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ON THE NOMURA ALGEBRAS OF FORMALLY SELF-DUAL ASSOCIATION SCHEMES OF CLASS 2

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Abstract. In this paper, the type-II matrices on (negative) Latin square graphs are considered and it is proved that, under certain conditions, the Nomura algebras of such type-II matrices are trivial. In addition, we construct type-II matrices on doubly regular tournaments and show that the Nomura algebras of such matrices are also trivial.

1. Introduction

In [3], Chan and Hosoya have considered the type-II matrices in the Bose-Mesner algebra of conference graphs and have proved that the Nomura algebras of such matrices are trivial when $n > 9$. In this paper, we show that the Nomura algebras obtained from some of the (negative) Latin square graphs are trivial. Moreover, we determine the type-II matrices attached to doubly regular tournaments. Then we show that the Nomura algebras obtained from these type-II matrices are trivial.

In the rest of the section, we remind some concepts and notations about type-II matrices, strongly regular graphs and association schemes.

1.1. **Type-II matrices.** In this subsection, we drive some notations and concepts in the type-II matrices and the Nomura algebras according to [2]. Let I_n and J denote the identity matrix of order n and the matrix of order n whose entries are all 1, respectively. Denote by $\mathbf{M}_n(\mathbb{C})$ the set of $n \times n$

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complex matrices. Let $W = (w_{ij})$ be an $n \times n$ complex matrix whose entries are all nonzero. We can define an associated $n \times n$ matrix by $W^{(-)} = (w_{ij}^{-1})$. An $n \times n$ complex matrix W is called *type*-II if

$$WW^{(-)\mathrm{t}} = nI.$$

As an example, the matrix $I+xJ$ of order $n \geq 2$ is a type-II where x is one of the roots of $nx^2+nx+1 = 0$, i. e., $x = \frac{1}{2}(2-n \pm \sqrt{n^2-4n})$. Such a matrix is called the *Potts model*.

Let X be a nonempty finite set with $|X| = n$. Let $W = (w_{ij})$ be a complex matrix whose rows and columns are indexed by X and whose entries are all nonzero. For each $u, v = 1, \ldots, n$, we define a column vector $\mathbf{e}_{uv}$ whose i-th entry is

$$\mathbf{e}_{uv}(i) = \frac{w_{iu}}{w_{iv}}.$$

The *Nomura algebra* of W is defined by

$$\mathcal{N}_W = \{M \in \mathbf{M}_n(\mathbb{C}) \mid \mathbf{e}_{uv} \text{ is an eigenvector of } M, \forall u, v \in X\}.$$

From [2, Corollary 4.1] it follows that the Nomura algebra of any type-II matrix is the Bose-Mesner algebra of a commutative association scheme. Since $\mathcal{N}_W$ contains I it is nonempty. The Nomura algebra $\mathcal{N}_W$ is said to be *trivial* if $\dim \mathcal{N}_W = 2$. For $M \in \mathcal{N}_W$, we define the *matrix of eigenvalues* of M denoted by $\Theta_W(M)$ to be an $n \times n$ matrix whose (u, v)-th entry is equal to the eigenvalue of W on $\mathbf{e}_{uv}$.

The following lemma determines whether or not two eigenvectors $\mathbf{e}_{uv}$'s belong to the same eigenspace of $\mathcal{N}_W$.

Lemma 1.1. [2, Lemma 3.2] *Let* $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} \neq 0$. *Then* $(\Theta_W(M))_{vu} = (\Theta_W(M))_{uz}$ *where* $M \in \mathcal{N}_W$.

The following lemma gives a sufficient condition for the Nomura algebra of a type-II matrix being trivial.

Lemma 1.2. *Let W be a type*-II *matrix. If* $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} \neq 0$ *for any distinct* $u, v \in X$ *and for all* $z \neq u$, *then* $\mathcal{N}_W$ *is trivial.*

Proof. By Lemma 1.1 we see that $(\Theta_W(M))_{vu} = (\Theta_W(M))_{uz}$ for any matrix M from $\mathcal{N}_W$. Put $\lambda := (\Theta_W(M))_{vu}$. Then $\mathbf{e}_{vu}$ and $\mathbf{e}_{uz}$ belong to the same eigenspace denoted by $\mathcal{E}_\lambda$ for all $z \neq u$. Put $V_\lambda := \{\mathbf{e}_{uz} \mid z \neq u\}$. Clearly, $V_\lambda \subseteq \mathcal{E}_\lambda$. From the definition of W we conclude that all vectors in V_λ are in $\mathbf{1}^\perp$. From [2, Lemma 2.1] it follows that V_λ is the set of $n-1$ linearly independent vectors. Hence, $\dim(\mathcal{E}_\lambda)$ is either $n-1$ or n. If $\dim(\mathcal{E}_\lambda) = n-1$ then the Hermitian space $\mathbb{C}^n$ has the form

$$\mathbf{1} \oplus \mathcal{E}_\lambda.$$

Thus, the Bose-Mesner algebra $\mathcal{N}_W$ has only two principle idempotents which implies that its rank is equal to 2.

If $\dim(\mathcal{E}_\lambda) = n$, then each vector in $\mathbb{C}^n$ is an eigenvector of M, especially $(1, 0, 0, \ldots, 0)^{\mathrm{t}}$. This implies that $M = \lambda I_n$. Hence, $\mathcal{N}_W = \{I_n\}$ which implies that $n = 1$. □

Theorem 1.3. [2, Theorem 6.4] *Suppose W is a type-II matrix of the form $J+(t-1)A$, where A is the incidence matrix of a symmetric $(n,k,\lambda)-\mathrm{BIBD}$ and*

$$t=\frac{1}{2(k-\lambda)}\big(2(k-\lambda)-n\pm\sqrt{n(n-4(k-\lambda))}\big). \tag{1.1}$$

If $n>3$ and $t\neq -1$, then the Nomura algebra of W is trivial.

1.2. **Strongly regular graphs.** A *strongly regular graph* Γ with parameters (n,k,λ,μ) is a simple graph on n vertices which is regular with valency k such that if the vertices α and β are adjacent then there are exactly λ vertices adjacent to both α and β and otherwise μ there are. It is known that if Γ is connected then it has three distinct eigenvalues k,θ and τ. The strongly regular graph Γ is called (*positive*) *Latin square* (resp. *negative Latin square*) if

$$n=m^2,\quad k=g(m-\epsilon),\quad \lambda=\epsilon m+g^2-3\epsilon g,\quad \mu=g(g-\epsilon)$$

for some positive integers m and g where $\epsilon=1$ (resp. $\epsilon=-1$). For such a graph $\theta=m-g$ and $\tau=-g$ (resp. $\theta=g$ and $\tau=g-m$).

In [2, Section 8], Chan and Godsil constructed the type-II matrices on the strongly regular graphs. Moreover, they showed that if Γ is formally self-dual then there are at most six type-II matrices, up to equivalence, in the Bose-Mesner algebra of Γ. In this paper, we consider one of the six cases and investigate its Nomura algebras. So we assume that Γ is a (negative) Latin square graph. Let A_1 be the adjacency matrix of Γ and let A_2 be its complement adjacency matrix. Again in [2, Section 8], it was shown that a matrix $W:=I+xA_1+yA_2$ is type-II if

$$\begin{aligned} x&=\frac{1}{2\tau}\big(\theta^2-\tau^2+2\theta+\epsilon_1\sqrt{(\theta-\tau)(\theta-\tau+2)(\theta+\tau)(\theta+\tau+2)}\big)\quad\text{and}\\ y&=\frac{1}{2(\theta+1)}\big(\theta^2-\tau^2+2(\theta+1)+\epsilon_2\sqrt{(\theta-\tau)(\theta-\tau+2)(\theta+\tau)(\theta+\tau+2)}\big),\end{aligned} \tag{1.2}$$

or

$$\begin{aligned} x&=\frac{1}{2\theta}\big(\tau^2-\theta^2+2\tau+\epsilon_3\sqrt{(\theta-\tau)(\theta-\tau-2)(\theta+\tau)(\theta+\tau+2)}\big)\quad\text{and}\\ y&=\frac{1}{2(\tau+1)}\big(\tau^2-\theta^2+2(\tau+1)+\epsilon_4\sqrt{(\theta-\tau)(\theta-\tau-2)(\theta+\tau)(\theta+\tau+2)}\big),\end{aligned} \tag{1.3}$$

where $\epsilon_i=\pm1$ for each i. We show that the Nomura algebras obtained from such a matrix are trivial when $m-2\nmid\lambda(g-1),\lambda(g-2)$ and $m\geq 3g$ with $g>2$. We shall assume that $\epsilon=\epsilon_i=\epsilon_{i+1}=\pm1$ in (1.2) and (1.3) for $i=1,3$ and prove the main theorem. Substituting $\theta=m-g$ and $\tau=-g$ into (1.2), we can write x and y in terms of m and g as follows.

$$x=\frac{e+\epsilon\sqrt{c}}{2g},\qquad y=\frac{gx-1}{g-m-1}, \tag{1.4}$$

where $e=2mg-m^2+2g-2m$ and $c=m(m+2)(m-2g)(m-2g+2)$. Similarly, (1.3) can be written as the form

$$x=\frac{e'+\epsilon\sqrt{c'}}{2(g-m)},\qquad y=\frac{(g-m)x-1}{g-1}, \tag{1.5}$$

where $e' = -2mg + m^2 + 2g$ and $c' = m(m-2)(m-2g)(m-2g+2)$. In (1.4) and (1.5), if we set $m = 2g-1$ then we see that x and y are complex and so Γ is a conference graph whose Nomura algebras investigated in [3], otherwise they are real. Therefore, the type-II matrices constructed on (negative) Latin square graphs are real when $m \geq 3g$.

Our main result is as follows:

Theorem 1.4. *Suppose that $W = I + xA_1 + yA_2$ is a type-*II *matrix attached to a Latin square graph where x and y satisfy* (1.4) *or* (1.5). *Let $m \geq 3g$ and $m-2 \nmid \lambda(g-1), \lambda(g-2)$ where $g > 2$. Then $\mathcal{N}_W$ is trivial.*

1.3. **Association schemes.** Let X be a nonempty finite set. Define $\Delta(X) := \{(x,x) \mid x \in X\}$. For each subset R of $X \times X$, we define by R^{t} the set of all pairs (x,y) with $(y,x) \in R$. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a partition of $X \times X$ and put $R_0 := \Delta(X)$. Then $\mathcal{X} = (X, \mathcal{R})$ is called an *association scheme* of class d if it satisfies the following conditions (**a**) $R_i^{\mathrm{t}} \in \mathcal{R}$ for each $R_i \in \mathcal{R}$; we denote R_i^{t} by $R_{i'}$, (**b**) for each $R_i, R_j, R_k \in \mathcal{R}$ there exists a number p_{ij}^k called the *intersection number* such that $|R_i(x) \cap R_{j'}(y)| = p_{ij}^k$ for all $(x,y) \in R_k$ where $R(x) := \{y \in X \mid (x,y) \in R\}$. An association scheme is called *symmetric* if $R_i = R_{i'}$, for all i.

It is known that each of the basis relations R_i is associated with a matrix A_i called the *adjacency matrix*. A subalgebra of $\mathbf{M}_n(\mathbb{C})$ spanned by $\mathcal{B} = \{I = A_0, A_1, \ldots, A_d\}$ is called the *Bose-Mesner algebra* of an association scheme if it satisfies the following conditions (**a**) $A_{i'} \in \mathcal{B}$ for each $A_i \in \mathcal{B}$, (**b**) the sum of the elements of $\mathcal{B}$ is J, (**c**) $\mathcal{B}$ is a basis for a $(d+1)$-dimensional semisimple subalgebra of $\mathbf{M}_n(\mathbb{C})$ whose structure constants are nonnegative.

Let $\mathcal{X}$ be an association scheme of class d with the basis relations R_i's and the primitive idempotents E_i's. Then from [1, (3.6) and (3.8)] we have

$$A_i = \sum_{j=0}^{d} p_i(j) E_j, \qquad E_i = \frac{1}{n} \sum_{j=0}^{d} q_i(j) A_j.$$

The matrices $P = (p_i(j))$ and $Q = (q_i(j))$ are called the first and the second eigenmatrices of $\mathcal{X}$, respectively, in which j and i denote row and column, respectively. An association scheme is said to be *formally self-dual* if $P = Q$ for some ordering of the primitive idempotents if necessary.

Association schemes of class 2 arise in connection with some combinatorial structures. The basis relations of a symmetric association scheme of class 2 are the edge sets of complementary strongly regular graphs. Conversely, each of the edge set of a strongly regular graph and the edge set of its complement forms a symmetric association scheme of class 2. The Bose-Mesner algebra of a strongly regular graph Γ is formally self-dual if and only if $n = (\theta - \tau)^2$ if and only if Γ is a conference graph, a Latin square graph, or a negative Latin square graph, see [4].

A *doubly regular tournament* is a loopless directed graph of order $2k+1$ and of valency k whose adjacency matrix A satisfies $A + A^{\mathrm{t}} + I = J$ and $A^2 = \frac{k-1}{2}A + \frac{k+1}{2}A^{\mathrm{t}}$. It follows that k must be an odd number. This definition follows that A is the adjacency matrix of a doubly regular tournament if and only if so is $A^{\rm t}$. It is known that the subalgebra of spanned by $\{I, A, A^{\rm t}\}$ is the *Bose-Mesner algebra* of a nonsymmetric association scheme of class 2. Conversely, any nonsymmetric association scheme of class 2 arises from a doubly regular tournament. It is easy to show that the Bose-Mesner algebra of a doubly regular tournament is formally self-dual.

2. Nomura algebras

2.1. **The Nomura algebras constructed on some of the Latin square graphs.** In this subsection, we follow the notations in subsection 1.2 and show that under certain conditions the Nomura algebras of the type-II matrices attached to (negative) Latin square graphs are trivial. To do so, we first show that c and c' cannot be perfect squares when $m \geq 3g$.

Lemma 2.1. *If $m \geq 3g$, then c and c' both cannot be perfect squares.*

Proof. Since $m \geq 3g$, we have $m > 3g-1$. Then $m = 3g+i-1$ for some positive integer i. Therefore, we have

$$\begin{aligned} c &= m(m+2)(m-2g)(m-2g+2) \\ &= (3g+i-1)(3g+i+1)(g+i-1)(g+i+1) \\ &= \big((3g+i)^2-1\big)\big((g+i)^2-1\big) \end{aligned}$$

and

$$\begin{aligned} c' &= m(m-2)(m-2g)(m-2g+2) \\ &= (3g+i-1)(3g+i-3)(g+i-1)(g+i+1) \\ &= \big((3g+i-2)^2-1\big)\big((g+i)^2-1\big). \end{aligned}$$

Let $X_1 = 3g+i, X_2 = 3g+i-2$ and $Y = g+i$. Then $c = (X_1^2-1)(Y^2-1)$ and $c' = (X_2^2-1)(Y^2-1)$. Let $X \in \{X_1, X_2\}$. We show that

$$(XY-2)^2 < (X^2-1)(Y^2-1) < (XY-1)^2. \tag{2.1}$$

It implies that since $XY-1$ and $XY-2$ are consecutive natural numbers, $(X^2-1)(Y^2-1)$ is not perfect square and hence c and c' are not both perfect squares. We have

$$(X^2-1)(Y^2-1) < (XY-1)^2 \iff 0 < (X-Y)^2.$$

Since $X \neq Y$, the last inequality is always true. Then the right-hand side of (2.1) is also true. On the other hand,

$$(XY-2)^2 < (X^2-1)(Y^2-1) \iff 0 < 4XY - X^2 - Y^2 - 3.$$

Let $T = 4XY - X^2 - Y^2 - 3$. If $X = X_1$, then $T = 2g^2 + 12ig + 2i^2 - 3$ and if $X = X_2$, then $T = 2g^2+4g+8ig+2i^2-4i-7$. Clearly, in each case $T > 0$ and hence $(XY-2)^2 < (X^2-1)(Y^2-1)$. This shows that c and c' both lie between the squares of two consecutive integers. Therefore, they cannot be perfect squares. □

Let $W = (w_{st})$ be a matrix with entries $\{1, x, y\}$ whose rows and columns are indexed by the set X. Set $\Lambda = \{u, v, z\}$ be a subset of X with $|\Lambda| = 3$. Define $\Omega_{uvz} = \{s \in X \mid w_{us} = w_{vs} = w_{zs} = x\}$. Similarly, for instance, if we replace u by $\bar{u}$, then we may define the subset $\Omega_{\bar{u}vz}$ of X as the elements

s in X such that $w_{us}=y$ and $w_{vs}=w_{zs}=x$. Therefore, by the similar way one can define $\Omega_{u\bar{v}z}$, $\Omega_{uv\bar{z}}$ or $\Omega_{\bar{u}\bar{v}z}$ and so on. Also, we define Ω_s to be the set of elements t of X such that $w_{st}=x$.

The proof of Theorem 1.4. Let w_{ij} denote the (i,j)-th entry of W. Suppose that Γ is a Latin square graph with parameters

$$n=m^2, \quad k=g(m-1), \quad \lambda=g^2-3g+m, \quad \mu=g(g-1)$$

for some positive integers m and g. By definition of $\mathbf{e}_{uv}$'s, we have

$$\mathbf{e}_{uv}(t)\mathbf{e}_{uz}(t)=\frac{w_{tu}^2}{w_{tv}w_{tz}}$$

for each $t\in X$ and then we have

$$\mathbf{e}_{uv}(t)\mathbf{e}_{uz}(t)=\begin{cases} 1 & \text{if } t\in\Omega_{uvz}\cup\Omega_{\bar{u}\bar{v}\bar{z}} \\ xy^{-1} & \text{if } t\in\Omega_{u\bar{v}z}\cup\Omega_{uv\bar{z}} \\ yx^{-1} & \text{if } t\in\Omega_{\bar{u}\bar{v}z}\cup\Omega_{\bar{u}v\bar{z}} \\ x^2y^{-2} & \text{if } t\in\Omega_{u\bar{v}\bar{z}} \\ y^2x^{-2} & \text{if } t\in\Omega_{\bar{u}vz} \end{cases}$$

for each $t\in X\setminus\Lambda$. Since W is symmetric and the entries on its main diagonal are 1, the Hermitian product of the vectors $\mathbf{e}_{uv}$ and $\mathbf{e}_{uz}$ can be written as the following form

$$\begin{aligned} \mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} &= \sum_{t\in X}\mathbf{e}_{uv}(t)\mathbf{e}_{uz}(t) \\ &= (w_{uv}w_{uz})^{-1}+(w_{uv}^2+w_{uz}^2)w_{vz}^{-1}+|\Omega_{uvz}\cup\Omega_{\bar{u}\bar{v}\bar{z}}|+|\Omega_{u\bar{v}z}\cup\Omega_{uv\bar{z}}|xy^{-1}+ \\ &\quad |\Omega_{\bar{u}\bar{v}z}\cup\Omega_{\bar{u}v\bar{z}}|yx^{-1}+|\Omega_{u\bar{v}\bar{z}}|x^2y^{-2}+|\Omega_{\bar{u}vz}|y^2x^{-2}. \end{aligned} \tag{2.2}$$

Let $\hat{u}\in\{u,\bar{u}\}, \hat{v}\in\{v,\bar{v}\}, \hat{z}\in\{z,\bar{z}\}$ and $s,t\in\Lambda$. Define $\Lambda_t=\{h\in\Lambda \mid w_{ht}=x\}$ and $\Lambda_{st}=\{h\in\Lambda \mid w_{ht}=w_{hs}=x\}$. By the definitions of Ω_t's and $\Omega_{\hat{u}\hat{v}\hat{z}}$'s we have the following sets.

$$\begin{aligned} &\Omega_u=\bigcup_{\hat{v},\hat{z}}\Omega_{u\hat{v}\hat{z}}\cup\Lambda_u, && \Omega_u\cap\Omega_v=\bigcup_{\hat{z}}\Omega_{uv\hat{z}}\cup\Lambda_{uv}, && X=\bigcup_{\hat{u},\hat{v},\hat{z}}\Omega_{\hat{u}\hat{v}\hat{z}}\cup\Lambda, \\ &\Omega_v=\bigcup_{\hat{u},\hat{z}}\Omega_{\hat{u}v\hat{z}}\cup\Lambda_v, && \Omega_u\cap\Omega_z=\bigcup_{\hat{v}}\Omega_{u\hat{v}z}\cup\Lambda_{uz}, && \\ &\Omega_z=\bigcup_{\hat{u},\hat{v}}\Omega_{\hat{u}\hat{v}z}\cup\Lambda_z, && \Omega_v\cap\Omega_z=\bigcup_{\hat{u}}\Omega_{\hat{u}vz}\cup\Lambda_{vz}. && \end{aligned}$$

By definition, $|\Omega_s\cap\Omega_t|$ is equal to λ if $w_{st}=x$ and μ otherwise. Then we have

$$\begin{cases} \sum_{\hat{v},\hat{z}}|\Omega_{u\hat{v}\hat{z}}|+|\Lambda_u|=k, & \sum_{\hat{z}}|\Omega_{uv\hat{z}}|+|\Lambda_{uv}|\in\{\lambda,\mu\}, \quad \sum_{\hat{u},\hat{v},\hat{z}}|\Omega_{\hat{u}\hat{v}\hat{z}}|=n-3, \\ \sum_{\hat{u},\hat{z}}|\Omega_{\hat{u}v\hat{z}}|+|\Lambda_v|=k, & \sum_{\hat{v}}|\Omega_{u\hat{v}z}|+|\Lambda_{uz}|\in\{\lambda,\mu\}, \\ \sum_{\hat{u},\hat{v}}|\Omega_{\hat{u}\hat{v}z}|+|\Lambda_z|=k, & \sum_{\hat{u}}|\Omega_{\hat{u}vz}|+|\Lambda_{vz}|\in\{\lambda,\mu\}. \end{cases} \tag{2.3}$$

There are eight distinct cases to consider depending on whether w_{st} is x or y. In each case, we calculate $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz}$. This calculation shows that if x and y satisfy (1.4) and (1.5), $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz}$ can be written in the form $a+\epsilon b\sqrt{c}$ and $a'+\epsilon b'\sqrt{c'}$, respectively for some real numbers a, a', b and b'. It follows from Lemma 2.1 that to show that $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz}$ is nonzero it is sufficient to prove that one of a, a', b and b' is nonzero. In what follows, we describe a method of computing $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz}$ for one case. The other cases are done similarly. By hypothesis, $m = 3g + i$ for some nonnegative integer i. Also, let $|\Omega_{uvz}| = f$.

Suppose first that $w_{uv} = w_{uz} = w_{vz} = x$. Then $|\Omega_s \cap \Omega_t| = \lambda$ for all $s, t \in \Lambda$. Also, we have $\Lambda_u = \{v, z\}, \Lambda_v = \{u, z\}, \Lambda_z = \{u, v\}, \Lambda_{uv} = \{z\}, \Lambda_{vz} = \{u\}$ and $\Lambda_{uz} = \{v\}$. We substitute these back into (2.3) and solve it to obtain

$$|\Omega_{\bar{u}vz}| = |\Omega_{u\bar{v}z}| = |\Omega_{uv\bar{z}}| = \lambda - f - 1, \qquad |\Omega_{\bar{u}\bar{v}\bar{z}}| = n + 3(\lambda - k) - f,$$
$$|\Omega_{u\bar{v}\bar{z}}| = |\Omega_{\bar{u}v\bar{z}}| = |\Omega_{\bar{u}\bar{v}z}| = k - 2\lambda + f.$$

Substituting into (2.2), it follows that

$$\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} = x^{-2} + 2x + n - 3(k-\lambda) + (\lambda - f - 1)(2xy^{-1} + y^2x^{-2}) + (k - 2\lambda + f)(2yx^{-1} + x^2y^{-2}).$$

We substitute x and y from (1.4) and (1.5) into the equation above and get

$$a = \frac{mc}{2g^2(m-g+1)}, \qquad a' = \frac{mc'}{2(m-g)^2(g-1)}.$$

Clearly, $a, a' \neq 0$. Similarly, if $w_{uv} = w_{uz} = x$ and $w_{vz} = y$, then

$$\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} = x^{-2} + 2x^2y^{-1} + n - 3k + 2\lambda + \mu + 2(\lambda - f)xy^{-1} + (\mu - f - 1)y^2x^{-2} + 2(k - \lambda - \mu + f)yx^{-1} + (k - 2\lambda + f - 2)x^2y^{-2}.$$

It follows that

$$a = \frac{m(m-g+2)c}{2g^2(m-g+1)^2}, \qquad a' = \frac{m(g-2)c'}{2(m-g)^2(g-1)^2}.$$

Clearly, $a, a' \neq 0$. If $w_{uv} = w_{vz} = x$ and $w_{uz} = y$ or $w_{uv} = y$ and $w_{uz} = w_{vz} = x$, we have

$$\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} = x^{-1}y^{-1} + y^2x^{-1} + x + n - 3k + 2\lambda + \mu + (2k - 3\lambda - \mu + 2f - 2)yx^{-1} + (\lambda + \mu - 2f - 1)xy^{-1} + (k - \lambda - \mu + f)x^2y^{-2} + (\lambda - f)y^2x^{-2}.$$

It implies that

$$b = \frac{m(m^2 + 3mg^2 - m^2g - 5mg - 2f - mf + 2m - g^3 + 5g^2 - 6g)}{g^2(m-g+1)^2},$$
$$b' = \frac{m(g^3 - 5g^2 + mg - mf - 2m + 6g + 2f)}{(m-g)^2(g-1)^2}.$$

If $b = 0$, then $f = -\frac{5mg + m^2g - 3mg^2 - m^2 + 6g - 5g^2 + g^3 - 2m}{m+2}$. We substitute $m = 3g + i$ into the last equality to obtain $f = \frac{g^3 + g^2 + 3ig^2 - ig - 2i + i^2g - i^2}{-3g - i - 2}$. Let d denote the numerator of this fraction. Since $g > 1$, we see that $3g^2 > g + 2$. Then

$$\begin{aligned} d &= g^3 + g^2 + 3ig^2 - ig - 2i + i^2g - i^2 \\ &= g^3 + g^2 + i\underbrace{(3g^2 - g - 2)}_{>0} + i^2\underbrace{(g-1)}_{>0}. \end{aligned}$$

This shows that $d > 0$ and so $f < 0$, a contradiction. Hence, $b \neq 0$. If $b' = 0$, then $f = \frac{\lambda(g-2)}{m-2}$. This contradicts the hypothesis that $m - 2 \nmid \lambda(g-2)$. Thus, $b' \neq 0$. If $w_{uv} = x$ and $w_{uz} = w_{vz} = y$ or $w_{uv} = w_{vz} = y$ and $w_{uz} = x$, then

$$\begin{aligned}\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} &= x^{-1}y^{-1} + x^2y^{-1} + y + n - 3k + \lambda + 2\mu - 1 + (\lambda + \mu - 2f)xy^{-1} + \\ &\quad (2k - \lambda - 3\mu + 2f - 1)yx^{-1} + (k - \lambda - \mu + f - 1)x^2y^{-2} + (\mu - f)y^2x^{-2}.\end{aligned}$$

From this we deduce that

$$\begin{aligned}b &= \frac{m(4g^2 - mf - g^3 + 3mg^2 - 3mg - m^2g - 3g - 2f)}{g^2(m - g + 1)^2}, \\ b' &= \frac{m(g^3 - 4g^2 - m - mf + 3g + mg + 2f)}{(m - g)^2(g - 1)^2}.\end{aligned}$$

If $b = 0$, then $f = -\frac{g(m^2 - 3mg + 3m + g^2 - 4g + 3)}{m+2}$. Substituting $m = 3g + i$ into the last equality, we get $f = -\frac{g(i^2 + g^2 + 5g + 3i + 3ig + 3)}{3g + i + 2} < 0$ which is a contradiction and hence $b \neq 0$. If $b' = 0$, then $f = \frac{\lambda(g-1)}{m-2}$ which contradicts the hypothesis of the theorem. If $w_{vz} = x$ and $w_{uv} = w_{uz} = y$, then

$$\begin{aligned}\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} &= y^{-2} + 2y^2x^{-1} + n - 3k + \lambda + 2\mu - 1 + 2(\mu - f)xy^{-1} + (\lambda - f)y^2x^{-2} + \\ &\quad 2(k - \lambda - \mu + f - 1)yx^{-1} + (k - 2\mu + f)x^2y^{-2}.\end{aligned}$$

It follows that

$$a = \frac{m(g+1)c}{2g^2(m - g + 1)^2}, \qquad a' = \frac{m(m - g - 1)c'}{2(m - g)^2(g - 1)^2}.$$

Clearly, $a, a' \neq 0$. If $w_{uv} = w_{uz} = w_{vz} = y$, then

$$\begin{aligned}\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} &= y^{-2} + 2y + n - 3k + 3\mu - 3 + (\mu - f)(2xy^{-1} + y^2x^{-2}) + \\ &\quad (k - 2\mu + f)(2yx^{-1} + x^2y^{-2}).\end{aligned}$$

It implies that

$$a = \frac{mc}{2g(m - g + 1)^2}, \qquad a' = \frac{mc'}{2(m - g)(g - 1)^2}.$$

It follows that $a, a' \neq 0$. In each case, we see that one of a, a', b and b' is nonzero and so $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} \neq 0$ by Lemma 2.1. Now from Lemma 1.2 we imply that $\mathcal{N}_W$ is trivial. This completes the proof of the theorem. □

From the proof of Theorem 1.4, we conclude that if x and y only satisfy (1.4) we can eliminate the hypotheses $m - 2 \nmid \lambda(g-1), \lambda(g-2)$ and $g > 2$ from the theorem. Therefore, we can state the following corollary.

Corollary 2.2. *Suppose that $W = I + xA_1 + yA_2$ is a type-*II *matrix attached to a Latin square graph where x and y satisfy* (1.4)*. If $m \geq 3g$, then $\mathcal{N}_W$ is trivial.*

The following corollary is a consequence of Theorem 1.4 for negative Latin square graphs.

Corollary 2.3. *Let $W = I + xA_1 + yA_2$ be a type-*II *matrix attached to a negative Latin square graph and let $m \geq 3g$. If x and y satisfy* (1.5) *or x and y satisfy* (1.4) *with $m + 2 \nmid \lambda(g+1), \lambda(g+2)$, then $\mathcal{N}_W$ is trivial.*

Proof. Let Γ be a negative Latin square graph with parameters

$$\big(m^2, g(m+1), g^2+3g-m, g(g+1)\big)$$

for some positive integers m and g. It is known that by replacing m and g by their opposites, the parameters of a negative Latin square graph obtain from a (positive) Latin square graph. Therefore, by replacing m and g by their opposites and substituting into (1.4) and (1.5), we can express (1.2) and (1.3) in terms of m and g for a negative Latin square graph. It follows that (1.2) (resp. (1.3)) for a negative Latin square graph when $\epsilon = \pm 1$ is the same (1.5) (resp. (1.4)) which we compute for a Latin square graph when $\epsilon = \mp 1$. This means that to show that $\mathbf{e}_{uv}^{\mathrm{t}}\mathbf{e}_{uz} \neq 0$, it is sufficient to replace m and g by their opposites in a, a', b and b' given in the proof of Theorem 1.4 and using a similar argument it can be shown that a, a', b or b' are nonzero in each case. Note that, using techniques similar to those used in the proof of Lemma 2.1 we can show that c and c' both cannot be perfect squares by replacing m and g by their opposites when $m \geq 3g$. This completes the proof of the corollary. $\square$

2.2. **The Nomura algebras constructed on doubly regular tournaments.** In this subsection, we first determine type-II matrices attached to nonsymmetric association schemes of class 2. Then, we show that the Nomura algebras of these type-II matrices are trivial.

Theorem 2.4. *Let A be the adjacency matrix of a doubly regular tournament on n vertices with valency k. Suppose that*

$$W = I + xA + y(J - I - A).$$

Then W is a type-II matrix if and only if one of the following holds:

(1) $x = y = \frac{1}{2}(2 - n \pm \sqrt{n^2 - 4n})$ *and W is the Potts model,*
(2) $x = 1, y = -\dfrac{k \pm i\sqrt{2k+1}}{k+1}$,
(3) $x = -\dfrac{k \pm i\sqrt{2k+1}}{k+1}, y = 1.$

Proof. We first note that the eigenvalues of a doubly regular tournament Γ are k, α and $\overline{\alpha}$, where $\alpha = \frac{1}{2}(-1 + i\sqrt{2k+1})$. Using the first eigenmatrix of Γ, we see that $WW^{(-)\mathrm{t}} = nI$ is equivalent to $\big(1 + xp_1(j) + yp_2(j)\big)\big(1 + x^{-1}p_1(j') + y^{-1}p_2(j')\big) = n$ for all $j = 0, 1$ and 2. Then,

$$(1 + kx + ky)(1 + kx^{-1} + ky^{-1}) = n,$$

$$(1 + \alpha x + \overline{\alpha} y)(1 + \overline{\alpha} x^{-1} + \alpha y^{-1}) = n,$$

$$(1 + \overline{\alpha} x + \alpha y)(1 + \alpha x^{-1} + \overline{\alpha} y^{-1}) = n,$$

where $n = 2k + 1$. Using Maple, these equations may be solved for x and y in terms of k to obtain one of the following cases that may arise.

(1) x and y are one of the roots of $z^2 + (2k-1)z + 1 = 0$, that is, $x, y \in \{\frac{1}{2}(1 - 2k - \sqrt{4k^2 - 4k - 3}), \frac{1}{2}(1 - 2k + \sqrt{4k^2 - 4k - 3})\}$. If $x \neq y$, then W is not type-II and so $x = y$. Using $n = 2k+1$, we can express x and y in terms of n, i. e., $x = y = \frac{1}{2}(2 - n \pm \sqrt{n^2 - 4n})$.
(2) $x = 1$ and y is one of the roots of $(1+k)z^2 + 2kz + 1 + k = 0$, i. e., $y = -\frac{k \pm i\sqrt{2k+1}}{k+1}$.

(3) $y = 1$ and x is one of the roots of $(1+k)z^2 + 2kz + 1 + k = 0$, i. e., $x = -\frac{k \pm i\sqrt{2k+1}}{k+1}$.

This completes the proof. □

Theorem 2.5. *Let W be a type-*II *matrix attached to doubly regular tournaments obtained in Theorem* 2.4 *with* $k > 1$. *Then* $\mathcal{N}_W$ *is trivial.*

Proof. Let x and y satisfy case (1) of Theorem 2.4. It is well known that the Nomura algebra of a Potts model of order $n \geq 5$ is trivial. Let x and y satisfy two other cases of Theorem 2.4 and let $\lambda \geq 1$. In [5], it has been proved that there exists a doubly regular tournament of order $4\lambda + 3$ if and only if there exists a skew Hadamard matrix of order $4\lambda + 4$. On the other hand, in [6], it has been proved that there exists a Hadamard matrix of order $4\lambda + 4$ if and only if there exists a symmetric $(4\lambda + 3, 2\lambda + 1, \lambda) -$ BIBD. Therefore, the adjacency matrix of a doubly regular tournament is the incidence matrix of a symmetric $(4\lambda+3, 2\lambda+1, \lambda)-$BIBD and so $n = 2k+1$ and $\lambda = \frac{k-1}{2}$. Substituting into (1.1), we see that $t = -\frac{k \pm i\sqrt{2k+1}}{k+1}$. If x and y satisfy case (2) of Theorem 2.4, then we have

$$\begin{aligned} W &= I + xA + y(J - I - A) \\ &= I + A + t(J - I - A) \\ &= J + (t-1)A^{\mathrm{t}}. \end{aligned}$$

If x and y satisfy case (3) of Theorem 2.4, then we have

$$\begin{aligned} W &= I + xA + y(J - I - A) \\ &= I + tA + (J - I - A) \\ &= J + (t-1)A. \end{aligned}$$

Therefore, in each case we see that the conditions of Theorem 2.4 satisfies the hypotheses of Theorem 1.3 and so $\mathcal{N}_W$ is trivial. □

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