

ECCENTRIC CONNECTIVITY INDEX AND ECCENTRIC DISTANCE SUM OF SOME GRAPH OPERATIONS

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ABSTRACT. Let $G = (V, E)$ be a connected graph. The eccentric connectivity index of G , $\xi^c(G)$, is defined as $\xi^c(G) = \sum_{v \in V(G)} \deg(v)ec(v)$, where $\deg(v)$ is the degree of a vertex v and $ec(v)$ is its eccentricity. The eccentric distance sum of G is defined as $\xi^d(G) = \sum_{v \in V(G)} ec(v)D(v)$, where $D(v) = \sum_{u \in V(G)} d(u, v)$. In this paper, we calculate the eccentric connectivity index and eccentric distance sum of generalized hierarchical product of graphs. Moreover, we present the exact formulae for the eccentric connectivity index of F -sum graphs in terms of some invariants of the factors.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For vertices $u, v \in V(G)$ the distance $d_G(u, v)$ between u and v is defined as the length of a shortest path connecting u and v in G . The eccentricity $ec(v)$ of v is the maximum distance from v to any other vertex. We use $\deg(v)$ to denote the degree of v . The total eccentricity of a graph G is defined as $\zeta(G) = \sum_{v \in V(G)} ec(v)$ [5].

In general, a topological index, sometimes also known as a graph-theoretic index, is a numerical invariant of a graph. There are several topological indices having been defined such as Wiener index, Zagreb index and PI-index etc. Recently, a lot of results on the eccentric connectivity index and eccentric distance sum have been obtained and some of them have been applied as means for modeling chemical, pharmaceutical and other properties of molecules, for details see [10, 13, 19, 20].

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The eccentric connectivity index (ECI) of a graph G , denoted by $\xi^c(G)$, is defined as $\xi^c(G) = \sum_{v \in V(G)} \deg(v)ec(v)$. In [19], the eccentric connectivity index has been studied and used for mathematical models of chemical and biological activities, and the exact lower and upper bounds are obtained in [17]. In [1], the author achieved the ECI of nanotubes and nanotori and in [7] the ECI of hexagonal belts and chains are studied. Moreover, in [12] the ECI of chemical trees are determined. For the complete development on the study of the ECI of graphs, see the comprehensive survey [13].

The eccentric distance sum (EDS) of a graph G is defined as $\xi^d(G) = \sum_{v \in V(G)} ec(v)D(v)$, where $D(v) = \sum_{u \in V(G)} d(u, v)$, it can also be defined alternatively as $\xi^d(G) = \sum_{\{u,v\} \subseteq V(G)} (ec(u)+ec(v))d(u, v)$ ([14]). For the recent survey on EDS see [11, 21]. The Wiener index $W(G)$ of a graph G is one of the most studied indices, it is defined as the sum of all distance between unordered pairs of vertices, i.e., $W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v)$. For further results on the Wiener index see [6] and the references therein. In [10, 19], the relationships between the ECI and Wiener index are investigated.

In [2, 3], as an extension of the Cartesian product of graphs, the authors introduce the generalized hierarchical product of graphs, and in [9] F -sum graphs are introduced. By definition, Cartesian product is a special case of the generalized hierarchical product and some well-known properties of the Cartesian product are inherited by the generalized hierarchical product. Moreover, the F -sum products are also special cases of the generalized hierarchical product, and therefore the results on the generalized hierarchical product can be used to obtain some properties of the F -sum products. In fact, in Section 3, we shall use the formula on ECI of generalized hierarchical product to deduce the ECI of F -sum graphs. In this paper we compute the ECI and EDS of generalized hierarchical product of graphs. Also as an application, we compute the ECI of the F -sum graphs by presenting exact formulae for the ECI of the F -sum graphs in terms of some invariants of the factors.

2. ECI and EDS of generalized hierarchical product of graphs

Definition 2.1. Let G and H be two graphs with nonempty vertex subset $U \subseteq V(G)$. Then the generalized hierarchical product $G(U) \square H$ is the graph with the vertex set $V(G) \times V(H)$ and two vertices (u, v) and (u', v') adjoined by an edge under the following condition:

$$(u, v) \sim (u', v') \iff \begin{cases} u = u' \in U \text{ and } v \sim v' \text{ in } H \text{ or} \\ v = v' \text{ and } u \sim u' \text{ in } G. \end{cases}$$

From the definition, it is obvious that

$$\deg_{G(U) \square H}(u, v) = \begin{cases} \deg_G(u) + \deg_H(v) & u \in U \\ \deg_G(u) & u \in V(G) \setminus U. \end{cases}$$

For $\emptyset \neq U \subseteq V(G)$, a path between vertices $u, v \in V(G)$ through U is a uv -path in G containing some vertex $z \in U$ (vertex z could be the vertex u or v). The distance between u and v through U , denoted by $d_{G(U)}(u, v)$, is the length of a shortest path between u and v through U . Note that, if

one of the vertices u and v belongs to U , then $d_{G(U)}(u, v) = d_G(u, v)$, see [8]. Similar to the distance through U , we define the following invariants related to U in G :

$$\begin{aligned}
 ec_{G(U)}(u) &= \max_{v \in V(G)} d_{G(U)}(u, v) \text{ (see [3])} \\
 W(G(U)) &= \sum_{\{u,v\} \subseteq V(G)} d_{G(U)}(u, v) \text{ (see [8])} \\
 \zeta(G(U)) &= \sum_{v \in V(G)} ec_{G(U)}(v) \\
 \xi^c(G(U)) &= \sum_{v \in V(G)} deg_G(v) ec_{G(U)}(v) \\
 \xi^d(G(U)) &= \sum_{\{u,v\} \subseteq V(G)} (ec_{G(U)}(u) + ec_{G(U)}(v)) d_{G(U)}(u, v) \\
 \varepsilon(G(U)) &= \sum_{u_i \in U} ec_{G(U)}(u_i) .
 \end{aligned}$$

Theorem 2.2. [3] *Let G and H be graphs and $U \subseteq V(G)$. Then*

$$\begin{aligned}
 (a) \ d_{G(U) \cap H}((u, v), (u', v')) &= \begin{cases} d_{G(U)}(u, u') + d_H(v, v') & v \neq v' \\ d_G(u, u') & v = v' \end{cases} \\
 (b) \ ec_{G(U) \cap H}(u, v) &= ec_{G(U)}(u) + ec_H(v).
 \end{aligned}$$

Now we give our first theorem of this section.

Theorem 2.3. *Let G and H be two connected graphs and $U \subseteq V(G)$. Then we have*

$$\xi^c(G(U) \cap H) = |V(H)|\xi^c(G(U)) + 2|E(G)|\zeta(H) + 2|E(H)|\varepsilon(G(U)) + |U|\xi^c(H) .$$

Proof. Set $V(G) = \{u_1, u_2, \dots, u_n\}, V(H) = \{v_1, v_2, \dots, v_m\}$. Then

$$\begin{aligned}
 \xi^c(G(U) \cap H) &= \sum_{i=1}^n \sum_{j=1}^m deg_{G(U) \cap H}(u_i, v_j) ec_{G(U) \cap H}(u_i, v_j) \\
 &= \sum_{u_i \in U} \sum_{j=1}^m deg_{G(U) \cap H}(u_i, v_j) ec_{G(U) \cap H}(u_i, v_j) \\
 &\quad + \sum_{u_i \in V(G) \setminus U} \sum_{j=1}^m deg_{G(U) \cap H}(u_i, v_j) ec_{G(U) \cap H}(u_i, v_j) \\
 &= \sum_{u_i \in U} \sum_{j=1}^m (deg_G(u_i) + deg_H(v_j))(ec_{G(U)}(u_i) + ec_H(v_j)) \\
 &\quad + \sum_{u_i \in V(G) \setminus U} \sum_{j=1}^m deg_G(u_i)(ec_{G(U)}(u_i) + ec_H(v_j)) \\
 &= |V(H)| \sum_{u_i \in U} deg_G(u_i) ec_{G(U)}(u_i) + \sum_{u_i \in U} deg_G(u_i) \zeta(H) \\
 &\quad + 2|E(H)| \sum_{u_i \in U} ec_{G(U)}(u_i) + |U| \xi^c(H) \\
 &\quad + |V(H)| \sum_{u_i \in V(G) \setminus U} deg_G(u_i) ec_{G(U)}(u_i) + \sum_{u_i \in V(G) \setminus U} deg_G(u_i) \cdot \zeta(H) \\
 &= |V(H)|\xi^c(G(U)) + 2|E(G)|\zeta(H) + 2|E(H)|\varepsilon(G(U)) + |U|\xi^c(H).
 \end{aligned}$$

□

Corollary 2.4. Let G and H be two connected graphs. Then

$$\xi^c(G \square H) = |V(H)|\xi^c(G) + 2|E(G)|\zeta(H) + 2|E(H)|\zeta(G) + |V(G)|\xi^c(H).$$

Theorem 2.5. Let G and H be graphs with $U \subseteq V(G)$. Then

$$\xi^d(G(U) \sqcap H) = |V(H)|^2\xi^d(G(U)) + |V(G)|^2\xi^d(H) + 2|V(G)|\zeta(G(U)) \cdot W(H) + 2|V(H)| \cdot \zeta(H) \cdot W(G(U))$$

Proof. Set $V(G) = \{u_1, u_2, \dots, u_n\}$, $V(H) = \{v_1, v_2, \dots, v_m\}$ and $\emptyset \neq U \subseteq V(G)$. Then

$$\begin{aligned} \xi^d(G(U) \sqcap H) &= \sum_{(u_i, v_j), (u_k, v_l) \in V(G(U) \sqcap H)} (ec_{G(U) \sqcap H}(u_i, v_j) + ec_{G(U) \sqcap H}(u_k, v_l)) \\ &\cdot d_{G(U) \sqcap H}((u_i, v_j), (u_k, v_l)) \\ &= \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j \neq l)} (ec_{G(U)}(u_i) + ec_H(v_j) + ec_{G(U)}(u_k) \\ &+ ec_H(v_l))(d_{G(U)}(u_i, u_k) + d_H(v_j, v_l)) + \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j=l)} \\ &(ec_{G(U)}(u_i) + ec_H(v_j) + ec_{G(U)}(u_k) + ec_H(v_l))d_{G(U)}(u_i, u_k) \\ &= \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j \neq l)} (ec_{G(U)}(u_i) + ec_{G(U)}(u_k))d_{G(U)}(u_i, u_k) \\ &+ \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j \neq l)} (ec_H(v_j) + ec_H(v_l))d_H(v_j, v_l) \\ &+ \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j \neq l)} (ec_{G(U)}(u_i) + ec_{G(U)}(u_k))d_H(v_j, v_l) \\ &+ \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j \neq l)} (ec_H(v_j) + ec_H(v_l))d_{G(U)}(u_i, u_k) \\ &+ \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j=l)} (ec_{G(U)}(u_i) + ec_{G(U)}(u_k))d_{G(U)}(u_i, u_k) \\ &+ \sum_{u_i, u_k \in V(G)} \sum_{v_j, v_l \in V(H) (j=l)} (ec_H(v_j) + ec_H(v_l))d_{G(U)}(u_i, u_k) \\ &= |V(H)|^2\xi^d(G(U)) + |V(G)|^2\xi^d(H) + 2|V(G)|\zeta(G(U)) \cdot W(H) \\ &+ 2|V(H)| \cdot \zeta(H) \cdot W(G(U)). \end{aligned}$$

□

Corollary 2.6. ([14])

$$\xi^d(G \square H) = |V(H)|^2 \cdot \xi^d(G) + |V(G)|^2 \cdot \xi^d(H) + 2|V(G)|\zeta(G) \cdot W(H) + 2|V(H)|\zeta(H) \cdot W(G).$$

Example 1. Let $P_n (n \geq 1)$ and $C_n (n \geq 3)$ be a path and a cycle of order n , respectively.

$$\xi^c(P_n) = \begin{cases} \frac{1}{2}(3n^2 - 6n + 4), & n \text{ is even,} \\ \frac{3}{2}(n-1)^2, & n \text{ is odd.} \end{cases} \text{ and } \xi^c(C_n) = \begin{cases} n^2, & n \text{ is even,} \\ n(n-1), & n \text{ is odd.} \end{cases}$$

$$\zeta(P_n) = \begin{cases} \frac{3}{4}n^2 - \frac{1}{2}n, & n \text{ is even,} \\ \frac{3}{4}n^2 - \frac{1}{2}n - \frac{1}{4}, & n \text{ is odd.} \end{cases} \text{ and } \zeta(C_n) = \begin{cases} \frac{1}{2}n^2, & n \text{ is even,} \\ \frac{1}{2}n(n-1), & n \text{ is odd.} \end{cases}$$

Then using the above results, one can obtain the following:

$$(1)\xi^c(P_m \square P_n) = \begin{cases} 3m^2n + 3mn^2 - \frac{3}{2}n^2 - \frac{3}{2}m^2 - 8mn + 3n + 3m, & m, n \text{ are even} \\ 3m^2n + 3mn^2 - \frac{3}{2}n^2 - \frac{3}{2}m^2 - 8mn + 2n + 2m + 1, & m, n \text{ are odd} \\ 3m^2n + 3mn^2 - \frac{3}{2}n^2 - \frac{3}{2}m^2 - 8mn + 3n + 2m + \frac{1}{2}, & m \text{ is even, } n \text{ is odd} \\ 3m^2n + 3mn^2 - \frac{3}{2}n^2 - \frac{3}{2}m^2 - 8mn + 2n + 3m + \frac{1}{2}, & m \text{ is odd, } n \text{ is even.} \end{cases}$$

$$(2)\xi^c(C_m \square C_n) = \begin{cases} 2m^2n + 2mn^2, & m, n \text{ are even} \\ 2m^2n + 2mn^2 - 4mn, & m, n \text{ are odd} \\ 2m^2n + 2mn^2 - 2mn, & m \text{ is even, } n \text{ is odd} \\ 2m^2n + 2mn^2 - 2mn, & m \text{ is odd, } n \text{ is even.} \end{cases}$$

$$(3)\xi^c(P_m \square C_n) = \begin{cases} 3m^2n + 2mn^2 - n^2 - 4mn + 2n, & m, n \text{ are even} \\ 3m^2n + 2mn^2 - n^2 - 5mn + 2n, & m, n \text{ are odd} \\ 3m^2n + 2mn^2 - n^2 - 6mn + 3n, & m \text{ is even, } n \text{ is odd} \\ 3m^2n + 2mn^2 - n^2 - 4mn + n, & m \text{ is odd, } n \text{ is even.} \end{cases}$$

3. ECI of F -sum graphs

First we recall some notation ([8, 9]). Let G be a connected graph.

- (a) $S(G)$ is obtained from G by replacing each edge of G by a path of length two.
- (b) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G , then joining each new vertex to the end vertices of the corresponding edge.
- (c) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G , then joining with edges those pairs of new vertices on adjacent edges of G .
- (d) $T(G)$ has its vertices the edges and vertices of G . Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G , $T(G)$ is also called the total graph of G .

Definition 3.1. Let F be one of the symbols S, R, Q or T . The F -sum $G +_F H$ of G and H is a graph with vertex set $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) of $G +_F H$ are adjacent if and only if $[u_1 = u_2 \text{ and } v_1 \sim v_2 \text{ in } H]$ or $[v_1 = v_2 \text{ and } u_1 \sim u_2 \text{ in } F(G)]$.

Note that if we set $U = V(G) \subseteq V(F(G))$, then $G +_F H = F(G)(U) \square H$. So by using Theorem 2.3, we have more easier way to compute the ECI of $G +_F H$.

In [9], F -sum graphs are introduced and the Wiener indices of the resulting graphs is studied. For the explicit expressions for the vertex PI indices of four sums of two graphs see [15]. In [16], the

authors determine the hyper and reverse Wiener indices of the F -sum graphs, and subject to some condition, they present some exact expressions for the reverse Wiener indices of the F -sum graphs. Among the results of F -sums, in this paper we will use the following one, where $L(G)$ is the line graph of G .

Theorem 3.2. [8] *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two connected graphs. Suppose that $U = V(G) \subseteq V(F(G))$, $F \in \{S, R, Q, T\}$. Then*

$$W(F(G)(U)) = W(F(G)) + |E_1|, F = S, R$$

$$W(F(G)(U)) = W(F(G)) + W(L(G)) + |E_1| + |E_1|^2, F = Q, T.$$

Lemma 3.3. *Let $F \in \{S, R, Q, T\}$ and let $G +_F H = F(G)(U) \cap H$, where $U = V(G)$. Then*

$$(a) |V(F(G))| = |V(G)| + |E(G)|, |E(S(G))| = 2|E(G)|, |E(R(G))| = 3|E(G)|, |E(Q(G))| = 2|E(G)| + |E(L(G))|, |E(T(G))| = 3|E(G)| + |E(L(G))|.$$

(b) *For every vertex $v \in U$, we have*

$$\deg_{S(G)}(v) = \deg_{Q(G)}(v) = \deg_G(v), \deg_{R(G)}(v) = \deg_{T(G)}(v) = 2\deg_G(v),$$

$$ec_{S(G)(U)}(v) = 2ec_G(v), ec_{R(G)(U)}(v) = ec_{T(G)(U)}(v) = ec_{Q(G)(U)}(v) - 1 = ec_G(v).$$

(c) *For every vertex $v \in V(F(G)) \setminus U$, we have*

$$\deg_{S(G)}(v) = \deg_{R(G)}(v) = 2, \deg_{Q(G)}(v) = \deg_{T(G)}(v) = \deg_{L(G)}(v) + 2,$$

$$ec_{S(G)(U)}(v) = 2ec_{L(G)}(v) + 1, ec_{R(G)(U)}(v) = ec_{Q(G)(U)}(v) = ec_{T(G)(U)}(v) = ec_{L(G)}(v) + 1.$$

Theorem 3.4. *Let $G(n \geq 2)$ and H be two connected graphs. Then*

$$(1) \xi^c(G +_S H) = 2|V(H)|[\xi^c(G) + 2\zeta(L(G)) + |E(G)|] + 4|E(G)|\zeta(H) + 4|E(H)|\zeta(G) + |V(G)|\xi^c(H),$$

$$(2) \xi^c(G +_R H) = 2|V(H)|[\xi^c(G) + 2\zeta(L(G)) + |E(G)|] + 6|E(G)|\zeta(H) + 2|E(H)|\zeta(G) + |V(G)|\xi^c(H),$$

$$(3) \xi^c(G +_Q H) = |V(H)|[\xi^c(G) + 2\zeta(L(G)) + 4|E(G)| + \xi^c(L(G)) + 2|E(L(G))|] + 2(2|E(G)| + |E(L(G))|)\zeta(H) + 2|E(H)| \cdot (\zeta(G) + |V(G)|) + |V(G)|\xi^c(H),$$

$$(4) \xi^c(G +_T H) = |V(H)|[2\xi^c(G) + 2\zeta(L(G)) + 2|E(G)| + \xi^c(L(G)) + 2|E(L(G))|] + 2(3|E(G)| + |E(L(G))|)\zeta(H) + 2|E(H)|\zeta(G) + |V(G)|\xi^c(H).$$

Proof. Let $U = V(G) \subseteq V(F(G))$. (1) By using the facts in lemma 3.3, we obtain

$$\begin{aligned} \xi^c(S(G)(U)) &= \sum_{v \in V(S(G))} \deg_{S(G)}(v) ec_{S(G)(U)}(v) \\ &= \sum_{v \in U} \deg_{S(G)}(v) ec_{S(G)(U)}(v) + \sum_{v \in V(S(G)) \setminus U} \deg_{S(G)}(v) ec_{S(G)(U)}(v) \\ &= 2 \sum_{v \in U} \deg_G(v) ec_G(v) + \sum_{v \in V(S(G)) \setminus U} 2 \cdot ec_{S(G)(U)}(v) \\ &= 2\xi^c(G) + 2 \sum_{v \in V(L(G))} (2ec_{L(G)}(v) + 1) \\ &= 2\xi^c(G) + 4\zeta(L(G)) + 2|E(G)|. \end{aligned}$$

$$\varepsilon(S(G)(U)) = \sum_{u \in U} ec_{S(G)(U)}(u) = \sum_{u \in V(G)} 2 \cdot ec_G(u) = 2\zeta(G).$$

Combining these with Theorem 2.3, we obtain the desired result.

(2) Still using the Lemma 3.3, we can achieve that

$$\begin{aligned} \xi^c(R(G)(U)) &= \sum_{v \in V(R(G))} deg_{R(G)}(v)ec_{R(G)(U)}(v) \\ &= \sum_{v \in U} deg_{R(G)}(v)ec_{R(G)(U)}(v) + \sum_{v \in V(R(G)) \setminus U} deg_{R(G)}(v)ec_{R(G)(U)}(v) \\ &= 2 \sum_{v \in U} deg_G(v)ec_G(v) + \sum_{v \in V(R(G)) \setminus U} 2 \cdot ec_{R(G)(U)}(v) \\ &= 2\xi^c(G) + 2 \sum_{v \in V(L(G))} (ec_{L(G)}(v) + 1) \\ &= 2(\xi^c(G) + \zeta(L(G)) + |E(G)|). \end{aligned}$$

$$\varepsilon(R(G)(U)) = \sum_{u \in U} ec_{R(G)(U)}(u) = \sum_{u \in V(G)} ec_G(u) = \zeta(G).$$

Using these with Theorem 2.3, we obtain the desired result.

(3) Now let $F = Q$, and combining with Lemma 3.3, we have

$$\begin{aligned} \xi^c(Q(G)(U)) &= \sum_{v \in V(Q(G))} deg_{Q(G)}(v)ec_{Q(G)(U)}(v) \\ &= \sum_{v \in U} deg_{Q(G)}(v)ec_{Q(G)(U)}(v) + \sum_{v \in V(Q(G)) \setminus U} deg_{Q(G)}(v)ec_{Q(G)(U)}(v) \\ &= \sum_{v \in U} deg_G(v)(ec_G(v) + 1) + \sum_{v \in V(Q(G)) \setminus U} deg_{Q(G)}(v)ec_{Q(G)(U)}(v) \\ &= \xi^c(G) + 2|E(G)| + \sum_{v \in V(L(G))} (deg_{L(G)}(v) + 2)(ec_{L(G)}(v) + 1) \\ &= \xi^c(G) + 4|E(G)| + \xi^c(L(G)) + 2|E(L(G))| + 2\zeta(L(G)). \end{aligned}$$

Also, $\varepsilon(Q(G)(U)) = \sum_{u \in U} ec_{Q(G)(U)}(u) = \sum_{u \in V(G)} (ec_G(u) + 1) = \zeta(G) + |V(G)|.$

Again by Theorem 2.3, we can obtain the desired result.

(4) Still considering the facts in Lemma 3.3, then we have

$$\begin{aligned} \xi^c(T(G)(U)) &= \sum_{v \in V(T(G))} deg_{T(G)}(v)ec_{T(G)(U)}(v) \\ &= \sum_{v \in U} deg_{T(G)}(v)ec_{T(G)(U)}(v) + \sum_{v \in V(T(G)) \setminus U} deg_{T(G)}(v)ec_{T(G)(U)}(v) \\ &= \sum_{v \in U} 2 \cdot deg_G(v)ec_G(v) + \sum_{v \in V(L(G))} (deg_{L(G)}(v) + 2)(ec_{L(G)}(v) + 1) \\ &= 2\xi^c(G) + 2|E(G)| + \xi^c(L(G)) + 2|E(L(G))| + 2\zeta(L(G)). \end{aligned}$$

Moreover, $\varepsilon(T(G)(U)) = \sum_{u \in U} ec_{T(G)(U)}(u) = \sum_{u \in V(G)} ec_G(u) = \zeta(G).$

Combining these results with Theorem 2.3, we can obtain the desired result.

□

Example 2. If $G = S(C_n(U) \square P_2(n \geq 3))$, the zig-zag polyhex nanotube $TUHC_6$, then

$$\xi^c(G) = \begin{cases} 10n^2 + 14n & n \text{ is even,} \\ 10n^2 + 4n & n \text{ is odd.} \end{cases}$$

Example 3. ([7]) Let L_n be a hexagonal chain with n hexagonal ($L_n = P_{n+1} +_S P_2, n \geq 2$). Then

$$\xi^c(L_n) = \begin{cases} 15n^2 + 14n + 2 & n \text{ is even,} \\ 15n^2 + 14n + 3 & n \text{ is odd.} \end{cases}$$

4. Concluding remark

In this paper, we compute the ECI and EDS of the generalized hierarchical product of graphs, and then using the obtained results we get the ECI of the F -sum graphs. As the applications, we deduce the ECI of the grids $P_m \square P_n$, the torus $C_m \square C_n$, the zig-zag polyhex nanotube $TUHC_6$ and the hexagonal chain L_n . Actually, by using the result on the EDS of the generalized hierarchical product of graphs, it is also possible to give some formulae for the EDS of the F -sum graphs, but it turns out that the formulae would be much more complicated and the deduction process is very tedious, for this reason we omit it.

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