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THE COMMON MINIMAL COMMON NEIGHBORHOOD DOMINATING SIGNED GRAPHS

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Abstract. In this paper, we define the common minimal common neighborhood dominating signed graph (or common minimal CN-dominating signed graph) of a given signed graph and offer a structural characterization of common minimal CN-dominating signed graphs. In the sequel, we also obtained switching equivalence characterization: $\overline{\Sigma} \sim CMCN(\Sigma)$, where $\overline{\Sigma}$ and $CMCN(\Sigma)$ are complementary signed graph and common minimal CN-signed graph of Σ respectively.

1. Introduction

For standard terminology and notation in graph theory we refer Harary [8] and Zaslavsky [34] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Signed graphs, in which the edges of a graph are labelled positive or negative, have developed many applications and a flourishing literature (see [34]) since their first introduction by Harary in 1953 [9]. Their natural extension to multisigned graphs, in which each edge gets an n-tuple of signs—that is, the sign group is replaced by a direct product of sign groups—has received slight attention, but the further extension to gain graphs (also known as voltage graphs), which have edge labels from an arbitrary group such that reversing the edge orientation inverts the label, have been well studied [34]. Note that in a multisigned group every element is its own inverse, so the question of edge reversal

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does not arise with multisigned graphs.

A *signed graph* $\Sigma = (\Gamma, \sigma)$ is a graph $\Gamma = (V, E)$ together with a function $\sigma : E \to \{+, -\}$, which associates each edge with the sign + or -. In such a signed graph, a subset A of $E(\Gamma)$ is said to be *positive* if it contains an even number of negative edges, otherwise is said to be *negative*. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced* [9] if in every cycle the product of the edge signs is positive. Σ is antibalanced [10] if in every even (odd) cycle the product of the edge signs is positive (resp., negative); equivalently, the negated signed graph $-\Sigma = (\Gamma, -\sigma)$ is balanced. A *marking* of Σ is a function $\mu : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking μ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\mu(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking μ of Σ is called *canonical marking* of Σ. In a signed graph $\Sigma = (\Gamma, \sigma)$, for any $A \subseteq E(\Gamma)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Proposition 1.1. *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

- **(i):** *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary* [9]*).*
- **(ii):** *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \mu(u)\mu(v)$. (Sampathkumar* [15]*).*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Proposition 1. For more new notions on signed graphs refer the papers ([12, 13, 16, 17], [19]-[30]).

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [34].

If $\mu : V(\Gamma) \to \{+, -\}$ is *switching function*, then *switching* of the signed graph $\Sigma = (\Gamma, \sigma)$ by μ means changing σ to σ^μ defined by:

$$\sigma^\mu = \mu(u)\sigma(uv)\mu(v).$$

The signed graph obtained in this way is denoted by Σ^μ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *isomorphic*, written as $\Sigma_1 \cong \Sigma_2$ if there exists a graph isomorphism $f : \Gamma_1 \to \Gamma_2$ (that is a bijection

$f : V(\Gamma_1) \to V(\Gamma_2)$ such that if uv is an edge in Γ_1 then $f(u)f(v)$ is an edge in Γ_2) such that for any edge $e \in E(\Gamma_1)$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $\Sigma_1 = (\Gamma_1, \sigma_1)$ *switches* to a signed graph $\Sigma_2 = (\Gamma_2, \sigma_2)$ (or that Σ_1 and Σ_2 are *switching equivalent*) written $\Sigma_1 \sim \Sigma_2$, whenever there exists a marking μ of Σ_1 such that $\Sigma_1^{\mu} \cong \Sigma_2$. Note that $\Sigma_1 \sim \Sigma_2$ implies that $\Gamma_1 \cong \Gamma_2$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [31]) or *cycle isomorphic* (see [33]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (See [33]):

Proposition 1.2. (**T. Zaslavsky** [33]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [18], the authors introduced the switching and cycle isomorphism for signed digraphs.

2. Common Minimal Common Neighborhood Dominating Signed Graphs

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [6] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [14] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950s and 1960s, beginning with Berge [4] in 1958. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability", which is now known as the domination number of a graph. Oystein Ore [11] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [32]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [5] published a survey paper, in which the notation $\gamma(\Gamma)$ was first used for the domination number of a graph Γ. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Let $\Gamma = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of Γ, if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of Γ is minimal, if for any vertex $v \in D$, $D - \{v\}$ is not a dominating set of Γ (See, Ore [11]).

Let Γ be simple graph with vertex set $V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$. For $i \neq j$, the common neighborhood of the vertices v_i and v_j is the set of vertices different from v_i and v_j which are adjacent to both

v_i and v_j and is denoted by $\Upsilon(v_i, v_j)$. Further, a subset D of V is called the *common neighborhood dominating set* (or *CN-dominating set*) if every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(\Gamma)$ and $|\Upsilon(u,v)| \geq 1$, where $|\Upsilon(u,v)|$ is the number of common neighborhoods between u and v. This concept was introduced by Alwardi et al. [2].

A common neighborhood dominating set D is said to be *minimal common neighborhood dominating set* if no proper subset of D is common neighborhood dominating set (see [2]).

Alwardi and Soner [3] introduced a new class of intersection graphs in the field of domination theory. The commonality minimal CN-dominating graph is denoted by $CMCN(\Gamma)$ is the graph which has the same vertex set as Γ with two vertices are adjacent if and only if there exist minimal CN-dominating in Γ containing them.

Motivated by the existing definition of complement of a signed graph, we extend the notion of common minimal CN-dominating graphs to signed graphs as follows: The *common minimal CN-dominating signed graph* $CMCN(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $CMCN(\Gamma)$ and sign of any edge uv in $CMCN(\Sigma)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called common minimal CN-dominating signed graph, if $\Sigma \cong CMCN(\Sigma')$ for some signed graph Σ'. In the following section, we shall present a characterization of common minimal CN-dominating signed graphs. The purpose of this paper is to initiate a study of this notion.

We now gives a straightforward, yet interesting, property of common minimal CN-dominating signed graphs.

Proposition 2.1. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its common minimal CN-dominating signed graph $CMCN(\Sigma)$ is balanced.*

Proof. Since sign of any edge uv in $CMCN(\Sigma)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ, by Proposition 1.1, $CMCN(\Sigma)$ is balanced. □

For any positive integer k, the k^{th} iterated common minimal CN-dominating signed graph $CMCN(\Sigma)$ of Σ is defined as follows:

$$CMCN^0(\Sigma) = \Sigma, \; CMCN^k(\Sigma) = CMCN(CMCN^{k-1}(\Sigma))$$

Corollary 2.2. *For any signed graph $\Sigma = (\Gamma, \sigma)$ and any positive integer k, $CMCN^k(\Sigma)$ is balanced.*

In [3], the authors characterized graphs for which $CMCN(\Gamma) \cong \overline{\Gamma}$.

Proposition 2.3. (**Anwar Alwardi et al.** [3])
For any graph $\Gamma = (V, E)$, $CMCN(\Gamma) \cong \overline{\Gamma}$ if and only if every minimal CN-dominating set of Γ is independent.

We now characterize signed graphs whose common minimal CN-dominating signed graphs and complementary signed graphs are switching equivalent.

Proposition 2.4. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\Sigma} \sim CMCN(\Sigma)$ if and only if every minimal CN-dominating set of Γ is independent.*

Proof. Suppose $\overline{\Sigma} \sim CMCN(\Sigma)$. This implies, $\overline{\Gamma} \cong CMCN(\Gamma)$ and hence by Proposition 2.3, every minimal CN-dominating set of Γ is independent.

Conversely, suppose that every minimal CN-dominating set of Γ is independent. Then $\overline{\Gamma} \cong CMCN(\Gamma)$ by Proposition 2.3. Now, if Σ is a signed graph with every minimal CN-dominating set of underlying graph Γ is independent, by the definition of complementary signed graph and Proposition 2.1, $\overline{\Sigma}$ and $CMCN(\Sigma)$ are balanced and hence, the result follows from Proposition 1.2. □

Proposition 2.5. *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their common minimal CN-dominating signed graphs are switching equivalent.*

Proof. Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Proposition 2.1, $CMCN(\Sigma_1)$ and $CMCN(\Sigma_2)$ are balanced and hence, the result follows from Proposition 1.2. □

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [10] as follows: $\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

Proposition 2.4 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 2.6. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\eta(\Sigma)} \sim CMCN(\Sigma)$ (or $\overline{\Sigma} \sim CMCN(\eta(\Sigma))$) if, and only if, every minimal CN-dominating set of Γ is independent.*

Corollary 2.7. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\eta(\Sigma)} \sim CMCN(\eta(\Sigma))$ if, and only if, every minimal CN-dominating set of Γ is independent.*

Corollary 2.8. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $CMCN(\Sigma) \sim CMCN(\eta(\Sigma))$.*

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $CMCN(\Sigma)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation of $CMCN(\Sigma)$ is balanced.

Proposition 2.9. *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $CMCN(\Gamma)$ is bipartite then $\eta(CMCN(\Sigma))$ is balanced.*

Proof. Since, by Proposition 2.1, $CMCN(\Sigma)$ is balanced, each cycle C in $CMCN(\Sigma)$ contains even number of negative edges. Also, since $CMCN(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $CMCN(\Sigma)$ is also even. Hence $\eta(CMCN(\Sigma))$ is balanced. □

3. Characterization of Common Minimal CN-Dominating Signed Graphs

The following result characterize signed graphs which are common minimal CN-dominating signed graphs.

Proposition 3.1. *A signed graph $\Sigma = (\Gamma, \sigma)$ is a common minimal CN-dominating signed graph if and only if Σ is balanced signed graph and its underlying graph Γ is a $CMCN(\Gamma)$.*

Proof. Suppose that Σ is balanced and its underlying graph Γ is a common minimal CN-dominating graph. Then there exists a graph Γ' such that $CMCN(\Gamma') \cong \Gamma$. Since Σ is balanced, by Proposition 1.1, there exists a marking μ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $CMCN(\Sigma') \cong \Sigma$. Hence Σ is a common minimal CN-dominating signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a common minimal CN-dominating signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $CMCN(\Sigma') \cong \Sigma$. Hence by Proposition 2.1, Σ is balanced. □

Problem 3.2. Characterize signed graphs for which $\overline{\Sigma} \cong CMCN(\Sigma)$.

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