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TOEPLITZ GRAPH DECOMPOSITION

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Abstract. Let n, t_1, …, t_k be distinct positive integers. A Toeplitz graph $G = (V, E)$ is a graph with $V = \{1, \dots, n\}$ and $E = \{(i, j) \mid |i-j| \in \{t_1, \dots, t_k\}\}$. In this paper, we present some results on decomposition of Toeplitz graphs.

1. Introduction

A *Toeplitz matrix* which is named after Otto Toeplitz (1881–1940), is an $n \times n$ matrix $A = (a_{i,j})$ where for i and j, $1 \le i, j \le n-1$, $a_{ij} = a_{(i+1)(j+1)}$. Toeplitz matrices are precisely those matrices such that all their diagonals parallel to the main diagonal have constant values. Thus, Toeplitz matrices are defined by their first row and first column.

Let $n, t_1, \dots, t_k$ be integers such that $1 \le t_1 < t_2 < \cdots < t_k < n$. A *Toeplitz graph*, denoted by $T_n\langle t_1, t_2, \dots, t_k\rangle$ is a graph with vertex set $\{1, \dots, n\}$ and edge set $\{(i, j) \mid |i-j| \in \{t_1, \dots, t_k\}\}$. The name of this class of graphs is due to the fact that their adjacency matrices are Toeplitz matrices. For example, see the graph $T_7\langle 3, 4, 5\rangle$, in Figure 1. Obviously, any such graph is uniquely defined by the first row of its adjacency matrix, i.e. by a 0–1 sequence whose first element is zero. Moreover, the number of edges in Toeplitz graph $T_n\langle t_1, \dots, t_k\rangle$ is equal to $\sum_{i=1}^{k}(n - t_i)$.

Properties of Toeplitz graphs, such as bipartiteness, planarity, colourability and Hamiltonicity have been studied in [1–11]. Now some usual graph notations: Let $G = (V(G), E(G))$ be a simple graph. For any vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood*, denoted by $N[v]$ is $N(v) \cup \{v\}$. An open neighborhood of set $S \subseteq V$ is $N(S) = \bigcup_{v\in S} N(v)$ and its closed neighborhood is $N[S] = N(S) \cup S$. We denote the *degree* of v

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by $d(v)$ which is $|N(v)|$ and the *minimum degree* of G is denoted by $\delta(G)$. The vertex v is called a *leaf* if $d(v) = 1$. A *complete graph* with n vertices, denoted K_n, is a graph where any two vertices are adjacent. A graph G is r-regular if $d(v) = r$, for all $v \in V(G)$. A graph G is connected if there is a path between u and v for all $u, v \in V(G)$. An *induced subgraph* of G is a subgraph obtained by deleting a set of vertices.

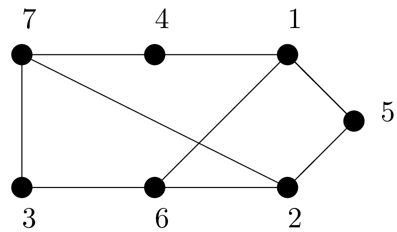

FIGURE 1. The Toeplitz graph $T_7\langle 3, 4, 5\rangle$

Let $M \subseteq E(G)$. The graph $G \setminus M$ is defined to be $(V(G), E(G) \setminus M)$. Also, an induced subgraph of G on $V(G) \setminus S$, where $S \subseteq V(G)$ is denoted by $G \setminus S$. The cycle of order n is denoted by C_n. The *diameter* of a graph G is the largest distance between two vertices of G, denoted by diam(G). In a graph G with at least one cycle, the length of a shortest cycle is called the *girth* of G and denoted by grith(G). A decomposition of a graph G is a family $\mathfrak{F}$ of edge-disjoint subgraphs of G such that $\bigcup_{F\in\mathfrak{F}} E(F) = E(G)$. If the family $\mathfrak{F}$ consists entirely of paths or entirely of cycles, we call $\mathfrak{F}$ a path decomposition or a cycle decomposition of G. A k- *factor* of G is a k- regular spanning subgraph. In particular, a 1- factor is a spanning subgraph whose edge set is a perfect matching and a 2- factor is a spanning subgraph whose components are cycles. A graph G is k- *factorable* if it admits a decomposition into k- factors.

Theorem 1.1. [12] A graph G is 2- factorable if and only if it is a $2k$- regular graph.

2. Bipartite Toeplitz Graphs

When we talk about graph decompositions, perhaps a natural problem is investigating whether the graph is decomposable to matchings, paths, or cycles. First, we address the matching problem. Hall theorem states that a bipartite graph with color classes X and Y has a matching which saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$. Clearly, graph $T_n\langle t_1\rangle$ is a bipartite graph. In [4], a complete characterization of $T_n\langle t_1, t_2\rangle$ and partial results for $T_n\langle t_1, t_2, t_3\rangle$ to be bipartite graphs is presented. The following procedure recognizes whether a finite 0–1 sequence S with two 1s defines a bipartite Toeplitz graph or not.

Theorem 2.1 ([4]). *Let be $G = T_n\langle t_1, t_2\rangle$. Determine r from $t_1 = (2\beta+1)2^r$. If $t_2/2^r$ is an odd integer then S defines a bipartite Toeplitz graph; if $t_2/2^r$ is not an odd number and if $n \leq t_1 + t_2 - \gcd(t_1, t_2)$, then again S defines a bipartite Toeplitz graph; else S defines a non-bipartite Toeplitz graph.*

Suppose $G = T_n\langle t_1, t_2, t_3\rangle$. For convenience, let $\alpha = t_1$, $\delta = t_2 - t_1$, and $\omega = t_3 - t_2$, so such a sequence has the following form:

$$B^{\alpha,\delta,\omega} = (\underbrace{0\ldots0}_{\alpha}1\underbrace{0\ldots01}_{\delta}\underbrace{0\ldots01}_{\omega}0\ldots0).$$

Let δ^* be the remainder of the division of δ by ω. For convenience and throughout the following, let $t_2 = \alpha + \delta^* + k\omega$ and $t_3 = \alpha + \delta^* + (k+1)\omega$ with $k \in \mathbb{N}$ such that $t_3 \le 2\alpha$.
In the following a complete solution for three special cases: (i) $u = 0$, (ii) δ^* divides u, and (iii) divides u is presented.

Theorem 2.2 ([4]). *Suppose that $G = T_n\langle t_1, t_2, t_3\rangle$.*

- **(i):** *Let $\alpha \equiv u \pmod{\omega}$. If $\gcd(\delta, \omega)$ does not divide u (and thus α), then $B^{\delta,\omega}$ induces a bipartite Toeplitz graph.*
- **(ii):** *Let $\alpha \equiv 0 \bmod \omega$, i.e. $\alpha = \gamma\omega$ with $\gamma \in \mathbb{N}$.*
  - (a)**:** *If $\delta \equiv \bmod \omega$, then $B^{\delta,\omega}$ defines a non-bipartite Toeplitz graph if and only if $0 \le k \le \gamma - 1$.*
  - (b)**:** *Let $1 \le \delta^* \le \omega - 1$. If $\alpha = 2\beta\omega$ with $\beta \in \mathbb{N}$, or if $\alpha = (2\beta+1)\omega$ with $\beta \in \mathbb{N}$ and δ^* does not divide ω, then $B^{\delta,\omega}$ defines a non-bipartite Toeplitz graph if and only if $0 \le k \le (\beta - 1)$.*
  - (c)**:** *If $\alpha = (2\beta+1)\omega$ with $\beta \in \mathbb{N}$ and δ^* divides ω, then $B^{\delta,\omega}$ defines a non-bipartite Toeplitz graph if and only if $0 \le k \le \beta$.*
- **(iii):** *Let $\alpha \equiv u \bmod \omega$, i.e. $\alpha = \gamma\omega + u$ with $1 \le u \le \omega$ and let δ^* be a divisor of u, then*
  - (a)**:** *If $\delta^* = u$, then $B^{\delta,\omega}$ defines a non-bipartite Toeplitz graph if and only if $0 \le k \le \gamma - 1$.*
  - (b)**:** *If $\delta^* < u$, then $B^{\delta,\omega}$ defines a non-bipartite Toeplitz graph if and only if $0 \le k \le \lfloor \gamma/2 \rfloor$.*
- **(iv):** *Let $\alpha \equiv u \bmod \omega$ and let $(\omega - \delta^*)$ be a divisor of u, i.e $u = \rho(\omega - \delta^*)$. Then $B^{\delta,\omega}$ defines a non-bipartite Toeplitz graph if and only if $0 \le k \le \lfloor (\gamma - 2)/2 \rfloor$.*

Along of these lines of argument, we present some results.

Proposition 2.3. *If $G = T_n\langle t_1, \ldots, t_k\rangle$ is a bipartite graph, then $k \le \lfloor \frac{n}{2} \rfloor$.*

Proof. Let

$$N(1) = \{1 + t_1, \ldots, 1 + t_k\}.$$

Assume $A = \{t_k - t_{k-1}, t_k - t_{k-2}, \ldots, t_k - t_1,\}$ and $\{t_1, \ldots, t_k\}$ to be two subsets of $\{1, \ldots, n\}$. Since G is bipartite, $A \cap B = \varnothing$. Clearly, $|A| = k - 1$ and $|B| = k$. Since $|A \cup B| \le n - 1$ and $A \cap B = \varnothing$, $2k - 1 \le n - 1$, so $k \le \frac{n-1}{2}$, providing the result. □

Proposition 2.4. *If for each i, $1 \le i \le k$, t_i is odd, then $T_n\langle t_1, \ldots, t_k\rangle$ is a bipartite graph.*

Proof. Let X and Y be the set of odd and even integers of $\{1, \ldots, n\}$, respectively. Then clearly, no two vertices in X or Y are adjacent. □

Proposition 2.5. *The graph $G = T_n\langle n - \lceil \frac{n-1}{2} \rceil, n - \lceil \frac{n-1}{2} \rceil + 1, \ldots, n - 1\rangle$ is a bipartite graph.*

Proof. Assume that $X = \{1, \dots, n - \lceil \frac{n-1}{2} \rceil - 1\}$ and $Y = \{n - \lceil \frac{n-1}{2} \rceil, \dots, n\}$. No two vertices in X or in Y are adjacent. □

Corollary 2.6. *If for each i, $t_i \in \{n - \lceil \frac{n-1}{2} \rceil, n - \lceil \frac{n-1}{2} \rceil + 1, \dots, n-1\}$, then $T_n\langle t_1, \dots, t_k \rangle$ is a bipartite graph.*

It is clear that there are exactly 2^{n-1} symmetric $(0,1)$ Toeplitz matrices of order n. This is a very small portion in comparison with the number of all $(0,1)$ symmetric matrices of order n. This allows us to investigate bipartitness of Toeplitz graphs by empirical search. Through a computation, we found all non-isomorphic bipartite Toplitze graphs up to order 20. It turns out that all the bipartite Toeplitz graph up to order 13 are obtained by Theorems 2.1, 2.2, Propositions 2.4, 2.5. Therefore these results all together give a characterization of all bipartite Toeplitz graphs up to order 13. Up to order 16, there are only 7 bipartite Toeplitz graph, namely

$$T_{14}\langle 2,6,10,13 \rangle, T_{14}\langle 4,11,12,13 \rangle, T_{15}\langle 2,6,10,14 \rangle, T_{15}\langle 4,12,13,14 \rangle,$$

$$T_{16}\langle 6,11,13,14 \rangle, T_{16}\langle 2,6,10,14,15 \rangle, T_{16}\langle 4,12,13,14,15 \rangle,$$

which do not come from Theorems 2.1 and 2.2, Propositions 2.4 and 2.5. Moreover, Table 1 shows the distribution of bipartite Toeplitz graphs of order up to 20. The characterization of bipartite Toeplitz graphs remains an open problem.

3. Factors and Decomposition of Toeplitz Graphs

We follow up the subject with a definition.

Definition 3.1. *If $A = [a_{ij}]$ is a $n \times n$ matrix whose entries come from a field, then the permanent of A is defined as,*

$$\mathrm{per}(A) = \sum a_{1\delta(1)} \dots a_{n\delta(n)},$$

where the summation is on all the permutations δ on $\{1, \dots, n\}$.

Theorem 3.2. *If A is the adjacency matrix of a graph G, then G has $\{1,2\}$- factor if and only if* $\mathrm{per}(A) > 0$.

Theorem 3.3. *Suppose $G = T_n\langle t_1, \dots, t_k \rangle$ and $S = \{t_1, \dots, t_k\}$. If $n - t \in S$ for some $t \in S$, then* $\mathrm{per}(A) > 0$ *and G has a $\{1,2\}$-factor.*

Proof. Let A be the adjacency matrix of G. The main diagonal contains only zeros. The $n-1$ distinct diagonals above the main diagonal will be labeled $1, 2, \dots, n-1$. Since t and $n - t \in S$, the diagonals with labels t and $n-t$ are 1. Therefore $a_{1(t+1)}a_{2(t+2)} \dots a_{(n-t)n}a_{(n-t+1)1} \dots a_{nt} = 1$. Hence $\mathrm{per}(A) > 0$. □

Theorem 3.4. *Let $G = T_n\langle t_1, t_2, t_3 \rangle$, $n = t_1 + t_3$, and $t_3 = \frac{n}{2}$. If n is even, then G is decomposable to P_4.*

TABLE 1. The distribution of bipartite Toeplitz graphs of order up to 20

Order \ # of arguments	1	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0	0	0
4	3	1	0	0	0	0	0	0	0	0
5	4	2	0	0	0	0	0	0	0	0
6	5	4	2	0	0	0	0	0	0	0
7	6	6	2	0	0	0	0	0	0	0
8	7	9	9	2	0	0	0	0	0	0
9	8	12	7	2	0	0	0	0	0	0
10	9	17	21	10	2	0	0	0	0	0
11	10	23	19	9	2	0	0	0	0	0
12	11	26	47	31	12	2	0	0	0	0
13	12	31	38	27	11	2	0	0	0	0
14	13	40	73	69	40	14	2	0	0	0
15	14	48	76	67	37	13	2	0	0	0
16	15	54	133	148	114	53	16	2	0	0
17	16	63	118	133	107	49	15	2	0	0
18	17	71	188	261	250	166	70	18	2	0
19	18	83	181	251	241	162	65	17	2	0
20	19	93	275	441	513	415	240	87	20	2

Proof. Let $M = \{i, i+\frac{n}{2} | 1 \leq i \leq \frac{n}{2}\} \subseteq E(G)$. Clearly, M is a perfect matching. It is straightforward to check that $G \backslash M \simeq \bigcup_{i=0}^{d-1} C_i$, where $d = \gcd(t_1, t_3)$ and

$$C_i = (d-i, 2d-i, \ldots, \frac{n}{d}d - i)$$

for $i = 0, 1, \ldots, d-1$. We consider a clockwise direction for each C_i for $i = 0, 1, \ldots, d-1$. So each vertex has exactly one in-neighbor and one out-neighbor. For each $e = \{i, j\} \in M$, consider out-neighbors of i and j as u and v, respectively. The path $P_e = (u, i, j, v)$ contains e and $P_e \simeq P_4$. Now, we show $\bigcup_{e \in M} P_e$ is a decomposition of G. Clearly, $M \subseteq \bigcup_{e \in M} P_e$. Suppose that an edge $\{r, s\} \in E(G \backslash M)$. Therefore, there is an i, $0 \le i \le (d-1)$, such that $\{r, s\} \in E(C_i)$. Now, we assume that $r < s$ and $r = dj - i$. So $s = d(j+1) - i$. First let that $r < \frac{n}{2}$ and let $e = \{r, r + \frac{n}{2}\}$. We have $e \in M$ and $\{r, s\} \in P_e$. Next, suppose that $r > \frac{n}{2}$. Let $e = \{r - \frac{n}{2}, r\}$. We have $e \in M$ and $\{r, s\} \in P_e$. Thus $E(G) \subseteq \bigcup_{e \in M} P_e$. Now, we claim that for each edge $\{r, s\} \in G \backslash M$, there exists a unique edge $e \in M$ such that $\{r, s\} \in E(P_e)$. The claim is correct since, in the chosen direction, each $\{r, s\} \in G \backslash M$ is an out-edge of the unique vertex. Hence the proof is complete. $\square$

Lemma 3.5. *(See* [14, p. 147]*) Let G be a $2m$-regular graph. Suppose that T is a tree with m edges. If* $\mathrm{diam}(T) \le \mathrm{grith}(G)$*, then G admits a decomposition to copies of T.*

Corollary 3.6. *Let $G = T_n\langle t_1, \ldots, t_{2m}\rangle$ such that $n = t_i + t_{2m-i+1}$ for each i, $1 \le i \le m$. Let T be a tree with m edges. If* $\mathrm{diam}(T) \le \mathrm{grith}(G)$*, then G admits a decomposition into some copies of T.*

Proof. First suppose that $p \in V(T_n\langle t_1, \ldots, t_{2m}\rangle)$ and $p \le t_1$. Since $t_1 + t_{2m} = n$, $N(p) = \{p + t_1, \ldots, p + t_{2m}\}$. Hence $d(p) = 2m$. Next, assume that $t_{i-1} < p \le t_i$, for some i, $1 \le i \le 2m$. Now, we have $N(p) = \{p - t_1, \ldots, p - t_{i-1}, p + t_1, \ldots, p + t_{2m-i+1}\}$ since $t_i + t_{2m-i+1} = n$. Therefore, $d(p) = 2m$. Finally, suppose that $t_2m < p \le n$. Clearly, $N(p) = \{p - t_1, \ldots, p - t_{2m}\}$. Thus $T_n\langle t_1, \ldots, t_{2m}\rangle$ is a $2m$-regular graph. By Theorem 3.5, the assertion holds. $\square$

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