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# TWO-OUT DEGREE EQUITABLE DOMINATION IN GRAPHS

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Abstract. An equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly. In this paper, we introduce new variant of equitable domination of a graph. Basic properties and some interesting results have been obtained.



## 1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [2]. Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of $v$ are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A subset $S$ of $V$ is called a dominating set if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)$ $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [4]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [5]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [4]. A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

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**Definition 1.1.** *Let $G = (V, E)$ be a graph, $D \subseteq V(G)$ and $v$ be any vertex in $D$. The out degree of $v$ with respect to $D$ denoted by $od_D(v)$, is defined as $od_D(v) = |N(v) \cap (V - D)|$.*

**Definition** 1.2[1]. Let $D$ be a dominating set of a graph $G = (V, E)$. For $v \in D$, let $od_D(v) = |N(v) \cap (V - D)|$. Then $D$ is called an equitable dominating set of type 1 if $|od_D(v_1) - od_D(v_2)| \leq 1$ for all $v_1, v_2 \in D$. The minimum cardinality of such a dominating set is denoted by $\gamma_{eq1}(G)$ and is called the 1- equitable domination number of $G$.

In this paper we make the equitable dominating set of a graph.

## 2. Two-Out Degree Equitable Domination Number in Graphs

**Definition 2.1.** *A dominating set $D$ in a graph $G$ is called a two-out degree equitable dominating set if for any two vertices $u, v \in D$, $|od_D(u) - od_D(v)| \leq 2$. The minimum cardinality of a two-out degree equitable dominating set is called the two-out degree equitable domination number of $G$, and is denoted by $\gamma_{2oe}(G)$. A subset $D$ of $V$ is a minimal two-out degree equitable dominating set if no proper subset of $D$ is a two-out degree equitable dominating set.*

It is obvious that any two-out degree dominating set in a graph $G$ is also a dominating set, and thus we obtain the obvious bound $\gamma(G) \leq \gamma_{2oe}(G)$. Also, it is easy to see that, $\gamma_{2oe}(G) = 1$ if and only if $\gamma(G) = 1$.

The following results are straightforward.

**Proposition 2.2.**

(1) For the complete bipartite graph $K_{n,m}$, $1 < m \leq n$, the two-out degree equitable domination number is:

$$\gamma_{2oe}(K_{n,m}) = \begin{cases} 2, & \text{if } n - m \leq 2; \\ r, & \text{if } n - m = r,\ 3 \leq r < m; \\ m, & \text{if } n - m = r,\ 3 \leq m \leq r. \end{cases}$$

(2) For the double star $S_{n,m}$, the two-out degree equitable domination number is:

$$\gamma_{2oe}(S_{n,m}) = \begin{cases} 2, & \text{if } |n - m| \leq 2\ ; \\ n + m - 1, & \text{if } |n - m| \geq 3,\ n \text{ or } m = 1; \\ n + m - 2, & \text{if } |n - m| \geq 3,\ n, m \geq 2. \end{cases}$$

**Theorem 2.3.** *For any connected graph $G$, if $\Delta - \delta \leq 2$, then $\gamma_{2oe}(G) = \gamma(G)$.*

*Proof.* Let $G$ be a connected graph such that $\Delta - \delta \leq 2$ and let $D$ be a minimum dominating set of $G$. Then $|D| = \gamma(G)$. Since $\Delta - \delta \leq 2$, it follows that for any two vertices $u, v \in D$, $|od_D(u) - od_D(v)| \leq 2$. Hence $\gamma_{2oe}(G) = \gamma(G)$. □

**Theorem 2.4.** *Let $D$ be a two-out degree equitable dominating set of a graph $G$. Then $D$ is a minimal two-out degree equitable dominating set of $G$ if and only if one of the following holds:*

(1) *$D$ is minimal dominating set.*

(2) *For any vertex $v \in D$, the set $U_v$ is nonempty, where $U_v = \{x, y \in D, |od_D(x) - od_D(y)| = 2$, and $v$ is adjacent to $x$ but not adjacent to $y\}$.*

*Proof.* Suppose that $D$ is a minimal two-out degree equitable dominating set of $G$. Then for any $v \in D$, $D - \{v\}$ is not two-out degree equitable dominating set. If $D$ is a minimal dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x, y \in D, |od_D(x) - od_D(y)| = 2$, and $v$ is adjacent to $x$ but not adjacent to $y\}$. There exist $x, y \in D - \{v\}$ such that $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| > 2$. If both $x, y$ are adjacent to $v$, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 2$, a contradiction. If both $x, y$ are not adjacent to $v$, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 2$, a contradiction. So, $v$ is adjacent to precisely one vertex of $\{x, y\}$. Without loss of generality, assume that $v$ is adjacent to $x$ and $v$ not adjacent to $y$.
Then,

$$2 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) + 1 - od_D(y)| \quad \leq \quad |od_D(x) - od_D(y)| + 1$$

So, $|od_D(x) - od_D(y)| > 1$. But $|od_D(x) - od_D(y)| \leq 2$.

So, $|od_D(x) - od_D(y)| = 2$. Hence $U_v$ is not empty.
Conversely, let $D$ be a two-out degree equitable dominating set and suppose that $D$ is a minimal two-out degree equitable dominating set. Suppose to the contrary $D$ is not a minimal two-out degree equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a two-out degree equitable dominating set. So, $D$ is not a minimal dominating set, a contradiction. Next, suppose that $D$ is a two-out degree equitable dominating set and (2) holds. Then for every $v \in D$, $U_v$ is not empty. So, for every $v \in D$, there exist $x, y \in D$ such that $v$ is adjacent to precisely one vertex of $\{x, y\}$, and $|od_D(x) - od_D(y)| = 2$. Suppose to the contrary $D$ is not a minimal two-out degree equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a two-out degree equitable dominating set. So, $2 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| \leq 2$ and thus

$$2 \geq |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \quad \leq \quad |od_D(x) - od_D(y)| + 1 = 3$$

Since $D - \{v\}$ is a two-out degree equitable dominating set, we have $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = 2$. Then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)|$, either $\{x, y\} \subseteq N(v)$, or $\{x, y\} \cap N(v) = \phi$. □

**Theorem 2.5.** *Let $G$ be a graph. Then $G$ has a unique minimal two-out degree equitable dominating set if and only if $G = \overline{K}_n$.*

*Proof.* Suppose that $D$ is a unique minimal two-out degree equitable dominating set of $G$. Suppose $G \neq \overline{K}_n$, then there exists $u \in D$ such that $deg(u) \geq 1$. Then $V - \{u\}$ is a two-out degree equitable dominating set of $G$. Hence there exists $D' \subseteq V - \{u\}$ such that $D'$ is a minimal two-out degree equitable dominating set. Since $u \notin D'$, $D \neq D'$. Hence $G$ has two minimal two-out degree equitable dominating sets, a contradiction. Thus $G = \overline{K}_n$ and $V(G)$ is the only two-out degree equitable dominating set of $G$. □

**Theorem 2.6.** *Let $G$ be a graph of order $n$. Let $u, v \in V(G)$ such that $N(u) \neq \phi$, $N(v) \neq \phi$ and $N[u] \cap N[v] = \phi$. Then $\gamma_{2oe}(G) \leq n-2$.*

*Proof.* Let $D = V - \{u, v\}$. Since $N[u] \cap N[v] = \phi$, $u$ and $v$ are not adjacent vertices. Since $u$ and $v$ are not isolated, there exist two distinct vertices $x, y \in D$ such that $x$ is adjacent to $u$ but not adjacent to $v$ and $y$ is adjacent to $v$ but not adjacent to $u$. Clearly the out degree of any vertex of $D$ is either 0 or 1. Hence $D$ is a two-out degree equitable dominating set of $G$. Thus $\gamma_{2oe}(G) \leq n-2$. □

**Lemma 2.7.** *Let $G = (V, E)$ be a connected graph and let $D = \{u, v\}$ be a subset of $V$ such that $N(u) \cap N(v) = \phi$ and $|od_D(u) - od_{V-D}(v)| \leq 2$. Then $\gamma_{2oe}(\overline{G}) = 2$.*

*Proof.* Let $D = \{u, v\}$ and let $x$ be any vertex of $V - D$. Since $u, v \in V(G)$ such that $N(u) \cap N(v) = \phi$, we consider the following cases.

**Case 1**: $x$ is adjacent to precisely one vertex of $\{u, v\}$ in $G$. Then $x$ is adjacent to precisely one vertex of $\{u, v\}$ in $\overline{G}$ also.

**Case 2**: $x$ is not adjacent to both $u$ and $v$ in $G$. Then $x$ is adjacent to both $u$ and $v$ in $\overline{G}$.

Since $G$ is connected, it follows from the above two cases that $\{u, v\}$ is a dominating set of $\overline{G}$ and $|od_D(u) - od_D(v)| \leq 2$ of $\overline{G}$. Hence there is no vertex with full degree and hence $\{u, v\}$ is a minimum two-out degree equitable dominating set. Thus $\gamma_{2oe}(\overline{G}) = 2$. □

**Theorem 2.8.** *Let $G = (V, E)$ be a connected graph, let $u$ and $v$ be any two vertices of $V(G)$ such that $N(u) \cap N(v) = \phi$ and $|deg(u) - deg(v)| \leq 2$. Then*

(1) $\gamma_{2oe}(G) + \gamma_{2oe}(\overline{G}) \leq n$.
(2) $\gamma_{2oe}(G)\gamma_{2oe}(\overline{G}) \leq 2(n-2)$.

*Proof.* Let $G = (V, E)$ be a connected graph. By Lemma 2.7, $\gamma_{2oe}(\overline{G}) = 2$ and by Theorem 2.6, $\gamma_{2oe}(G) \leq n-2$. Hence $\gamma_{2oe}(G) + \gamma_{2oe}(\overline{G}) \leq n$ and $\gamma_{2oe}(G)\gamma_{2oe}(\overline{G}) \leq 2(n-2)$. □

**Theorem 2.9.** *Let $G$ be an isolate-free graph of order $n$ and let $D$ be a maximum independent set of $G$ such that for $u \in V - D$, $|N(u) \cap D| \leq 2$. Then $\gamma_{2oe}(G) \leq n - \beta$.*

*Proof.* Let $G$ be an isolate-free graph of order $n$. Since $D$ is a maximum independent set of $G$, $V - D$ is dominating set of $G$. Then for any $u, v \in V - D$, $|od_{V-D}(u) - od_{V-D}(v)| \leq 2$. Hence $V - D$ is a two-out degree equitable dominating set of $G$. Thus $\gamma_{2oe}(G) \leq |V - D| \leq n - |D| \leq n - \beta$. □

**Theorem 2.10.** *For any graph $G$ of order $n$ and size $m$, $\gamma_{2oe}(G) = n - m$ if and only if $G = \cup_{i=1}^{\gamma_{2oe}} K_{1,r_i}$ such that $|r_i - r_j| \leq 2$, $1 \leq i, j \leq \gamma_{2oe}(G)$.*

*Proof.* Let $\gamma_{2oe}(G) = n - m$. Suppose that $G$ has $t$-components. The minimum number of edges in each component is $n_i - 1$, where $n_i$ is the number of vertices in that component. Since any dominating set of $G$ has at least one vertex from each component of $G$, $t \leq \gamma_{2oe}(G)$. But $m \geq n_1 - 1 + n_2 - 1 + \cdots + n_t - 1$.

So, $m \geq n-t$. Hence $t \geq \gamma_{2oe}(G)$. Thus $t = \gamma_{2oe}(G)$. If $G$ is not a forest, then $G$ contains a component $G_1$, say which is cyclic. Then $m \geq n$, so that $\gamma_{2oe}(G) \leq 0$ which is not possible. Hence $G$ is a forest. Since $t = \gamma_{2oe}(G)$, it follows that each component is a star. That is $G = \cup_{i=1}^{\gamma_{2oe}} K_{1,r_i}$. Since the centers of the stars constitute a minimum two-out degree equitable dominating set, it follows that if $G_i = K_{1,r_i}$ and $G_j = K_{1,r_j}$, then $|r_i - r_j| \leq 2$. □

**Theorem 2.11.** *For any positive integer $m$, there exists a graph $G$ such that $\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = m$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$.*

*Proof.* For $m = 1$, take $G = K_{4,4}$, $\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = 2 - 1 = 1$.
For $m = 2$, take $G = K_{3,6}$, $\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = 3 - 1 = 2$.
For $m \geq 3$, take $G = S_{r,s}$, where $r + s = m + 3$, $s \geq r + 3$
$\gamma_{2oe}(G) = r + s - 2 = m + 1$,
$\lfloor \frac{n}{\Delta+1} \rfloor = \lfloor \frac{r+s+2}{s+2} \rfloor = 1$,
$\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = r + s - 3 = m$. □

**Theorem 2.12.** *Let $G$ be a graph of order $n$ having $p_0$ isolated vertices. Then $\gamma_{2oe}(G) \geq \frac{n+2p_0}{3}$.*

*Proof.* Let $D$ be any minimum two-out degree equitable dominating set of $G$. Then for any $v \in D$, $v$ is dominating at most two vertices of $V - D$. Let $|D'| = |D| - p_0$. Then $2|D'| \geq |V - D|$. It follows that, $2|D'| + |D| \geq n$. Then $\gamma_{2oe}(G) \geq \frac{n+2p_0}{3}$. □

The bound is sharp for $\overline{K_n}$.

**Theorem 2.13.** *Let $G$ be a graph and let $D$ be a minimum two-out degree equitable dominating set of $G$ containing $t$ pendant vertices such that every vertex of $V - D$ is a pendant vertex. Then $\gamma_{2oe}(G) \geq \frac{n+t}{3}$.*

*Proof.* Let $D$ be any minimum two-out degree equitable dominating set of $G$ containing $t$ pendant vertices such that every vertex $v \in V - D$ is a pendant vertex. Then $2|D| - t \geq |V - D|$. It follows that, $3|D| - t \geq n$. Hence $\gamma_{2oe}(G) \geq \frac{n+t}{3}$. □

The bound is sharp for $lK_2$, $l \geq 1$.

**Theorem 2.14.** *Let $G$ be a graph, and $u, v, w \in V(G)$ such that $u, v, w$ are not isolates and $(uv, vw, uw) \notin E(G)$. Then $V - \{u, v, w\}$ is a two-out degree equitable dominating set if and only if one of the following conditions hold:*

(1) $N(u) \cap N(v) \cap N(w) = \phi$.
(2) $N(u) \cap N(v) \cap N(w) \neq \phi$ *and* $N[u] \cup N[v] \cup N[w] = V$.

*Proof.* Let $D = V - \{u, v, w\}$. Suppose one of the conditions (1),(2) holds. Since $u, v, w$ are not isolates and $(uv, vw, uw) \notin E(G)$, then $D$ is a dominating set. If (1) holds, then any vertex $x \in D$ is adjacent to at most two of $u, v, w$, such that the out degree of $x$ is at most two. Hence $D$ is a two-out degree equitable dominating set. If (2) holds, then there exists $x \in N(u) \cap N(v) \cap N(w)$ such that

$x \in D$. It follows that $od_D(x) \leq 3$. Since $N[u] \cup N[v] \cup N[w] = V$, $od_D(y) \geq 1$, for each $y \in D$. Hence $D$ is a two-out degree equitable dominating set.

Conversely, suppose $D = V - \{u, v, w\}$ is a two-out degree equitable dominating set. If (1) hold, then we are done. If not then there exists $x \in N(u) \cap N(v) \cap N(w)$. i.e, $x \in D$, and $od_D(x) = 3$. Since $D$ is a two-out degree equitable dominating set, then for any $y \in D$, $1 \leq od_D(y) \leq 3$.

Hence $N(u) \cap N(v) \cap N(w) \neq \phi$ and $N[u] \cup N[v] \cup N[w] = V$. □

## 3. Two-Out Degree Equitable Domatic Number in Graphs

**Definition 3.1.** *A partition $P = \{V_1, V_2, \ldots, V_l\}$ of $V(G)$ is called a two-out degree equitable domatic partition if $V_i$ is a two-out degree equitable dominating set for every $1 \leq i \leq l$.*

**Example** :

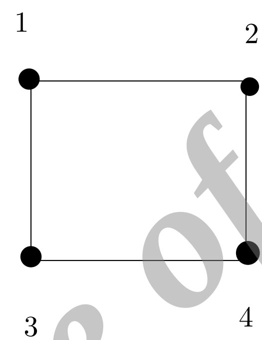


Figure 1

$\{\{1, 2\}, \{3, 4\}\}$ is a two-out degree equitable domatic partition of $G$.

**Definition 3.2.** *The two-out degree equitable domatic number of $G$ is the maximum cardinality of a two-out degree equitable domatic partition of $G$ and is denoted by $d_{2oe}(G)$.*

We now proceed to compute $d_{2oe}(G)$ for some standard graphs. It can be easily verified that

(1) For the complete graph $K_n$, $d_{2oe}(K_n) = n$.
(2) For the cycle $C_n$, $n \geq 4$, $d_{2oe}(C_n) = 2$.
(3) For the path $P_n$, $d_{2oe}(P_n) = 2$.
(4) For the star $K_{1,n}$, $d_{2oe}(K_{1,n}) = 2$.
(5) For the wheel $W_n$ with $n$ vertices, $d_{2oe}(W_n) = 2$.
(6) For the complete bipartite graph $K_{n,m}$, $m \leq n$ we have

$$d_{2oe}(K_{n,m}) = \begin{cases} m, & \text{if } n - m \leq 2,\ n, m \geq 2; \\ 2, & \text{otherwise.} \end{cases}$$

**Theorem 3.3.** *For any graph $G$, $d_{2oe}(G) \leq \delta(G) + 1$.*

*Proof.* Let $D$ be any two-out degree equitable dominating set of $G$. Then for any $v \in V(G)$, $D \cap N[v] \neq \phi$. Let $v \in V(G)$ such that $deg(v) = \delta(G)$ and $N[v] = \{v, u_1, u_2, \ldots, u_\delta\}$. If $d_{2oe}(G) > \delta(G) + 1$, then there exist at least $(\delta(G)+2)$ sets of a two-out degree equitable domatic partition of $G$, each containing at least one element of $N[v]$. Then $deg(v) \geq \delta(G) + 1$, a contradiction. Hence $d_{2oe}(G) \leq \delta(G) + 1$. □

**Theorem 3.4.** *For any graph $G$ of order $n$, $d_{2oe}(G) \leq \frac{n}{\gamma_{2oe}(G)}$.*

*Proof.* Suppose that $d_{2oe}(G) = t$, for some positive integer $t$. Let $P = \{D_1, D_2, \ldots, D_t\}$ be the two-out degree equitable domatic partition of $G$. Obviously, $|V(G)| = \sum_{i=1}^{t} |D_i|$ and from definition of the two-out degree equitable domination number $\gamma_{2oe}(G)$, we have $\gamma_{2oe}(G) \leq |D_i|$, $i = 1, 2, \ldots, t$. Hence $n = \sum_{i=1}^{t} |D_i| \geq t\gamma_{2oe}(G)$. Thus $d_{2e}(G) \leq \frac{n}{\gamma_{2oe}(G)}$. □

## 4. **Conclusion**

We can generalize the concept of a two-out degree equitable domination as follows: Let $G = (V, E)$ be a graph with dominating set $D$, then $D$ is called a k-out degree equitable dominating set if for any two vertices $u$ and $v$ in $D$, $|od_D(u) - od_D(v)| \leq k$. The similar results can be obtained from the two-out degree equitable domination.

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