www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 2 No. 2 (2013), pp. 1-7.

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ON THE COMPLEXITY OF THE COLORFUL DIRECTED PATHS IN VERTEX COLORING OF DIGRAPHS

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Communicated by Manouchehr Zaker

Abstract. The colorful paths and rainbow paths have been considered by several authors. A colorful directed path in a digraph G is a directed path with $\chi(G)$ vertices whose colors are different. A v-colorful directed path is such a directed path, starting from v. We prove that for a given 3-regular triangle-free digraph G determining whether there is a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful directed path is **NP**-complete.

1. Introduction

Graph coloring is a well-studied area of graph theory. For a graph G, a *proper k-coloring* of G is a function $c : V(G) \longrightarrow \{1, \cdots, k\}$ such that $c(u) \neq c(v)$ for every two adjacent vertices $u, v \in V(G)$. The chromatic number of G denoted by $\chi(G)$, is the smallest k for which G has a proper k-coloring. For a given coloring of a graph G, we say path P of G is a *rainbow path* if all vertices of P have different colors. A *v-rainbow path* is a rainbow path starting from the vertex v. A *v-colorful path* is a rainbow path starting from the vertex v with $\chi(G)$ vertices. Let G be a graph. We recall that a path in G is said to represent all $\chi(G)$ colors if all the colors $1, \ldots, \chi(G)$ appear on this path. A *colorful directed path* in a digraph G is a directed path with $\chi(G)$ vertices whose colors are different. A *v-colorful directed path* is such a directed path, starting from v. The colorful paths and rainbow paths have been considered by several authors, for instance see [1, 2, 3, 4, 6, 7]. In 2007, Lin posed the following problem [7].

MSC(2010): Primary: 05C15; Secondary: 05C20, 68Q25.

Keywords: Colorful Directed Paths, Computational Complexity, Vertex Coloring.

Received: 16 October 2012, Accepted: 7 May 2013.

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Problem 1.1. [7] *Let G be a connected graph. Does there always exist a proper vertex coloring of G with $\chi(G)$ colors such that every vertex of G is on a path with $\chi(G)$ vertices which represents all $\chi(G)$ colors?*

Afterwards, Akbari et al. proposed the following stronger conjecture [1].

Conjecture 1.2. [1] *Let $G \neq C_7$ be a connected graph. Then there exists a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful path.*

In [2] this was proved with $\lfloor \frac{\chi(G)}{2} \rfloor$ vertices instead of $\chi(G)$ vertices. Afterwards, Alishahi et al. strengthen this to $\chi(G)-1$ vertices [3]. Also in [2] it was proved that, there exists a proper $(\Delta(G)+1)$-coloring of G with a v-colorful path for every $v \in V(G)$. Furthermore, in [2] it was proved that this result is true if one replaces $(\Delta(G)+1)$ colors with $2\chi(G)$ colors.

A proper vertex coloring of a digraph D is defined, simply a vertex coloring of its underlying graph G, and its chromatic number $\chi(D)$ is defined to be the chromatic number $\chi(G)$ of G. The chromatic number of a digraph provides interesting information about its subdigraphs. The following well-known result, due to Gallai, gives a relationship between the length of the longest path and the chromatic number (for example see [9]).

Theorem 1.3. [Gallai Theorem] *Every digraph G has a directed path with at least $\chi(G)$ vertices.*

In 2001, Li generalized the Gallai Theorem by specifying the starting vertex of the directed path [6].

Theorem 1.4. [6] *If G is a digraph in which v is a vertex that can reach all other vertices, then G has a directed path starting at v with at least $\chi(G)$ vertices.*

Li gave the following conjecture for the digraph [6].

Conjecture 1.5. [6] *For any proper $\chi(G)$-coloring of a digraph G and any vertex $v \in V(G)$ that can reach all other vertices, there is a directed path starting at v whose vertices use all $\chi(G)$ colors.*

Chang et al. gave a counterexample to the above conjecture [4]. In this note, we are interested in the following problem.

Problem: *Colorful Directed Paths*
INPUT: A connected digraph G.
QUESTION: Is there a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there is a v-colorful directed path?

Our main result is that *Colorful Directed Paths* is **NP**-complete for 3-regular triangle-free digraphs. In contrast, we show that *Colorful Directed Paths* can be solved in polynomial time for 2-regular digraphs.

In [8] it was proved that, it is **NP**-complete to decide whether G is colorable with $\chi(G)$ colors in such a way that for a given vertex $v \in V(G)$ there is a path starting at v representing all $\chi(G)$ colors.

Next, by a similar argument, we prove that the following problem is **NP**-complete for disconnected graphs.

Problem: *Colorful Paths*
INPUT: A graph G
QUESTION: Is there a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful path?

We follow [5, 9] for terminology and notation not defined here, and we consider finite simple graphs and digraphs. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We denote the maximum degree and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. The union of simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a disjoint union, and generally denote it by $G + H$. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the join of G and H, denoted $G \vee H$. Also, for every $v \in V(G)$, $d(v)$ denotes the degree of v. For a natural number r, a graph G is called an *r-regular graph* if $d(v) = r$, for each $v \in V(G)$.

2. NP-completeness

Theorem 2.1. Colorful Directed Paths *is* **NP**-*complete for* 3-*regular triangle-free digraphs and it can be solved in polynomial time for* 2-*regular digraphs.*

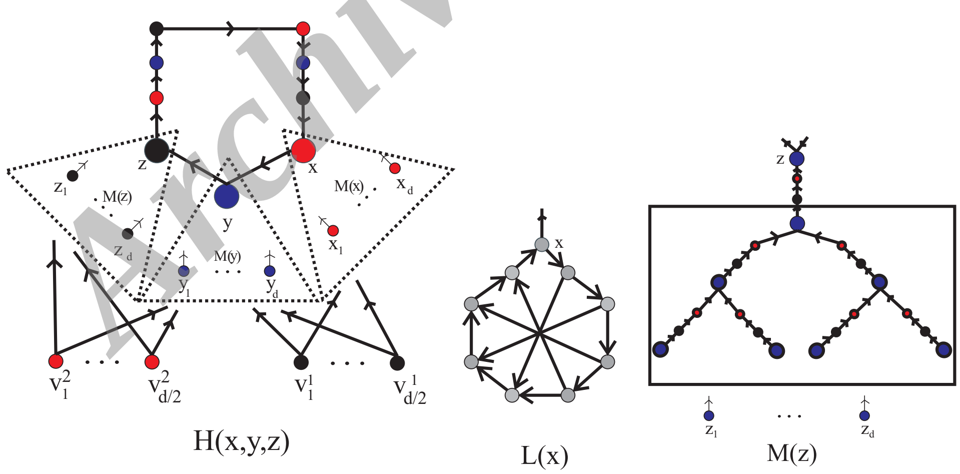

FIGURE 1. The auxiliary digraphs $H(x, y, z)$, $L(x)$ and $M(z)$.

Proof. First, we show that *Colorful Directed Paths* can be solved in polynomial time for 2-regular digraphs. Let G be a connected 2-regular digraph. We have the following straightforward characterization. If G is a connected 2-regular digraph, then there exists a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful directed path, if and only if, for every vertex $v \in V(G)$,

$d^+(v) = 1$ and $|V(G)| = 2k$ or $3k$. Next, we prove that *Colorful Directed Paths* is **NP**-complete for 3-regular triangle-free digraphs. Clearly, the problem is in **NP**. We reduce 3-*Sat* to our problem. Let Φ be a 3-Sat formula with clauses $C = \{c_1, \cdots, c_k\}$ and variables $X = \{x_1, \cdots, x_n\}$. Also let $d = 10(k + n)$. We use the auxiliary digraphs T, $M(z)$, $H(x, y, z)$, $A(x_j)$, $B(c_j)$ and $L(x)$, which are shown in Figure 1 and Figure 2. We construct a digraph $G(\Phi)$ as the digraph arising from the following construction:

Algorithm 1 : Construction of $G(\Phi)$.

1: We start $H(x, y, z)$ as the digraph $G(\Phi)$.

2: For each variable x_j, put a copy of $A(x_j)$ also put two directed edges $x_j v^1_{2j-1}$, $\neg x_j v^1_{2j}$ from x_j and $\neg x_j$ to v^1_{2j-1} and v^1_{2j}, respectively.

3: For each clause c_j, put a copy of $B(c_j)$ also put five directed edges $v^2_j s^1_j$, $v^1_{4j+2n} s^1_j$, $c^1_j v^1_{4j-1+2n}$, $c^2_j v^1_{4j-2+2n}$ and $c^3_j v^1_{4j-3+2n}$.

4: For each clause $c_j = l_1 \vee l_2 \vee l_3$, for every i, $1 \leq i \leq 3$, add the directed edge $c^i_j a^j_{l_i}$ from c^i_j to $a^j_{l_i}$.

5: For each vertex v, if $d(v) = 1$, put two auxiliary graphs $L(v_x)$, $L(v_{x'})$ and also put two directed edges vv_x and $vv_{x'}$, from v to v_x and $v_{x'}$.

6: For each vertex v, if $d(v) = 2$, put the auxiliary graphs $L(v_x)$ and the directed edges vv_x from v to v_x.

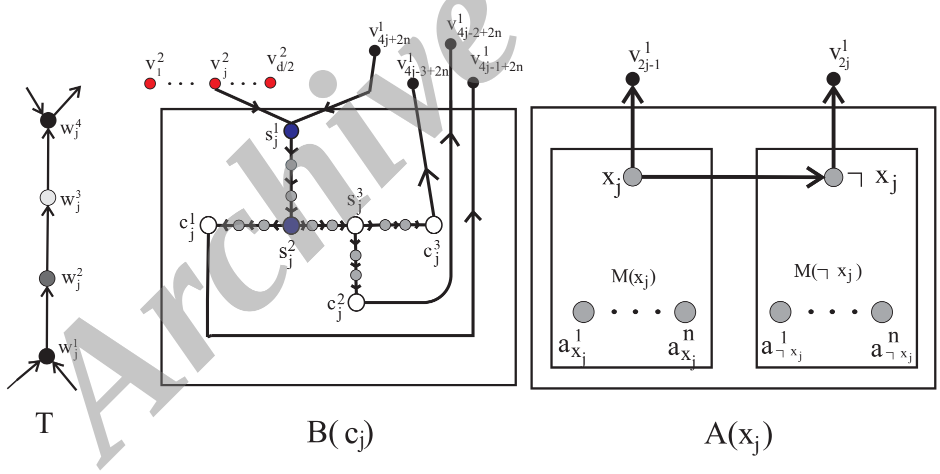

FIGURE 2. The auxiliary digraphs $A(x_j)$, $B(c_j)$ and T.

Next, we discuss basic properties of the digraph $G(\Phi)$. The digraph $G(\Phi)$ is 3-regular and triangle-free. Let f be a proper 3-coloring, such that for every vertex v, there exists a v-colorful directed path. We have:

$$\{f(\neg x_j), f(x_j)\} = \{Red, Blue\}, \qquad f(x_j) = f(a^1_{x_j}) = \cdots = f(a^n_{x_j}),$$
$$f(\neg x_j) = f(a^1_{\neg x_j}) = \cdots = f(a^n_{\neg x_j}), \qquad f(s^1_j) = f(s^2_j) = Blue,$$

Moreover, for every copy of T we have $f(w_j^1) = f(w_j^4)$ and for every copy of $M(z)$ we have $f(z) = f(z_1) = \cdots = f(z_d)$. Also for every copy of $H(x, y, z)$ we have $f(z) = f(z_1) = \cdots = f(z_d)$, $f(x) = f(x_1) = \cdots = f(x_d)$, $f(y) = f(y_1) = \cdots = f(y_d)$, $f(x) = f(v_1^2) = \cdots = f(v_{d/2}^2)$ and $f(z) = f(v_1^1) = \cdots = f(v_{d/2}^1)$. First, suppose that Φ is satisfiable with the satisfying assignment Γ. Now we present the proper 3-coloring f for $G(\Phi)$, such that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path. Let $f(x) = Red$, $f(y) = Blue$ and $f(z) = Black$. Now, for every vertex v, $v \in V(H(x, y, z))$, the color of v, is determined uniquely. For each variable x_i, if $x_i = True$, then let $f(x_i) = Red$ and $f(\neg x_i) = Blue$. Otherwise let $f(x_i) = Blue$ and $f(\neg x_i) = Red$. For every $c_j = l_1 \vee l_2 \vee l_3$, color the vertices of $B(c_j)$ according to the Figure 3. Now, for every vertex v, $v \in V(A(x_j))$, the color of v, is determined uniquely. Finally, color the vertices of every copy of $L(x)$. It is easy to see that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path.

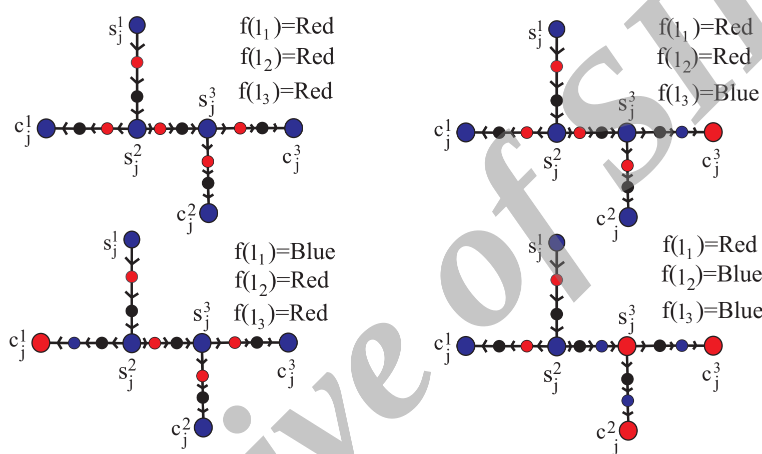

FIGURE 3. Four cases.

Next, suppose that $G(\Phi)$ has the proper 3-coloring f, such that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path. With no loss of generality suppose that $f(x) = Red$, $f(y) = Blue$ and $f(z) = Black$. For each variable x_i, let $x_i = True$ in Γ if and only if $f(x_i) = Red$. Let $c_j = l_1 \vee l_2 \vee l_3$ be an arbitrary clause. We have $f(s_j^1) = f(s_j^2) = Blue$, therefore $Blue \in \{f(c_j^1), f(s_j^3)\}$, so $Blue \in \{f(c_j^1), f(c_j^2), f(c_j^3)\}$. Consequently $Red \in \{f(a_{l_1}^j), f(a_{l_2}^j), f(a_{l_3}^j)\}$, so $Red \in \{f(l_1), f(l_2), f(l_3)\}$. Therefore; Γ is a satisfying assignment for Φ.

□

Theorem 2.2. *For every $r \geq 3$, the following problem is* **NP***-complete: "given an r-regular triangle-free digraph G with $\chi(G) = 3$, does there exist a proper 3-coloring of G such that for every $v \in V(G)$, there exists a v-colorful directed path?"*

Proof. The proof is similar to the proof of Theorem 2.1. Consider two disjoint copies of $G(\Phi)$ ($G(\Phi)$ is introduced in the proof of the previous theorem), then for every vertex v, put a directed edge from v to the corresponding vertex in the second copy of $G(\Phi)$. By repeating this procedure, we find an r-regular triangle-free digraph G' with $2^{r-3}|V(G(\Phi))|$ vertices. Clearly, there exists a proper 3-coloring of G' such that for every $v \in V(G')$, there exists a v-colorful directed path, if and only if, there exists a proper 3-coloring of $G(\Phi)$ such that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path.

□

Theorem 2.3. Colorful Paths *is* **NP**-*complete.*

Proof. Clearly, Colorful Paths is in **NP**. We reduce Hamilton Path to this problem (for a given graph G, does G have a Hamilton path? [5]). Consider a graph G with $|V(G)| = n$, as an instance of Hamilton Path. We construct a new graph G' with the property that, there exists a proper $\chi(G')$-coloring of G' such that for every $v \in V(G')$, there exists a v-colorful path, if and only if G has a Hamilton path. Let $G' = (G \vee K_1) + K_{n+1}$. If G has a Hamilton path then $G \vee K_1$ has a Hamilton cycle, so there exists a proper $\chi(G')$-coloring of G' such that for every $v \in V(G')$, there exists a v-colorful path. Next, suppose that there exists a proper $\chi(G')$-coloring f of G' such that for every $v \in V(G')$, there exists a v-colorful path. Now consider a u-colorful path $uv_1v_2 \dots v_n$ for G', clearly $v_1v_2 \dots v_n$ is a Hamilton path for G. So Colorful Paths is **NP**-complete.

□

Acknowledgments

This paper derived from research project "On the algorithmic complexity of directed colorful path in order to find an efficient algorithm for it" done in Abadan Branch, Islamic Azad University, Abadan, Iran. Also the authors would like express their deep gratitude to Arash Ahadi for reading the draft of this paper.

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