Then there exists a Hamilton path P_2 between $(v_1, y_2)(v_2, y_2)$ in $G\square(T-T_1)$. Hence

$$P = P_2 \cup (P_1 - \{(v_1, y_1)(v_2, y_1)\}) \cup \{(v_1, y_1)(v_1, y_2), (v_2, y_1)(v_2, y_2)\}$$

is a Hamilton path between (v_l, y_i) and (v_f, y_j) in $G\square T$.

Case 2. *Any* $(v_l, y_i), (v_f, y_j) \in G\Box(T - T_1),\ 1 \leq l, f \leq n.$

The proof of this case is similar to the proof of Case 1. So it is omitted.

Case 3. *Any* $(v_l, y_i) \in G\Box T_1,\ (v_f, y_j) \in G\Box(T - T_1),\ 1 \leq l, f \leq n.$

Let P_1 be a Hamilton path between (v_l, y_i) and (v_t, y_1) in $G\Box T_1$. Let P_2 be a Hamilton path between (v_t, y_2) and (v_f, y_j) in $G\Box(T - T_1)$. Then

$$P = P_1 \cup P_2 \cup \{(v_t, y_1)(v_t, y_2)\}$$

is a Hamilton path between (v_l, y_i) and (v_f, y_j) in $G\Box T$.

3. Proof of Theorem 1.6

In this section we will give the sufficient and necessary condition for $G\Box K_2$ being hamilton-connected. We know that prism over odd cycle is hamilton-connected[7], but the prism over even cycle is not hamilton-connected. We now can consider the case when G contains $\Theta(1, 2k+1, 2l+1)$ or $\Theta(1, 2k+1, 2l+1)$ as a spanning subgraph. Note that G a is hamiltonian graph with even order.

Lemma 3.1. *Let* $V(P_m) = \{x_1, x_2, \cdots, x_m\}$*, and let* X, Y *be the bipartite partition of bipartite graph* $G = P_m\Box K_2$*. Then there exists Hamilton path joining any two vertices* $(x_e, y_i) \in X$ *and* $(x_f, y_j) \in Y$ *for* $1 \leq e, f \leq m,\ 1 \leq i, j \leq 2$*, but no Hamilton path joining* $(x_l, y_1) \in X$ *and* $(x_l, y_2) \in Y$ *for* $1 < l < m$*.*

Proof. Let $V(P_m) = \{x_1, x_2, \cdots, x_m\}$, $V(K_2) = \{y_1, y_2\}$, and $V(P_m\Box K_2) = \{(x_1, y_1), (x_2, y_1), \cdots, (x_m, y_1), (x_1, y_2), (x_2, y_2), \cdots, (x_m, y_2)\}$. We can see that $P_m\Box K_2$ is a bipartite graph. Let X and Y be the bipartite partition of the G. We will show that for any $(x_e, y_i) \in X$ and $(x_f, y_j) \in Y$ for $1 \leq e, f \leq m$, $1 \leq i, j \leq 2$ there exists a Hamilton path joining them, but no Hamilton path joining $(x_l, y_1) \in X$ and $(x_l, y_2) \in Y$ for $1 < l < m$.

Obviously, there exists Hamilton path between (x_1, y_1) and (x_1, y_2) or (x_m, y_1) and (x_m, y_2) in G. But no Hamilton path between $(x_l, y_1) \in X$ and $(x_l, y_2) \in Y$ for $1 < l < m$.

Now consider any $(x_e, y_i) \in X$ and $(x_f, y_j) \in Y$, where $e \neq f$.

By induction on m. suppose that it is true for $P_k\Box K_2$ with $k < m$. Let $P_1 = \langle x_1, x_2, \cdots, x_{f-1}\rangle$, $P_2 = \langle x_f, x_{f+1}, \cdots, x_m\rangle$. We may assume $x_e \in V(P_1)$. By the induction hypothesis, there is a Hamilton path P' between (x_e, y_i) and (x_{f-1}, y_{j+1}) in $P_1\Box K_2$ and there exists Hamilton a path P'' between (x_f, y_j) and (x_f, y_{j+1}), where $(x_f, y_j)(x_f, y_{j+1}) \in E(P_m\Box K_2)$. Then

$$P = P' \cup P'' \cup \{(x_f, y_{j+1})(x_{f-1}, y_{j+1})\}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_j) in G. □

Proposition 3.2. *Let* $G = \Theta(1, 2k, 2l)$*. Then* $H = G\Box K_2$ *is not hamilton-connected.*

Proof. Let C_1 and C_2 be two even cycles, and let $V(C_1) = \{x_1,\ x_2,\ \cdots,\ x_{2k}\}$, $V(C_2) = \{x_1, x_2, x_3', \cdots, x_{2l}'\}$, $E(C_1 \cap C_2) = \{x_1x_2\}$, $V(K_2) = \{y_1, y_2\}$.

We suppose that $G\Box K_2$ is hamilton-connected. There exists a Hamilton path P between (x_1, y_1) and (x_2, y_2). Let $C = G \setminus \{x_1x_2\}$, and note that C is an cycle with even order. Then $C\Box K_2$ is not hamilton-connected, and there exists no Hamilton path between (x_1, y_1) and (x_2, y_2) in $C\Box K_2$. So edges $(x_1, y_1)(x_2, y_1)$ or $(x_1, y_2)(x_2, y_2)$ must be contained in P, and one of them must be the first or last edge of P.

Now let $P_1 = \langle x_3, x_4, \cdots, x_{2k}\rangle$, and let $P_2 = \langle x_3', x_4', \cdots, x_{2l}'\rangle$. Say edge $(x_1, y_1)(x_2, y_1)$ is first edge of P. From the argument earlier, there must exist Hamilton path between (x_3, y_1) and (x_{2k}, y_2) in $P_1\Box K_2$. Because (x_3, y_1) and (x_{2k}, y_2) is in same partite in $P_1\Box K_2$, by the Lemma 3.1, there exists no Hamilton path joining (x_3, y_1) and (x_{2k}, y_2), contradiction. □

Lemma 3.3. *Let $G = \Theta(1, 2k + 1, 2l + 1)$. Then $H = G\Box K_2$ is hamilton-connected.*

Proof. Let C_1 and C_2 be two odd cycles, and $V(C_1) = \{x_1, x_2,\ \cdots, x_{2k+1}\}$, $V(C_2) = \{x_1, x_2, x_3', \cdots, x_{2l+1}'\}$, $E(C_1 \cap C_2) = \{x_1x_2\}$, $V(K_2) = \{y_1, y_2\}$.

Case 1. *Any $(x_e, y_i), (x_f, y_j) \in V(C_1\Box K_2),\ 1 \le e, f \le 2k + 1,\ 1 \le i, j \le 2$.*

Let $P_1 = \langle x_3, \cdots, x_e\rangle$, $P_2 = \langle x_{e+1}, \cdots, x_f, \cdots, x_{2k+1}\rangle$.

Let $X_1,\ Y_1$ be bipartite partition of $P_1\Box K_2$; $X_2,\ Y_2$ be bipartite partition of of $P_2\Box K_2$. Without loss of generality, we may assume that $(x_3, y_1) \in X_1$, $(x_3, y_2) \in Y_1$ and $(x_{2k+1}, y_1) \in X_2$, $(x_{2k+1}, y_2) \in Y_2$.

If $(x_e, y_i) \in X_1$ (if $(x_e, y_i) \in Y_1$ we can choose the (x_3, y_1)), by Lemma 3.1, we can find a Hamilton path P_3 between (x_e, y_i) and (x_3, y_2) in $P_1\Box K_2$. Similarly if $(x_f, y_j) \in X_2$ (if $(x_f, y_j) \in Y_2$ we can choose the (x_{2k+1}, y_1)), we can find a Hamilton path P_4 between (x_f, y_j) and (x_{2k+1}, y_2) in $P_2\Box K_2$.

By Lemma 1.4, there is a Hamilton path P_5 between (x_1, y_2) and (x_2, y_2) in $C_2\Box K_2$. Then

$$P = P_3 \cup P_4 \cup P_5 \cup \{(x_1, y_2)(x_{2k+1}, y_2), (x_2, y_2)(x_3, y_2)\}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_j) in $G\Box K_2$.

Case 2. *Any $(x_e', y_i)(x_f', y_j) \in V(C_2\Box K_2),\ 3 \le e, f \le 2l + 1,\ 1 \le i, j \le 2$.*

The proof of this case is similar to that of Case 1. So it is omitted.

Case 3. *Any $(x_e, y_i) \in V(C_1\Box K_2),\ (x_f', y_j) \in V(C_2\Box K_2),\ 1 \le e \le 2k+1,\ 3 \le f \le 2l+1,\ 1 \le i, j \le 2$.*

Let $P_{2l-2} = \langle x_3', x_4', \cdots, x_{2l+1}'\rangle$ and let $X,\ Y$ be bipartite partition of $P_{2l-2}\Box K_2$. Consider the vertices $(x_{2l+1}', y_1), (x_{2l+1}', y_2)$ or $(x_3', y_1), (x_3', y_2)$ that are in different partite sets. Say the former, we can assume $(x_{2l+1}', y_1) \in X, (x_{2l+1}', y_2) \in Y$.

Without loss of generality, we may assume $(x_f', y_j) \in X$, by Lemma 3.1, there is a Hamilton path P_1 between (x_f', y_j) and (x_{2l+1}', y_2) in $P_{2l-2}\Box K_2$. Also there is a Hamilton path P_2 between (x_1, y_2) and (x_e, y_i) in $C_1\Box K_2$. Then

$$P = P_1 \cup P_2 \cup \{(x_{2l+1}', y_2)(x_1, y_2)\}$$

is a Hamilton path between (x_e, y_i) and (x'_f, y_j) in $G\Box K_2$. □

Proposition 3.4. *Suppose that $H = C_n\Box K_2$ is a hamilton-connected graph. Then n is an odd integer.*

Proof. Let $V(C_n) = \{x_1, x_2, \cdots, x_n\}$, $V(K_2) = \{y_1, y_2\}$. Let P_1 be a Hamilton path between (x_1, y_1) and (x_2, y_2), then $P_1 = \langle(x_1, y_1), (x_1, y_2), (x_n, y_1), (x_n, y_2), (x_{n-1}, y_1), \cdots, (x_3, y_1), (x_2, y_1), (x_2, y_2)\rangle$ and P_1 contains every pillar of the prism. Let $X = V(G_{1y_1})$, then $|\partial(X)| = 2t + 1$, where $\partial(X)$ is set of edges with one end in X.

Since $H = C_n\Box K_2$ is an odd graph, and by $|\partial(X)| + 2e(X) = \sum_{v\in X} d(v)$, we have $|\partial(X)| = |X|$ (mod 2), so $|X| = |V(G_{1y_1})| = n$ is an odd integer. □

Conclude all the above, we can obtain the sufficient and necessary condition for prism over hamiltonian graph being hamilton-connected, that is Theorem 1.6. Therefore we can give a sufficient condition for $G\Box P_m$ being hamilton-connected.

Theorem 3.5. *Let m be an integer, where $m \geq 2$. Then $H = G\Box P_m$ is hamiltonian-connected, if:*

(i) *G is a hamiltonian graph with odd order, or*

(ii) *G is a hamiltonian graph with even order, and G contains $\Theta(1, 2k + 1, 2l + 1)$ as a spanning subgraph, where $\Theta(1, 2k + 1, 2l + 1)$ is the graph union of two odd cycles with a common edge.*

Proof. Let C_n be a Hamilton cycle of G, $V(C_n) = \{x_1, x_2, \cdots, x_n\}$, $V(P_m) = \{y_1, y_2, \cdots, y_m\}$.

We prove it by induction on m.

When $m = 2$, by Theorem 1.6 it is right. Now we suppose that $G\Box P_{m-1}$ is hamilton-connected when G satisfies those conditions. We shall show that $H = G\Box P_m$ is hamilton-connected.

Case 1. *Any $(x_e, y_i), (x_f, y_j) \in V(C_n\Box P_{m-1}),\ 1 \leq e, f \leq n,\ 1 \leq i, j \leq m - 1$.*

By induction hypothesis there is a Hamilton path P_1 between (x_e, y_i) and (x_f, y_j) in $G\Box P_{m-1}$. Without loss of generality, assume $(x_i, y_{m-1})(x_{i+1}, y_{m-1}) \in E(C_n\Box P_{m-1} \cap P_1)$. Then

$$\begin{aligned} P \;=\; & (P_1 \setminus \{(x_i, y_{m-1})(x_{i+1}, y_{m-1})\}) \cup \{(x_i, y_{m-1})(x_i, y_m), (x_{i+1}, y_{m-1})(x_{i+1}, y_m)\} \cup \\ & \{(x_i, y_m)(x_{i-1}, y_m), (x_{i-1}, y_m)(x_{i-2}, y_m), \cdots, (x_{i+2}, y_m)(x_{i+1}, y_m)\} \end{aligned}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_j) in H.

Case 2. *Any $(x_e, y_i) \in V(C_n\Box P_{m-1}),\ (x_f, y_m) \in V(G_{1y_m}),\ 1 \leq e, f \leq n,\ 1 \leq i \leq m - 1$,*

By induction hypothesis there is a Hamilton path P_1 between (x_e, y_i) and (x_{f+1}, y_{m-1}) in $G\Box P_{m-1}$. Then

$$\begin{aligned} P \;=\; & P_1 \cup \{(x_{f+1}, y_{m-1})(x_{f+1}, y_m)\} \cup \\ & \{(x_{f+1}, y_m)(x_{f+2}, y_m), (x_{f+2}, y_m)(x_{f+3}, y_m), \cdots, (x_{f-1}, y_m)(x_f, y_m)\} \end{aligned}$$

is a Hamilton path between (x_e, y_i) and (x_f, y_m) in H.

Case 3. *Any* $(x_e, y_m),\ (x_f, y_m) \in V(G_{1y_m}),\ 1 \le e, f \le n$.

Let $P_1 = \langle (x_{e+1}, y_m), (x_{e+2}, y_m), \cdots, (x_f, y_m)\rangle,\ P_2 = \langle (x_{f+1}, y_m), (x_{f+2}, y_m), \cdots, (x_e, y_m)\rangle$. By induction hypothesis there is a Hamilton path P_3 between (x_{e+1}, y_{m-1}) and (x_{f+1}, y_{m-1}) in $G\Box P_{m-1}$. Then

$$P = P_3 \cup \{(x_{e+1}, y_{m-1})(x_{e+1}, y_m), (x_{f+1}, y_{m-1})(x_{f+1}, y_m)\} \cup P_1 \cup P_2$$

is a Hamilton path between (x_e, y_m) and (x_f, y_m) in H. □

By the proof of the Lemma 3.3 we can obtain the following corollary.

Corollary 3.6. *Let G be a graph such that $G\Box K_2$ is a hamilton-connected graph. If $H = G \cup P$ with $|E(G \cap P)| = 1$, where P is an ear of G, then $H\Box K_2$ is hamilton-connected.*

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Rushengul Hoshur

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, P.R. China

Email: Rushengul@yahoo.com.cn

Elkin Vumar

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, P.R. China

Email: vumar@xju.edu.cn, elkin1226@sina.com