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BOUNDING THE DOMINATION NUMBER OF A TREE IN TERMS OF ITS ANNIHILATION NUMBER

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Abstract. A set S of vertices in a graph G is a dominating set if every vertex of $V - S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The annihilation number $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of edges in G. In this paper, we show that for any tree T of order $n \geq 2$, $\gamma(T) \leq \frac{3a(T)+2}{4}$, and we characterize the trees achieving this bound.

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum degree* of a graph G is denoted by $\delta = \delta(G)$. We write P_n for a path of order n. For a subset $S \subseteq V(G)$, we let

$$\sum(S, G) = \sum_{v \in S} \deg_G(v).$$

A dominating set, abbreviated DS, of a graph G is a set S of vertices of G such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS of G. A DS of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The literature

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on the subject of domination parameters in graphs has been surveyed and detailed in the two books [7, 8].

Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a graph G arranged in non-decreasing order, and so $d_1 \le d_2 \le \ldots \le d_n$. The annihilation number of G, denoted $a(G)$, is the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. Equivalently, the annihilation number is the largest integer k such that the

$$\sum_{i=1}^{k} d_i \le \sum_{i=k+1}^{n} d_i.$$

It is clear from the definition that if G has m edges and annihilation number k, then $\sum_{i=1}^{k} d_i \le m$. As an immediate consequence of the definition of the annihilation number, Larson and Pepper [10] observed that for any graph G of order n,

$$a(G) \ge \lfloor \frac{n}{2} \rfloor. \tag{1.1}$$

The annihilation number was introduced by Pepper in [12] and has been studied by several authors (see for example [1, 2, 4, 5, 6, 9, 10, 13]). In [12] and [13], Pepper proved that the annihilation number is an upper bound on the independence number of a graph and in [10] the case for equality of the upper bound was characterized by Larson and Pepper. Since independence number is an upper bound on domination number, we deduce that for any graph G, $\gamma(G) \le a(G)$.

The relation between annihilation number and some graph parameters have been studied by several authors. For instance, DeLaViña et al. presented an upper bound on 2-domination number in terms of annihilation number for some classes of graphs [4], Aram et al. investigated the relation between the Roman domination number and the annihilation number of trees [1], Desormeaux et al. proved that for any tree T, $a(T) + 1$ is an upper bound on the total domination number [6].

Our purpose in this paper is to establish an upper bound on the domination number of a tree in terms of its annihilation number.

The domination and annihilation numbers are easy to compute for paths and we have the following Observations.

Observation 1.1. For $n \ge 2$,

$$\gamma(P_n) = \lceil \frac{n}{3} \rceil.$$

Observation 1.2. For $n \ge 2$,

$$a(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proposition 1.3. For $n \ge 2$, $\gamma(P_n) \le \frac{3a(T)+2}{4}$ with equality if and only if $n = 4$.
Furthermore, $\gamma(P_n) = \frac{3a(T)+1}{4}$ if and only if $n = 2$ or 10.

2. Main results

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_t. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider.

Proposition 2.1. If T is a spider different from P_4, then $\gamma(T) \leq \frac{3a(T)+1}{4}$ with equality if and only if T is the wounded spider obtained from $K_{1,4}$ by subdividing its exactly three edges.

Proof. Let T be a spider. If $T = S_t$ is a healthy spider for some $t \geq 2$, then obviously $\gamma(T) = t$ and $a(T) = t + \lfloor \frac{t}{2} \rfloor$, and hence $\gamma(T) < \frac{3a(T)}{4}$.

Now let T be a wounded spider obtained from $K_{1,t}$ $(t \geq 2)$ by subdividing $0 \leq s \leq t-1$ edges. If $s = 0$, then T is a star and we have $\gamma(T) = 1$ and $a(T) = t$. Hence $\gamma(T) < \frac{3a(T)}{4}$. Suppose $s > 0$. Since $T \neq P_4$, we have $s \neq 1$ or $t \neq 2$. Then $\gamma(T) = 1 + s$ and $a(T) = s + \lfloor \frac{s}{2} \rfloor + (t-s)$. If $s = 3$ and $t = 4$, then clearly $\gamma(T) = \frac{3a(T)+1}{4}$. Otherwise, it is easy to see that $\gamma(T) < \frac{3a(T)+1}{4}$.

If T is the wounded spider obtained from $K_{1,4}$ by subdividing its exactly three edges, then clearly $\gamma(T) = 4$ and $a(T) = 5$. Hence $\gamma(T) = \frac{3a(T)+1}{4}$, and the proof is complete. □

A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. A strong support vertex is said to be *end-strong support vertex* if all its neighbors except one of them are leaves. For $r, s \geq 1$, a double star $S(r,s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v. Let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. In the sequel, we denote by $T - T_v$, the tree obtained from a rooted tree T by deleting all vertices of $D[v]$.

Theorem 2.2. If T is a tree of order $n \geq 2$, then $\gamma(T) \leq \frac{3a(T)+2}{4}$.

Proof. The proof is by induction on n. The statement holds for all trees of order $n \leq 4$. For the inductive hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree T of order less than n the result is true. Let T be a tree of order n. If T is a path, then the result follows by Proposition 1.3. So, assume T is not a path. If $\text{diam}(T) = 2$, then T is a star and it follows from Proposition 2.1 that $\gamma(T) < \frac{3a(T)}{4}$. If $\text{diam}(T) = 3$, then T is a double star $S(r,s)$. In this case, $a(T) = r + s \geq 3$ and $\gamma(T) = 2$. Hence $\gamma(T) < \frac{3a(T)}{4}$. Thus, we may assume that $\text{diam}(T) \geq 4$.

In what follows, we will consider trees T' formed from T by removing a set of vertices. For such a tree T' of order n', let $d'_1, d'_2, \ldots, d'_{n'}$ be the non-decreasing degree sequence of T', and let S' be a set of vertices corresponding to the first $a(T')$ terms in the degree sequence of T'. In fact, if $u_1, u_2, \ldots, u_{n'}$ are the vertices of T' such that $\deg(u_i) = d'_i$ for each i, then $S' = \{u_1, u_2, \ldots, u_{a(T')}\}$. We denote the size of T' by m'. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. T has no end-strong support vertex.

Let T have a end-strong support vertex u and let u_1, u_2 be the two leaves adjacent to u and let w be a vertex of T with maximum distance from u. Root T at w and let v be the parent of u. Assume $T' = T - T_u$. Then obviously $\gamma(T) \leq \gamma(T') + 1$. If $v \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) - 1 \leq \sum(S', T') \leq m' = m - 3$, and hence $\sum(S', T) \leq m - 2$. Let $S = S' \cup \{u_1, u_2\}$. Then $\sum(S, T) = \sum(S', T) + 2 \leq m$ implying that $a(T) \geq a(T') + 2$. By inductive hypothesis, we obtain

$$\gamma(T) \leq \gamma(T') + 1 \leq \frac{3a(T') + 2}{4} + 1 \leq \frac{3(a(T) - 2) + 2}{4} + 1 = \frac{3a(T)}{4},$$

as desired. (■)

Let $v_1 v_2 \ldots v_D$ be a diametral path in T and root T at v_D. By Claim 1, we have $\deg(v_2) = 2$ and all neighbors of v_3, except v_4, are leaves or support vertices of degree 2. Similarly, by rooting T at v_1, we may assume $\deg(v_{D-1}) = 2$ and all neighbors of v_{D-2}, except v_{D-3}, are leaves or support vertices of degree 2. If $\mathrm{diam}(T) = 4$, then T is a spider and the result follows by Proposition 2.1. Assume $\mathrm{diam}(T) \geq 5$.

Claim 2. $\deg_T(v_3) \leq 3$.

Let $\deg_T(v_3) \geq 4$. First let v_3 be adjacent to a support vertex, say w_2, not in $\{v_2, v_4\}$. Suppose w_1 is the leaf adjacent to w_2 and let $T' = T - \{v_1, v_2, w_1, w_2\}$. Then every dominating set of T' can be extended to a dominating set of T by adding v_1, w_1 and hence $\gamma(T) \leq \gamma(T') + 2$. Suppose $v_3 \notin S'$. Then $\sum(S', T) = \sum(S', T')$. In this case, let $S = S' \cup \{v_1, v_2, w_1\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w_1) \leq m$ implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. It follows from inductive hypothesis that

$$\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T') + 2}{4} + 2 \leq \frac{3(a(T) - 3) + 2}{4} + 2 < \frac{3a(T) + 2}{4}.$$

Now assume $v_3 \in S'$. Then $\sum(S', T) = \sum(S', T') + 1$. In this case, let $S = (S' - \{v_3\}) \cup \{v_1, v_2, w_1, w_2\}$. Since $\deg_T(w_2) \leq \deg_{T'}(v_3)$, we have that $\sum(S, T) = \sum(S', T) - \deg_{T'}(v_3) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w_1) + \deg_T(w_2) \leq m$. Therefore, $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ and the result follows as above.

Now let all neighbors of v_3, except v_2, v_4, are leaves. Assume $T' = T - T_{v_3}$. Then every dominating set of T' can be extended to a domination set of T by adding v_1, v_3 and hence $\gamma(T) \leq \gamma(T') + 2$. Suppose z_1, z_2 are two leaves adjacent to v_3 and let $S = S' \cup \{v_1, z_1, z_2\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(z_1) + \deg_T(z_2) \leq m$ implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ and the result follows as above. (■)

Claim 3. $\deg_T(v_3) = 2$.

Assume $\deg_T(v_3) = 3$. First let v_3 be adjacent to a support vertex x_2 of degree 2, not in $\{v_2, v_4\}$. Suppose x_1 is the leaf adjacent to x_2 and let $T' = T - T_{v_3}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding v_2, x_2 and hence $\gamma(T) \leq \gamma(T') + 2$. If $v_4 \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v_4 \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 4$. Let $S = S' \cup \{v_1, v_2, x_1\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(x_1) \leq m$ implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. It follows from inductive hypothesis that $\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T')+2}{4} + 2 \leq \frac{3(a(T)-3)+2}{4} + 2 < \frac{3a(T)+2}{4}$.

Now let v_3 be adjacent to a leaf w. We consider the following cases.

Case 3.1. $\deg_T(v_4) \geq 4$.
Let $T' = T - T_{v_3}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding v_2, v_3. Hence $\gamma(T) \leq \gamma(T') + 2$. Suppose that $v_4 \notin S'$. In this case, let $S = S' \cup \{v_1, v_2, w\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) = \sum(S',T') + 4 \leq m' + 4 = m$, implying that $a(T) \geq a(T') + 3$. By inductive hypothesis we have $\gamma(T) < \frac{3a(T)+2}{4}$.

Now let $v_4 \in S'$. Assume $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3, w\}$. Then $\sum(S,T) = \sum(S',T') - \deg_{T'}(v_4) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w) \leq m$ and hence $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. By inductive hypothesis, we obtain $\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T')+2}{4} + 2 \leq \frac{3(a(T)-3)+2}{4} + 2 < \frac{3a(T)+2}{4}$.

Case 3.2. $\deg_T(v_4) = 2$.
Let $T' = T - T_{v_4}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding v_1, v_3 and so $\gamma(T) \leq \gamma(T') + 2$. If $v_5 \notin S'$, then $\sum(S',T) = \sum(S',T')$ and if $v_5 \in S'$, then $\sum(S',T) = \sum(S',T') + 1$. Thus, $\sum(S',T) \leq \sum(S',T') + 1 \leq m' + 1 = m - 4$. Let $S = S' \cup \{v_1, v_2, w\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) \leq m$ implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. By inductive hypothesis, we obtain $\gamma(T) < \frac{3a(T)+2}{4}$.

Case 3.3. $\deg_T(v_4) = 3$ and there exists a path $v_4 z_3 z_2 z_1$ in T such that $\deg_T(z_3) = 2$, $\deg_T(z_1) = 1$ and $z_3 \notin \{v_3, v_5\}$.
By Claim 1, we have $\deg_T(z_2) = 2$. Let $T' = T - T_{z_3}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding z_2 and so $\gamma(T) \leq \gamma(T') + 1$. Assume that $v_4 \notin S'$. In this case, let $S = S' \cup \{z_1, z_2\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(z_1) + \deg_T(z_2) \leq m' + 3 = m$, implying that $a(T) \geq |S| = |S'| + 2 = a(T') + 2$. Applying inductive hypothesis we obtain $\gamma(T) \leq \gamma(T') + 1 \leq \frac{3a(T')+2}{4} + 1 \leq \frac{3(a(T)-2)+2}{4} + 1 \leq \frac{3a(T)}{4} < \frac{3a(T)+2}{4}$.

Now suppose $v_4 \in S'$. In this case, let $S = (S' - \{v_4\}) \cup \{z_1, z_2, z_3\}$. Since $\deg_T(z_3) \leq \deg_{T'}(v_4)$, we have $\sum(S,T) = \sum(S',T') - \deg_{T'}(v_4) + \deg_T(z_1) + \deg_T(z_2) + \deg_T(z_3) \leq \sum(S',T') + 3 \leq m' + 3 \leq m$. Therefore, $a(T) \geq |S| = |S'| + 2 = a(T') + 2$ and the result follows by inductive hypothesis as above.

Case 3.4. $\deg_T(v_4) = 3$ and there exists a path $z_4 z_3 z_2 z_1$ in T such that $v_4 z_3 \in E(T)$, all neighbors of z_3, except z_2, v_4, are leaves, $\deg(z_1) = \deg(z_4) = 1$ and $z_3 \notin \{v_3, v_5\}$.
By Claim 1, we may assume $\deg_T(z_2) = 2$. If $\deg_T(z_3) \geq 4$, then the result follows as Claim 2. Thus, we assume $\deg_T(z_3) = 3$. Let $T' = T - T_{v_4}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding z_3, z_1, v_3, v_1, implying that $\gamma(T) \leq \gamma(T') + 4$. If $v_5 \notin S'$, then $\sum(S',T) = \sum(S',T')$ and if $v_5 \in S'$, then $\sum(S',T) = \sum(S',T') + 1$. Thus, $\sum(S',T) \leq m - 8$. Let $S = S' \cup \{v_1, v_2, w, z_1, z_2, z_4\}$. Then $\sum(S,T) = \sum(S',T) + 8 \leq m$ implying that $a(T) \geq |S| = a(T') + 6$. By inductive hypothesis, we have $\gamma(T) \leq \gamma(T') + 4 \leq \frac{3a(T')+2}{4} + 4 \leq \frac{3(a(T)-6)+2}{4} + 4 \leq \frac{3a(T)}{4} < \frac{3a(T)+2}{4}$.

Case 3.5. $\deg(v_4) = 3$ and v_4 is adjacent to a leaf, say w'.
Assume $T' = T - T_{v_4}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding v_1, v_3, v_4 and so $\gamma(T) \leq \gamma(T') + 3$. As above, we have $\sum(S',T) \leq m - 5$. Let $S = S' \cup \{v_1, v_2, w, w'\}$.

Then $\sum(S,T) \leq m$ and hence $a(T) \geq |S| = a(T') + 4$. Applying inductive hypothesis we obtain $\gamma(T) \leq \gamma(T') + 3 \leq \frac{3a(T')+2}{4} + 3 \leq \frac{3(a(T)-4)+2}{4} + 3 = \frac{3a(T)+2}{4}$.

Case 3.6. $\deg(v_4) = 3$ and v_4 is adjacent to a support vertex $z_2 \neq v_5$.
By Claim 1, we may assume $\deg_T(z_2) = 2$. Let z_1 be the leaf adjacent to z_2 and let $T' = T - T_{v_4}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding z_2, v_1, v_3 and so $\gamma(T) \leq \gamma(T') + 3$. Clearly $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 6$. Let $S = S' \cup \{v_1, v_2, w, z_1\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(z_1) \leq m$ and hence $a(T) \geq |S| = |S'| + 4 = a(T') + 4$. Applying inductive hypothesis we obtain $\gamma(T) \leq \gamma(T') + 3 \leq \frac{3a(T')+2}{4} + 3 \leq \frac{3(a(T)-4)+2}{4} + 3 = \frac{3a(T)+2}{4}$. (■)

Similarly, by rooting T at v_1, we may assume that $\deg(v_{D-2}) = 2$.

We now return to the proof of theorem. If $\mathrm{diam}(T) = 5$ or 6 then $T = P_6$ or P_7, respectively, and the result is immediate by Proposition 1.3. Let $\mathrm{diam}(T) \geq 7$ and $T' = T - \{v_1, v_2, v_3, v_D, v_{D-1}, v_{D-2}\}$. Then every $\gamma(T')$-set can be extended to a dominating set of T by adding v_2, v_{D-1} and hence $\gamma(T) \leq \gamma(T') + 2$. Suppose $S = S' \cup \{v_1, v_2, v_D\}$. Then $\sum(S,T) \leq m$ implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. Applying inductive hypothesis, we obtain $\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T')+2}{4} + 2 \leq \frac{3(a(T)-3)+2}{4} + 2 < \frac{3a(T)+2}{4}$. This completes the proof. □

Theorem 2.3. Let T be a tree of order $n \geq 2$. Then $\gamma(T) = \frac{3a(T)+2}{4}$ if and only if $T = P_4$.

Proof. If $T = P_4$, then clearly $\gamma(T) = \frac{3a(T)+2}{4}$.

Conversely, let $\gamma(T) = \frac{3a(T)+2}{4}$. By Proposition 1.3, we have $n \geq 4$. Suppose to the contrary that $T \neq P_4$. Among all trees with these properties, let T be chosen so that its order is minimum. Let $v_1 v_2 \ldots v_D$ be a diametral path in T and root T at v_D. By the proof of Theorem 2.2, we may assume $\mathrm{diam}(T) \geq 5$ and we need to consider two cases.

Case 1. $\deg(v_2) = 2, \deg(v_3) = \deg(v_4) = 3$, v_3 is adjacent to a leaf w and v_4 is adjacent to a leaf w'. Let $T' = T - T_{v_4}$. By the Case 3.5, we have $\gamma(T) \leq \gamma(T') + 3$ and $a(T) \geq a(T') + 4$. It follows from Theorem 2.2 that

$$\gamma(T) \leq \gamma(T') + 3 \leq \frac{3a(T')+2}{4} + 3 \leq \frac{3(a(T)-4)+2}{4} + 3 = \frac{3a(T)+2}{4}. \tag{2.1}$$

Since $\gamma(T) = \frac{3a(T)+2}{4}$, the inequalities occurring in (2.1) become equalities. In particular, we have $\gamma(T) = \gamma(T') + 3$ and $\gamma(T') = \frac{3a(T')+2}{4}$. By the choice of T, we deduce that $T' = P_4$. If v_4 is adjacent to a leaf of $T' = P_4$, then clearly $\gamma(T) = \gamma(T') + 2 < \gamma(T') + 3$, a contradiction. If v_4 is adjacent to a support vertex of $T' = P_4$, then it is easy to see that $\gamma(T) = 5$ and $a(T) = 7$ and hence $\gamma(T) = 5 < \frac{3a(T)+2}{4}$, which is a contradiction.

Case 2. $\deg(v_2) = 2, \deg(v_3) = \deg(v_4) = 3$ and v_3 is adjacent to a leaf w and v_4 is adjacent to a support vertex z_2 of degree 2.
Assume $T' = T - T_{v_4}$. An argument similar to that described in Case 1, shows that $T' = P_4$. It is easy to see that $\gamma(T) = 5$ and $a(T) = 7$. Hence $\gamma(T) = 5 < \frac{3a(T)+2}{4}$, a contradiction. This completes the proof. □

We conclude this paper with two open problems.

Problem 1. Characterize the trees T for which $\gamma(T) = \frac{3a(T)+1}{4}$.

If G is a connected graph of order n with minimum degree at least three, then it is known ([14]) that $\gamma(G) \leq \frac{3n}{8}$. Hence if G is a connected graph of order n with minimum degree at least 3, then it follows from (1.1) that $\gamma(G) \leq \frac{3a(T)+1}{4}$.

Cockayne, Ko and Shepherd [3] proved that if a connected graph G of order n, is $K_{1,3}$-free and $K_3 o K_1$-free then $\gamma(G) \leq \lceil \frac{n}{3} \rceil$. Using (1.1), we deduce that if G is a connected $K_{1,3}$-free and $K_3 o K_1$-free graph of order n, then $\gamma(G) \leq \frac{3a(T)+2}{4}$. Hence we propose the following conjecture.

Conjecture 2. For any connected graph G, $\gamma(G) \leq \frac{3a(G)+2}{4}$.

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