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A SIMPLE APPROACH TO ORDER THE MULTIPLICATIVE ZAGREB INDICES OF CONNECTED GRAPHS

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Abstract. The first (Π_1) and the second (Π_2) multiplicative Zagreb indices of a connected graph G, with vertex set $V(G)$ and edge set $E(G)$, are defined as $\Pi_1(G) = \prod_{u\in V(G)} {d_u}^2$ and $\Pi_2(G) = \prod_{uv\in E(G)} d_u d_v$, respectively, where d_u denotes the degree of the vertex u. In this paper we present a simple approach to order these indices for connected graphs on the same number of vertices. Moreover, as an application of this simple approach, we extend the known ordering of the first and the second multiplicative Zagreb indices for some classes of connected graphs.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph (i.e. G does not have loops or multiple edges). Zagreb indices were first introduced in [6], and they are among oldest and most used molecular structure-descriptors [8]. The first Zagreb index is defined as the sum of the squares of the degrees of the vertices:

$$M_1(G) \quad = \sum_{u\in V(G)} {d_u}^2.$$

The second Zagreb index is defined as the sum of the product of the degrees of adjacent vertices

$$M_2(G) \quad = \sum_{uv\in E(G)} d_u\, d_v\ .$$

We encourage the reader to consult [1, 4, 11, 15, 14, 16] for historical background, computational techniques, and mathematical properties of Zagreb indices. A detailed bibliography on recent research

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of Zagreb indices is found in [2, 12].

Following an earlier idea of Narumi and Katayama [10], recently Gutman [3] introduced the multiplicative version of the Zagreb indices. In particular he put forward

$$\Pi_1 = \Pi_1(G) = \prod_{u \in V(G)} {d_u}^2,$$

$$\Pi_2 = \Pi_2(G) = \prod_{uv \in E(G)} d_u\, d_v.$$

In [3, 5], Gutman determined that among all trees with $n \geq 5$ vertices, the extremal trees with respect to these multiplicative Zagreb indices are path P_n and and star S_n. Also Gutman and Ghorbani [5] have obtained some properties of Narumi-Katayama index, defined as: $NK(G) = \prod_{u \in V(G)} d_u$ for a graph G.

In [13] Xu and Hua introduced some graph transformations which increase or decrease these two indices. As an application, they obtained a unified approach to characterize extremal (maximal and minimal) trees, unicyclic graphs and bicyclic graphs with respect to multiplicative Zagreb indices, respectively. In this paper by theory of majorization we obtain and extend their results and obtain ordering of the first and second multiplicative Zagreb indices for some class of connected graphs. Our idea and pictures are from [7] and this paper was very helpful for us.

2. Preliminaries results

Throughout this paper we only consider finite, undirected and simple graphs. A tree is a connected acyclic graph. The path of order n is denoted by P_n and the cycle of order n is denoted by C_n. A pendent vertex or leaf of a graph is a vertex of degree 1. Let $\mathcal{T}_n$, $\mathcal{U}_n$ and $\mathcal{B}_n$ be the class of trees of order n, the class of connected unicyclic graphs of order n, and the class of connected bicyclic graphs of order n, respectively. Also for a graph G, denoted by $\Delta(G)$ the maximum degree of G.

Let $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$, be two non-increasing sequences of real numbers. If they satisfy the conditions

$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, for $1 \leq k \leq n-1$ and $\sum_{i=1}^{i} x_n = \sum_{i=1}^{k} y_n$, then we say that $x = (x_1, x_2, \ldots, x_n)$ is **majorized** by $y = (y_1, y_2, \ldots, y_n)$ and write $x \preceq y$. Furthermore, by $x \prec y$ we mean that $x \preceq y$ and $x \neq y$.

We recall that if $I \subset R$ is an interval and $f: I \to R$ is a real-value function such that $f''(t) \geq 0$ on I, then f is convex on I. If $f''(t) > 0$, then f is strictly convex on I. Similarly, if $f''(t) \leq 0$ ($f''(t) < 0$) on I, then f is concave (strictly concave) on I.

A real-value function ϕ defined on a set $A \subset R^n$ is said to be Schur-convex on A if

$$x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in A \textit{ and } x \preceq y \ \Rightarrow \phi(x) \leq \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ where $x \prec y$, then ϕ is said to be strictly Schur-convex on A. Similarly, ϕ is said to be Schur-concave on A if

$$x, y \in A \text{ and } x \preceq y \Rightarrow \phi(x) \geq \phi(y),$$

and ϕ is strictly Schur-concave on A if strict inequality $\phi(x) > \phi(y)$ holds when $x \prec y$.

The following result has been established in [9]

Lemma 2.1. *[See [9]] Let $I \subset R$ be an interval and let $\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} g(x_i)$, where $g : I \to R$. Then if g is strictly convex on I, then ϕ is strictly Schur-convex on I^n. Similarly if g is strictly concave on I, then ϕ is strictly Schur-concave on I^n.*

Corollary 2.2. *Let G and $\acute{G}$ be two connected graphs with degree sequences $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, respectively. If $x \preceq y$, then*
(i) $\Pi_1(G) \geq \Pi_1(\acute{G})$, with the equality holds if an only if $x = y$.
(ii) $\Pi_2(G) \leq \Pi_2(\acute{G})$, with the equality holds if an only if $x = y$.

Proof: (i) Observe that for $t > 0$, $(2\ln t)'' = \frac{-2}{t^2} < 0$. So on interval $(0, +\infty)$, the function $2\ln t$ is strictly concave. If $x \preceq y$, then by Lemma 2.1, $\sum_{i=1}^{n} 2\ln x_i \geq \sum_{i=1}^{n} 2\ln y_i$ and thus $\ln \prod_{i=1}^{n} {x_i}^2 \geq \ln \prod_{i=1}^{n} {y_i}^2$. Since e^t is a strictly increasing function, so $\ln \prod_{i=1}^{n} {x_i}^2 \geq \ln \prod_{i=1}^{n} {y_i}^2$, and hence $\Pi_1(G) \geq \Pi_1(\acute{G})$. Now suppose that $x \preceq y$ and $\Pi_1(G) = \Pi_1(\acute{G})$. Then $\prod_{i=1}^{n} x_i^2 = \prod_{i=1}^{n} y_i^2$, and thus $\sum_{i=1}^{n} 2\ln x_i \geq \sum_{i=1}^{n} 2\ln y_i$. Hence by Lemma 2.1, we have $x = y$.

(ii) Note that for $t > 0$, $(t\ln t)'' = \frac{1}{t} > 0$. So on interval $(0, +\infty)$, the function $t\ln t$ is strictly convex. If $x \preceq y$, then by Lemma 2.1, $\sum_{i=1}^{n} x_i \ln x_i \leq \sum_{i=1}^{n} y_i \ln y_i$ and thus $\ln \prod_{i=1}^{n} {x_i}^{x_i} \leq \ln \prod_{i=1}^{n} {y_i}^{y_i}$. Since e^t is a strictly increasing function, so $\prod_{i=1}^{n} {x_i}^{x_i} \leq \prod_{i=1}^{n} {y_i}^{y_i}$. It is easy to see that $\Pi_2(G) = \prod_{i=1}^{n} {x_i}^{x_i}$ and $\Pi_2(\acute{G}) = \prod_{i=1}^{n} {y_i}^{y_i}$, and hence $\Pi_2(G) \leq \Pi_2(\acute{G})$. Now suppose that $x \preceq y$ and $\Pi_2(G) = \Pi_2(\acute{G})$. Then $\prod_{i=1}^{n} x_i^{x_i} = \prod_{i=1}^{n} y_i^{y_i}$, and thus $\sum_{i=1}^{n} x_i \ln x_i \leq \sum_{i=1}^{n} y_i \ln y_i$. Hence by Lemma 2.1, we have $x = y$.

3. Main Results

3.1. Trees.

Theorem 3.1. [5] *Let $T \in \mathcal{T}_n \setminus \{P_n, S_n\}$ be a tree with n vertices. Then*

$$\Pi_1(S_n) < \Pi_1(T) < \Pi_1(P_n), \tag{3.1}$$

$$\Pi_2(S_n) > \Pi_2(T) > \Pi_2(P_n). \tag{3.2}$$

Proof: (3.1) Since the degree sequence $(2,\ldots,2,1,1)$ is minimal in the class $\mathcal{T}_n$ (i.e., in the order $\prec$) and the degree sequence $(n-1,1,\ldots,1)$ is maximal in the class $\mathcal{T}_n$, we obtain the result by part (i) of Corollary 2.2.
The proof of claim (3.2) is similar and we omit the details.

Let $T_1 = S_n, T_2, \ldots, T_{13}$ be the trees on n vertices as shown in Fig. 1. Then we have

Theorem 3.2. *Let $T \in \mathcal{T}_n \setminus \{T_1, T_2, \ldots, T_6, T_{13}\}$ and $n \geq 13$. Then $\Pi_1(T_1) < \Pi_1(T_2) < \Pi_1(T_3) < \Pi_1(T_6) < \Pi_1(T_4) = \Pi_1(T_5) < \Pi_1(T_{13}) < \Pi_1(T)$.*

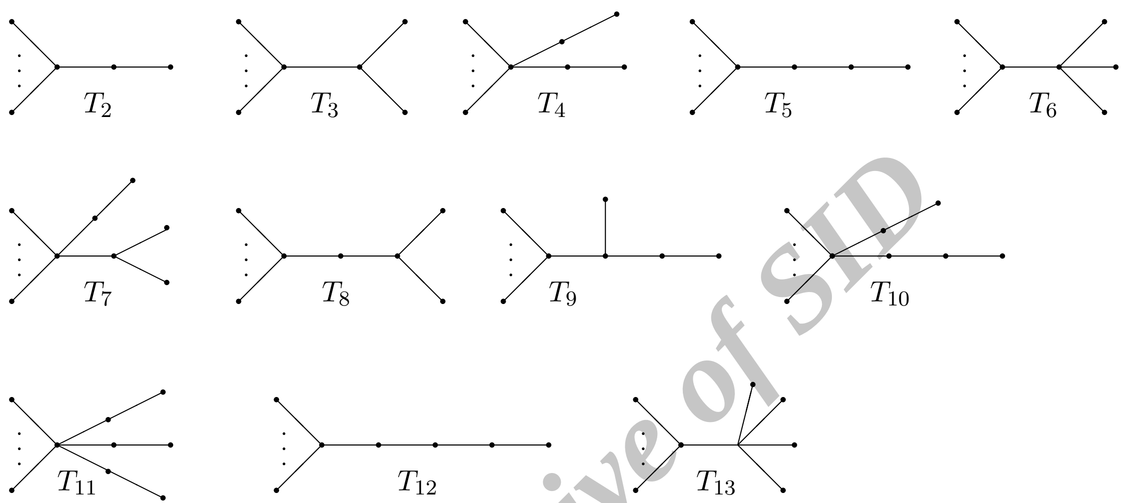

Fig. 1. The trees T_2, ..., T_{13}. The picture taken from [7]

Proof: By an elementary computation, we have $\Pi_1(T_1) = (n-1)^2$, $\Pi_1(T_2) = 4(n-2)^2$, $\Pi_1(T_3) = 9(n-3)^2$, $\Pi_1(T_4) = 16(n-3)^2 = \Pi_1(T_5)$, $\Pi_1(T_6) = 16(n-4)^2$, $\Pi_1(T_7) = 36(n-4)^2 = \Pi_1(T_8) = \Pi_1(T_9)$, $\Pi_1(T_{10}) = 64(n-4)^2 = \Pi_1(T_{11}) = \Pi_1(T_{12})$ and $\Pi_1(T_{13}) = 25(n-5)^2$. So we only need to show that if $T \in \mathcal{T}_n \setminus \{T_1, T_2, \ldots, T_{13}\}$, then $\Pi_1(T) > \Pi_1(T_{13})$.

Clearly, T_1 is the unique tree with $\Delta = n-1$, T_2 is the unique tree with $\Delta = n-2$, T_3, T_4, T_5 are all trees with $\Delta = n-3$, $T_6, \ldots, T_{12}$ are all trees with $\Delta = n-4$. Since $T \in \mathcal{T}_n \setminus \{T_1, T_2, \ldots, T_{13}\}$, then $\Delta(T) \leq n-5$.

Let $(a) = (d_1, d_2, \ldots, d_n)$ be the degree sequence of T. Since the degree sequence of T_{13} is $(b) = (n-5, 5, 1, \ldots, 1)$, it is easy to see that $(a) \prec (b)$, because T_{13} is the unique tree with (b) as its degree sequence. Thus, $\Pi_1(T_{13}) < \Pi_1(T)$ follows from Corollary 2.2.

Theorem 3.3. *Let $T \in \mathcal{T}_n \setminus \{T_1, T_2, \ldots, T_6, T_{13}\}$ and $n \geq 23$. Then $\Pi_2(T_1) > \Pi_2(T_2) > \Pi_2(T_3) > \Pi_2(T_4) = \Pi_2(T_5) > \Pi_2(T_6) > \Pi_2(T_7) = \Pi_2(T_8) = \Pi_2(T_9) > \Pi_2(T_{10}) = \Pi_2(T_{11}) = \Pi_2(T_{12}) > \Pi_2(T_{13}) > \Pi_2(T)$.*

Proof: By an elementary computation, we have $\Pi_2(T_1) = n^n$, $\Pi_2(T_2) = 4(n-2)^{n-2}$, $\Pi_2(T_3) = 27(n-3)^{n-3}$, $\Pi_2(T_4) = 16(n-3)^{n-3} = \Pi_2(T_5)$, $\Pi_2(T_6) = 256(n-4)^{n-4}$, $\Pi_2(T_7) = 108(n-4)^{n-4} =$

$\Pi_2(T_8) = \Pi_2(T_9)$, $\Pi_2(T_{10}) = 64(n-4)^{n-4} = \Pi_2(T_{11}) = \Pi_2(T_{12})$ and $\Pi_2(T_{13}) = 3125(n-5)^{n-5}$. So we only need to show that if $T \in \mathcal{T}_n \setminus \{T_1, T_2, \dots, T_{13}\}$, then $\Pi_2(T) > \Pi_2(T_{13})$.

Clearly, T_1 is the unique tree with $\Delta = n-1$, T_2 is the unique tree with $\Delta = n-2$, T_3, T_4, T_5 are all trees with $\Delta = n-3$, $T_6, \dots, T_{12}$ are all trees with $\Delta = n-4$. Since $T \in \mathcal{T}_n \setminus \{T_1, T_2, \dots, T_{13}\}$, then $\Delta(T) \le n-5$.

Let $a = (d_1, d_2, \dots, d_n)$ be the degree sequence of T. Since the degree sequence of T_{13} is $b = (n-5, 5, 1, \dots, 1)$, it is easy to see that $a \prec b$, because T_{13} is the unique tree with b as its degree sequence. Thus, $\Pi_2(T_{13}) > \Pi_2(T)$ follows from Corollary 2.2.

3.2. **Unicyclic graphs.** Let $U_1, U_2, \dots, U_{16}$ be the unicyclic graphs as shown in Fig. 2. Then we have the next result.

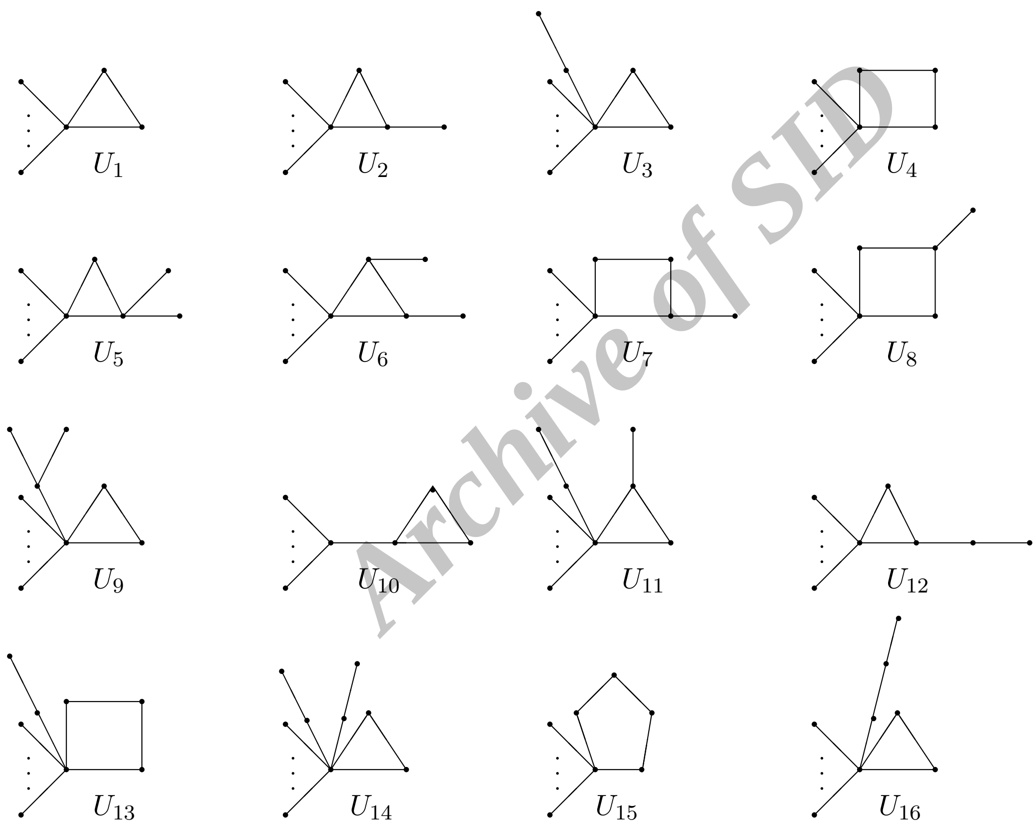

Fig. 2. The unicyclic graphs U_1, ..., U_{16}. The picture taken from [7]

Theorem 3.4. *Let $G \in \mathcal{U}_n \setminus \{U_1, U_2, \dots, U_{12}\}$ and $n \ge 13$. Then $\Pi_1(U_1) < \Pi_1(U_2) < \Pi_1(U_5) < \Pi_1(U_3) = \Pi_1(U_4) < \Pi_1(U_6) < \Pi_1(U_7) = \Pi_1(U_8) = \Pi_1(U_9) = \Pi_1(U_{10}) = \Pi_1(U_{11}) = \Pi_1(U_{12}) < \Pi_1(G)$.*

Proof: By an elementary computation, we have $\Pi_1(U_1) = 16(n-1)^2$, $\Pi_1(U_2) = 36(n-2)^2$, $\Pi_1(U_3) = 64(n-2)^2 = \Pi_1(U_4)$, $\Pi_1(U_5) = 64(n-3)^2$, $\Pi_1(U_6) = 81(n-3)^2$, $\Pi_1(U_7) = 144(n-3)^2 = \Pi_1(U_8) = \Pi_1(U_9) = \Pi_1(U_{10}) = \Pi_1(U_{11}) = \Pi_1(U_{12})$, and $\Pi_1(U_{13}) = 256(n-3)^2 = \Pi_1(U_{14}) = \Pi_1(U_{15}) = \Pi_1(U_{16})$.

So we only need to prove that if $G \in \mathcal{U}_n \setminus \{U_1, U_2, \dots, U_{16}\}$, then $\Pi_1(G) > \Pi_1(U_{12})$.

It is easy to check that U_1 is the unique unicyclic graph with $\Delta = n-1$, U_2, U_3, U_4 are all unicyclic graphs with $\Delta = n-2$, and $U_5, U_6, \ldots, U_{16}$ are all unicyclic graphs with $\Delta = n-3$. If $G \in \mathcal{U}_n \setminus \{U_1, U_2, \ldots, U_{16}\}$, then $\Delta(G) \leq n-4$. Suppose that degree sequence of G is $(a) = (d_1, d_2, \ldots, d_n)$. Since $G \in \mathcal{U}_n$, then G has only exactly one cycle. This implies that $n-4 \geq d_1 \geq d_2 \geq d_3 \geq 2$. Let $(b) = (n-4, 5, 2, 1, \ldots, 1)$. Then $(a) \preceq (b)$. By Corollary 2.2, we can conclude that

$$\Pi_1(G) \geq (n-4)^2 \times 5^2 \times 2^2 = 200(n-4)^2 > 144(n-3)^2 = \Pi_1(U_{12}).$$

This completes the proof.

Theorem 3.5. *Let $G \in \mathcal{U}_n \setminus \{U_1, U_2, \ldots, U_{16}\}$ and $n \geq 22$. Then $\Pi_2(U_1) > \Pi_2(U_2) > \Pi_2(U_3) = \Pi_2(U_4) > \Pi_2(U_5) > \Pi_2(U_6) > \Pi_2(U_7) = \Pi_2(U_8) = \Pi_2(U_9) = \Pi_2(U_{10}) = \Pi_2(U_{11}) = \Pi_2(U_{12}) > \Pi_2(U_{13}) = \Pi_2(U_{14}) = \Pi_2(U_{15}) = \Pi_2(U_{16}) > \Pi_2(G)$.*

Proof: By an elementary computation, we have $\Pi_2(U_1) = 16(n-1)^{n-1}$, $\Pi_2(U_2) = 108(n-2)^{n-2}$, $\Pi_2(U_3) = 64(n-2)^{n-2} = \Pi_2(U_4)$, $\Pi_2(U_5) = 1024(n-3)^{n-3}$, $\Pi_2(U_6) = 729(n-3)^{n-3}$, $\Pi_2(U_7) = 432(n-3)^{n-3} = \Pi_2(U_8) = \Pi_2(U_9) = \Pi_2(U_{10}) = \Pi_2(U_{11}) = \Pi_2(U_{12})$, and $\Pi_2(U_{13}) = 256(n-3)^{n-3} = \Pi_2(U_{14}) = \Pi_2(U_{15}) = \Pi_2(U_{16})$.

Next we only need to show that if $G \in \mathcal{U}_n \setminus \{U_1, U_2, \ldots, U_{16}\}$, then $\Pi_2(G) > \Pi_2(U_{12})$.

It is easy to check that U_1 is the unique unicyclic graph with $\Delta = n-1$, U_2, U_3, U_4 are all unicyclic graphs with $\Delta = n-2$, and $U_5, U_6, \ldots, U_{16}$ are all unicyclic graphs with $\Delta = n-3$. If $G \in \mathcal{U}_n \setminus \{U_1, U_2, \ldots, U_{16}\}$, then $\Delta(G) \leq n-4$. Suppose that degree sequence of G is $a = (d_1, d_2, \ldots, d_n)$. Since $G \in \mathcal{U}_n$, then G has only exactly one cycle. This implies that $n-4 \geq d_1 \geq d_2 \geq d_3 \geq 2$. Let $b = (n-4, 5, 2, 1, \ldots, 1)$. Then $a \preceq b$. By Corollary 2.2, we can conclude that

$$\Pi_2(G) \leq (n-4)^{n-4} \times 5^5 \times 2^2 = 12500(n-4)^{n-4} < 256(n-3)^{n-3} = \Pi_2(U_{16}).$$

This completes the proof.

3.3. **Bicyclic graphs.** Let $B_1, B_2, \ldots, B_{11}$ be the bicyclic graphs as shown in Fig. 3. Then we have the next result.

Theorem 3.6. *Let $G \in \mathcal{B}_n \setminus \{B_1, B_2, \ldots, B_5\}$ and $n \geq 12$. Then $\Pi_1(B_1) < \Pi_1(B_3) < \Pi_1(B_2) < \Pi_1(B_4) = \Pi_1(B_5) < \Pi_1(G)$.*

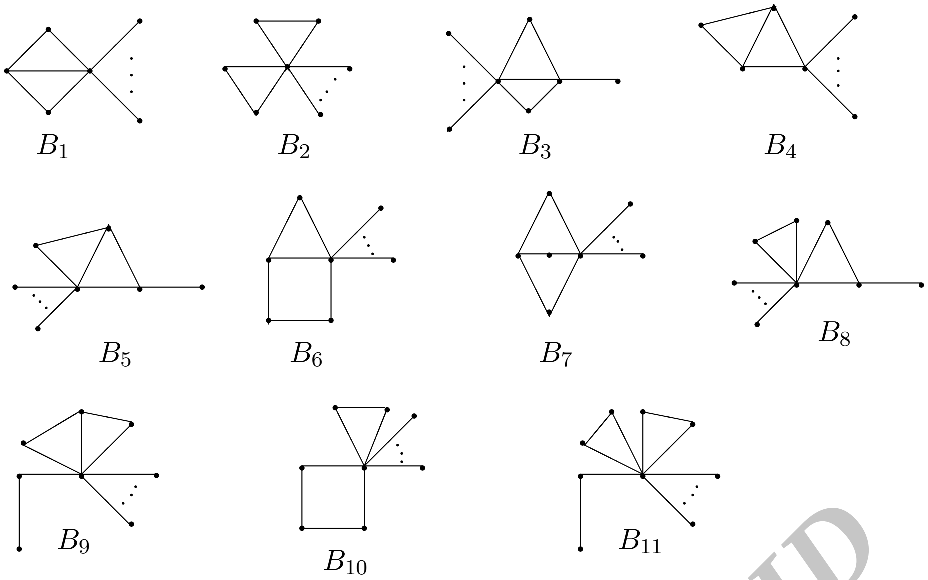

Fig. 3. The bicyclic graphs B_1,..., B_{11}.The picture taken from [7]

Proof: By an elementary computation, we have $\Pi_1(B_1) = 144(n-1)^2$, $\Pi_1(B_2) = 256(n-1)^2$, $\Pi_1(B_3) = 256(n-2)^2$, $\Pi_1(B_4) = 324(n-2)^2 = \Pi_1(B_5)$, $\Pi_1(B_6) = 576(n-2)^2 = \Pi_1(B_7) = \Pi_1(B_8) = \Pi_1(B_9)$ and $\Pi_1(B_{10}) = 1024(n-2)^2 = \Pi_1(B_{11})$. So we only need to prove that if $G \in \mathcal{B}_n \setminus \{B_1, B_2, \ldots, B_{11}\}$, then $\Pi_1(G) > \Pi_1(B_5)$

It is easy to check that B_1, B_2 are all bicyclic graphs with $\Delta = n-1$, $B_3, \ldots, B_{11}$ are all bicyclic graphs with $\Delta = n-2$. If $G \in \mathcal{B}_n \setminus \{B_1, B_2, \ldots, B_{11}\}$, then $\Delta(G) = n-3$. Suppose the degree sequence of G is $(a) = (d_1, d_2, d_3, \ldots, d_n)$. Since $G \in \mathcal{B}_n$, then $n-3 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq 2$. Let $(b) = (n-3, 5, 2, 2, 2, 1, \ldots, 1)$. Then $(a) \preceq (b)$. By Corollary 2.2, we can conclude that

$$\Pi_1(G) \geq (n-3)^2 \times 25 \times 4 \times 4 = 400(n-3)^2 > 324(n-1)^2 = \Pi_1(B_5).$$

This completes the proof.

Theorem 3.7. *Let $G \in \mathcal{B}_n \setminus \{B_1, B_2, \ldots, B_{11}\}$ and $n \geq 21$. Then* $\Pi_2(B_1) > \Pi_2(B_2) > \Pi_2(B_3) > \Pi_2(B_4) = \Pi_2(B_5) > \Pi_2(B_6) = \Pi_2(B_7) = \Pi_2(B_8) = \Pi_2(B_9) > \Pi_2(B_1) = \Pi_2(B_1) > \Pi_2(G)$.

Proof: By an elementary computation, we have $\Pi_2(B_1) = 432(n-1)^{n-1}$, $\Pi_2(B_2) = 256(n-1)^{n-1}$, $\Pi_2(B_3) = 4096(n-2)^{n-2}$, $\Pi_2(B_4) = 2916(n-2)^{n-2} = \Pi_2(B_5)$, $\Pi_2(B_6) = 1728(n-2)^{n-2} = \Pi_2(B_7) = \Pi_2(B_8) = \Pi_2(B_9)$ and $\Pi_2(B_{10}) = 1024(n-2)^{n-2} = \Pi_2(B_{11})$. So we only need to prove that if $G \in \mathcal{B}_n \setminus \{B_1, B_2, \ldots, B_{11}\}$, then $\Pi_2(G) > \Pi_2(B_5)$

It is easy to check that B_1, B_2 are all bicyclic graphs with $\Delta = n-1$, $B_3, \ldots, B_{11}$ are all bicyclic graphs with $\Delta = n-2$. If $G \in \mathcal{B}_n \setminus \{B_1, B_2, \ldots, B_{11}\}$, then $\Delta(G) = n-3$. Suppose the degree sequence of G is $a = (d_1, d_2, d_3, \ldots, d_n)$. Since $G \in \mathcal{B}_n$, then $n-3 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq 2$. Let $b = (n-3, 5, 2, 2, 1, \ldots, 1)$. Then $a \preceq (b)$. By Corollary 2.2, we can conclude that

$$\Pi_2(G) \leq (n-3)^{n-3} \times 5^5 \times 2^2 \times 2^2 = 50000(n-3)^{n-3} < 1024(n-2)^{n-2} = \Pi_2(B_{11}).$$

This completes the proof.

Theorem 3.8. *Let $\mathcal{H}$ be class of connected bicyclic graphs with $(3,3,2,\ldots,2)$ as theirs degree sequence. If $G \in \mathcal{B}_n \setminus \{\mathcal{H}\}$, then for each $H \in \mathcal{H}$ we have $\Pi_1(H) > \Pi_1(G)$ and $\Pi_2(H) < \Pi_2(G)$.*

Proof: The claim follows since the degree sequence $(3,3,2,\ldots,2)$ is minimal in the class of $\mathcal{B}_n$.

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