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NOTE ON DEGREE KIRCHHOFF INDEX OF GRAPHS

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ABSTRACT. The degree Kirchhoff index of a connected graph G is defined as the sum of the terms $d_i\,d_j\,r_{ij}$ over all pairs of vertices, where d_i is the degree of the i-th vertex, and r_{ij} the resistance distance between the i-th and j-th vertex of G. Bounds for the degree Kirchhoff index of the line and para-line graphs are determined. The special case of regular graphs is analyzed.

1. Introduction

Throughout this paper all graphs are assumed to be finite and simple. Let $G = (V, E)$ be such a graph of order n, having m edges. In other words, $|V(G)| = n$ and $|E(G)| = m$. By $\deg(v)$ is denoted the degree (= number of first neighbors) of the vertex $v \in V(G)$.

The eigenvalues of the adjacency matrix $\mathcal{A}(G)$ of G are called the eigenvalues of G and will be denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The Laplacian matrix of G is defined as $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ where $\mathcal{D}(G) = [d_{ij}]$ is the diagonal matrix with $d_{ii} = \deg(v_i)$, and $d_{ij} = 0$ for $i \neq j$. The eigenvalues of $\mathcal{L}(G)$ are called the Laplacian eigenvalues of G and will be denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where $\mu_n = 0$ for all graphs.

Provided the graph G has no isolated vertices, the normalized Laplacian matrix $\tilde{\mathcal{L}}(G)$ is defined as

$$\tilde{\mathcal{L}} = \mathcal{D}^{-1/2}\,\mathcal{L}\,\mathcal{D}^{-1/2}$$

which implies that its (i, j)-entry is equal to 1 if $i = j$, equal to $-1/\sqrt{\deg(v_i)\,\deg(v_j)}$ if $i \neq j$ and the vertices v_i, v_j are adjacent, and zero otherwise. The eigenvalues of $\tilde{\mathcal{L}}$ are called the normalized

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Laplacian eigenvalues of G [5, 6], and will be denoted by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$, where $\delta_n = 0$ for all graphs.

In [13], Klein and Randić introduced the notion of resistance-distance as a second distance function on the vertex set of a graph. If r_{ij} denotes the resistance–distance between vertices v_i and v_j in a graph G then the Kirchhoff index of G is defined as $Kf(G) = \sum_{i<j} r_{ij}$ [13].

Originally, the r_{ij} was conceived as the resistance between the nodes i and j in an electrical network corresponding to the graph G, in which all edges are replaced by resistors of unit resistance. Eventually, it was shown [1, 18–20] that the resistance distance can be expressed in terms of Laplacian matrix and its spectrum.

In [11], it was proven that $Kf(G) = n \sum_{i=1}^{n-1} 1/\mu_i$. We refer to [10, 14, 15, 23–25] for more information about the mathematical properties of the Kirchhoff index. Recently, Chen and Zhang [8], introduced the degree Kirchhoff index of G as $Kf^*(G) = \sum_{i<j} d_i\, d_j\, r_{ij}$ and proved that $Kf^*(G) = 2m \sum_{i=1}^{n-1} 1/\delta_i$, see [4, 9, 16, 17, 25] for details.

The line graph $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have a common vertex. Define $L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$, $k \geq 1$. A well–known result in graph theory states that the line graph $L^k(G)$ of an r-regular G of order n is a $(2^k\, r - 2^{k+1} + 2)$-regular graph of order $n \prod_{i=0}^{k-1}(2^{i-1}\, r - 2^i + 1)$, having $n \prod_{i=0}^{k}(2^{i-1}\, r - 2^i + 1)$ edges. Following Yan et al. [21], a para-line graph of G, denoted by $C(G)$, is defined as a line graph of the subdivision graph $S(G)$ of G. Here, $S(G)$ is the graph obtained from G by inserting a vertex to every edge of G. This graph is also called the clique-inserted graph [21, 22]. Let $C^0(G) = G$ and $C^k(G) = C(C^{k-1}(G))$, $k \geq 1$. Notice that for $k \geq 0$, $C^k(G)$ is r-regular with $n'_k = n\, r^k$ vertices and $m'_k = \frac{1}{2}\, n\, r^{k+1}$ edges.

It is well known that if G is connected, then $L(G)$ and $S(G)$ are also connected. For two graphs G_1 and G_2, $G_1 \cup G_2$ is the disjoint union of G_1 and G_2. The join $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by connecting all vertices of $V(G_1)$ with all vertices of $V(G_2)$. If $G_1, G_2, \ldots, G_k$ are graphs with mutually disjoint vertex sets, we denote $G_1 + G_2 + \cdots + G_k$ by $\sum_{j=1}^{k} G_j$. In the case that $G_1 = G_2 = \cdots = G_k = G$, we denote $\sum_{j=1}^{k} G_j$ by $k\, G$. The following results are crucial throughout this paper.

Lemma 1.1 [7]. Let G be a graph of order $n \geq 2$ without isolated vertices. Then 0 is a simple eigenvalue of $\tilde{\mathcal{L}}(G)$ if and only if G is connected. Moreover, $\sum_{i=1}^{n} \delta_i = n$.

Lemma 1.2 [22]. Let G be a simple r-regular graph with n vertices, $m = \frac{1}{2}\, nr$ edges, and eigenvalues $r = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues of the para-line graph $C(G)$ are: 0 with multiplicity $m-n$, -2 with multiplicity $m-n$ and simple eigenvalues $\frac{1}{2}\left[r + 2 \pm \sqrt{r^2 + 4(\lambda_i + 1)}\right]$, where $1 \leq i \leq n$.

As well known [7], if G is a connected r-regular graph, then $\delta(G) \leq 2$ with equality if and only if G is bipartite.

Lemma 1.3 [5]. Let G_1 be an r_1-regular graph of order n_1, and G_2 an r_2-regular graph of order n_2. Let $2 \geq \delta_1(G_1) \geq \delta_2(G_1) \geq \cdots \geq \delta_n(G_1) = 0$ be the normalized Laplacian eigenvalues of G_1 and $2 \geq \delta_1(G_2) \geq \delta_2(G_2) \geq \cdots \geq \delta_n(G_2) = 0$ the normalized Laplacian eigenvalues of G_2. Then the normalized Laplacian eigenvalues of the join graph $G_1 + G_2$ is computed as follows:

$$
\begin{pmatrix}
\dfrac{n_2}{n_2 + r_1} + \dfrac{n_1}{n_1 + r_2} & \dfrac{n_2 + r_1\,\delta_i(G_1)}{n_2 + r_1} & \dfrac{n_1 + r_2\,\delta_i(G_2)}{n_1 + r_2} & 0 \\
1 & 1 \leq i \leq n_1 - 1 & 1 \leq i \leq n_2 - 1 & 1
\end{pmatrix}.
$$

Throughout this paper our notation is standard. The complement of a graph G is denoted by $\overline{G}$. K_n and $\overline{K}_n$ are the complete and empty graphs on n vertices, respectively. The complete bipartite graph with bipartitions of size n_1 and n_2 is denoted by K_{n_1,n_2}. The Kneser graph $KG_{m,n}$ is the graph whose vertices are the m-subsets of an n-set, two such subsets being adjacent if only if their intersection is empty.

2. **Main results**

The aim of this section is to find bounds for degree Kirchhoff index of graphs. Some formulas for the degree Kirchhoff index of $L^k(G)$ and $C^k(G)$, $k \geq 1$, are also presented.

Proposition 2.1. Let G be a connected r-regular graph of order $n \geq 3$, having t spanning trees. Then $Kf^*(G) \geq n\,r^2(n-1)(t\,n)^{-1/(n-1)}$, with equality if and only if $G \cong K_n$.

Proof. By the arithmetic–geometric inequality [12],

$$
\frac{1}{n-1}Kf^*(G) = \frac{2m}{n-1}\sum_{i=1}^{n-1}\frac{1}{\delta_i} \;\geq\; 2m\left(\prod_{i=1}^{n-1}\frac{1}{\delta_i}\right)^{1/(n-1)}.
$$

Since $\delta_i = \mu_i/r$ and $m = \frac{1}{2}n\,r$, the matrix–tree theorem [3] implies that $t\,n = \prod\limits_{i=1}^{n-1}\mu_i$. Thus,

$$
Kf^*(G) \;\geq\; n\,r(n-1)\left(\prod_{i=1}^{n-1}\frac{r}{\mu_i}\right)^{1/(n-1)} = n\,r^2(n-1)\left(\frac{1}{t\,n}\right)^{1/(n-1)}.
$$

Equality holds if and only if all δ_i's for $i = 1, 2, \ldots, n-1$, are mutually equal. This happens only if $G \cong K_n$ [7]. $\qquad\square$

Corollary 2.2. Let G be a connected r-regular graph of order $n \geq 3$ and let $\overline{G}$ be also connected, having $\bar{t}$ spanning trees. Then

$$
Kf^*(\overline{G}) > n(n-r-1)^2\,(n-1)\left(\frac{1}{\bar{t}\,n}\right)^{1/(n-1)}.
$$

Proposition 2.3. Suppose that G_1 and G_2 are graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then

$$Kf^*(G_1 \cup G_2) = Kf^*(G_1) + Kf^*(G_2) + 2\left(m_2 \sum_{i=1}^{n_1-1} \frac{1}{\delta_i(G_1)} + m_1 \sum_{j=1}^{n_2-1} \frac{1}{\delta_j(G_2)}\right)$$

i.e.,

$$Kf^*(G_1 \cup G_2) = (m_1 + m_2)\left(\frac{1}{m_1} Kf^*(G_1) + \frac{1}{m_2} Kf^*(G_2)\right).$$

Proof. By definition, the normalized Laplacian eigenvalues of $G_1 \cup G_2$ are: $\delta_i(G_1)$, $\delta_j(G_2)$, 0, 0, where $1 \le i \le n_1 - 1$ and $1 \le j \le n_2 - 1$. This yields the result. $\square$

Proposition 2.4. Suppose G_1 and G_2 are r_1- and r_2-regular graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then,

$$
\begin{aligned}
Kf^*(G_1 + G_2) &= Kf^*(G_1 + \overline{K}_{n_2}) + Kf^*(G_2 + \overline{K}_{n_1}) - Kf^*(K_{n_1,n_2}) \\[2mm]
&+ \frac{n_2 r_2(n_2 + r_1)}{n_1 + n_2} Kf(G_1 + \overline{K}_{n_2}) + \frac{n_1 r_1(n_1 + r_2)}{n_1 + n_2} Kf(G_2 + \overline{K}_{n_1}) \\[2mm]
&- n_1 n_2 \left[r_1 + r_2 + 1 + \frac{r_1}{2n_2 + r_1} + \frac{r_2}{2n_1 + r_2} - 2B\right] \\[2mm]
&- n_1 r_1 \left[n_1 + r_2\left(\frac{1}{n_1 + n_2} + \frac{n_1 - 1}{n_2}\right) - \frac{n_2}{2n_2 + r_1} - B\right] \\[2mm]
&- n_2 r_2 \left[n_2 + r_1\left(\frac{1}{n_1 + n_2} + \frac{n_2 - 1}{n_1}\right) - \frac{n_1}{2n_1 + r_2} - B\right]
\end{aligned}
$$

where

$$B = \frac{(n_2 + r_1)(n_1 + r_2)}{n_2(n_1 + r_2) + n_1(n_2 + r_1)} \ .$$

Proof. Obviously, $m_1 = \frac{1}{2} n_1 r_1$, $m_2 = \frac{1}{2} n_2 r_2$, $r_1 \delta_i(G_1) = \mu_i(G_1)$, and $r_2 \delta_i(G_2) = \mu_i(G_2)$, where $\mu_i(G_j)$ is a Laplacian eigenvalue of G_j, $j = 1, 2$ and $1 \le i \le n_j$. So,

$$
\begin{aligned}
Kf^*(G_1 + G_2) &= (n_1 r_1 + n_2 r_2 + 2n_1 n_2)\left[\sum_{i=1}^{n_1-1} \frac{n_2 + r_1}{n_2 + \mu_i(G_1)} + \sum_{i=1}^{n_2-1} \frac{n_1 + r_2}{n_1 + \mu_i(G_2)}\right. \\[2mm]
(2.1) &\qquad\qquad \left. + \frac{(n_2 + r_1)(n_1 + r_2)}{n_2(n_1 + r_2) + n_1(n_2 + r_1)}\right].
\end{aligned}
$$

Thus,

$$
(2.2) \qquad Kf^*(G_1 + \overline{K}_{n_2}) = (n_1 r_1 + 2n_1 n_2)\left[n_2 - 1 + \frac{n_2 + r_1}{2n_2 + r_1} + \sum_{i=1}^{n_1-1} \frac{n_2 + r_1}{n_2 + \mu_i(G_1)}\right]
$$

and

$$(2.3) \qquad Kf^*(G_2 + \overline{K}_{n_1}) = (n_2\, r_2 + 2n_1\, n_2)\left[n_1 - 1 + \frac{n_1 + r_2}{2n_1 + r_2} + \sum_{i=1}^{n_2-1}\frac{n_1 + r_2}{n_2 + \mu_i(G_2)}\right]\ .$$

We further have:

$$(2.4) \qquad Kf(G_1 + \overline{K}_{n_2}) = 1 + (n_1 + n_2)\left(\frac{n_2 - 1}{n_1} + \sum_{i=1}^{n_1-1}\frac{1}{n_2 + \mu_i(G_1)}\right)$$

and

$$(2.5) \qquad Kf(G_2 + \overline{K}_{n_1}) = 1 + (n_1 + n_2)\left(\frac{n_1 - 1}{n_2} + \sum_{i=1}^{n_2-1}\frac{1}{n_1 + \mu_i(G_2)}\right).$$

Then the proof is obtained by substituting Eqs. (2.2)–(2.5) into (2.1). $\qquad\qquad\square$

Corollary 2.6. Let G be an r-regular graph of order n. Then

$$Kf^*(2G) = 4n(n + r)^2\left(\sum_{i=1}^{n-1}\frac{1}{n + \mu_i(G)} + \frac{1}{4n}\right).$$

Lemma 2.7. Let G be a connected r-regular graph with n vertices and m edges. Then,

$$Kf^*(L(G)) = (r-1)^2\left[2r\, Kf(G) + \frac{n^2}{2}(r-2)\right].$$

Proof. Suppose that the eigenvalues of G are $r = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Then 0, $\frac{r-\lambda_i(G)}{2r-2}$, $2 \leq i \leq n$, and $\frac{r-1}{r}$ with multiplicity $\frac{n(r-2)}{2}$ are the normalized Laplacian eigenvalues of $L(G)$. Since the number of edges of $L(G)$ is $\frac{n\, r(r-1)}{2}$,

$$
\begin{aligned}
Kf^*(L(G)) &= nr(r-1)\left[\sum_{i=2}^{n}\frac{2r-2}{r-\lambda_i} + \frac{n(r-2)(r-1)}{2r}\right] \\
&= 2nr(r-1)^2\sum_{i=2}^{n}\frac{1}{\mu_i} + \frac{n^2(r-1)^2(r-2)}{2} \\
&= (r-1)^2\left[2r\, Kf(G) + \frac{n^2}{2}(r-2)\right]
\end{aligned}
$$

proving the result. $\qquad\qquad\square$

Proposition 2.8. Let G be an r-regular graph with n vertices. Then

$$Kf^*(L^{k+1}(G)) \sim \frac{n^2}{2}(2^k r - 2^{k+1} + 1)^2\prod_{j=0}^{k-2}r_j^2$$

$$\times\left[\frac{1}{2}\prod_{j=k-2}^{k-1}(r_j - 1) + \frac{r-2}{2^k}(2^{k-1}r - 2^k + 2)^2\right]$$

where, $r_j = 2^j\, r - 2^{j+1} + 2$ and k is a sufficiently large integer.

Proof. Let r_k and n_k denote the degree and the number of vertices of $L^k(G)$, respectively. By Lemma 2.7, we get $Kf^*(L^{k+1}(G)) = (r_k-1)^2 \left[2r_k\,Kf(L^k(G)) + \frac{n_k^2}{2}(r_k-2)\right]$. From [21, Theorem 3.3], we have

$$Kf(L^k(G)) \sim \frac{n^2}{4}(2^{k-2}r - 2^{k-1} + 1)\prod_{j=0}^{k-2}(2^{j-1} - 2^j + 1)^2, \text{ when } k \to \infty \ .$$

Therefore,

$$Kf^*(L^{k+1}(G)) = \left[\frac{1}{2}(2^{k-1}r - 2^k + 1)(2^{k-2} - 2^{k-1} + 1)\prod_{j=0}^{k-2} r_j^2 + \frac{r-2}{2^k}\left(\prod_{j=0}^{k-1} r_j\right)^2\right]$$

$$\times \ \frac{n^2}{2}(2^k r - 2^{k+1} + 1)^2 = \left[\frac{1}{2}(2^{k-1}r - 2^k + 1)(2^{k-2} - 2^{k-1} + 1)\right.$$

$$+ \ \left.\frac{r-2}{2^k}(2^{k-1}r - 2^k + 2)^2\right]\frac{n^2}{2}(2^k r - 2^{k+1} + 1)^2\prod_{j=0}^{k-2} r_j^2$$

$$= \ \frac{n^2}{2}(2^k r - 2^{k+1} + 1)^2\prod_{j=0}^{k-2} r_j^2\left[\frac{1}{2}\prod_{j=k-2}^{k-1}(r_j - 1) + \frac{r-2}{2^k}(2^{k-1}r - 2^k + 2)^2\right]$$

where $r_j = 2^j r - 2^{j+1} + 2$ and k is an enough large integer. $\qquad\square$

Lemma 2.9. Let G be an r-regular graph of order n. Then

$$Kf^*(C(G)) = n\,r^2\left[\frac{r(r+2)}{n}\,Kf(G) + \frac{n(r-2)(r+1) + r}{r+2}\right].$$

Proof. By Lemma 1.2, since $C(G)$ is r-regular, the normalized Laplacian eigenvalues of $C(G)$ are:

$$\begin{pmatrix} \dfrac{r + 2 \pm \sqrt{r^2 + 4(\lambda_i + 1)}}{2r} & \dfrac{r+2}{r} & 1 & 0 \\[2mm] 2 \le i \le n & \dfrac{n(r-2)}{2} + 1 & \dfrac{n(r-2)}{2} & 1 \end{pmatrix}$$

Applying the definition of the degree Kirchhoff index we then have:

$$Kf^*(C(G)) = 2n\,r^3\sum_{i=2}^{n}\left(\frac{1}{r + 2 + \sqrt{r^2 + 4(\lambda_i + 1)}} + \frac{1}{r + 2 - \sqrt{r^2 + 4(\lambda_i + 1)}}\right)$$

$$+ \ n\,r^2\left[\frac{(n(r-2) + 2)r}{2(r+2)} + \frac{n(r-2)}{2}\right]$$

$$= \ n\,r^2\left[\sum_{i=2}^{n}\frac{r(r+2)}{r - \lambda_i} + \frac{n(r-2)(r+1) + r}{r+2}\right]$$

$$= \ n\,r^2\left[\frac{r(r+2)}{n}\,Kf(G) + \frac{n(r-2)(r+1) + r}{r+2}\right]$$

as desired. $\square$

Proposition 2.10. Let G be an r-regular graph of order n. Then

$$Kf^*(C^{k+1}(G)) \quad \sim \quad r^{k+2}\,(r+2)^k \left[n^2\,(r-2)(r+1) \left(\frac{1}{2} + \frac{r^k}{(r+2)^{k+1}} \right) \right.$$

$$+ \quad \frac{n\,r}{r+1} + r(r+2)Kf(G) \bigg] + \frac{n\,r^{k+3}}{r+2} \qquad \text{when } k \to \infty \ .$$

Moreover, if $k \to \infty, r \to \infty$ then

$$Kf^*(C^{k+1}(G)) \sim n\,r^{k+2} \left[(r+2)^{k+1} \left(\frac{n}{2} + \frac{Kf(G)}{n} \right) + n\,r^{k+1} + 1 \right].$$

Proof. Let $C^k(G)$ has exactly n'_k vertices. By Lemma 2.9,

$$Kf^*(C^{k+1}(G)) \quad = \quad n'_k\,r^2 \left[\frac{r(r+2)}{n'_k}\,Kf(C^k(G)) + \frac{n'_k\,(r-2)(r+1)+r}{r+2} \right]$$

$$= \quad r^3\,(r+2)\,Kf(C^k(G)) + \frac{n^2\,r^{2k+2}\,(r-2)(r+1) + n\,r^{k+3}}{r+2} \ .$$

From [21, Theorem 3.6], we have

$$Kf(C^k(G)) \sim \left[\frac{n^2\,(r-2)(r+1)}{2r(r+2)} + \frac{n}{(r+1)(r+2)} + Kf(G) \right] r^k\,(r+2)^k \qquad \text{for } k \to \infty \ .$$

Then,

$$Kf^*(C^{k+1}(G)) \quad \sim \quad r^{k+2}\,(r+2)^k \left[\frac{n^2\,(r-2)(r+1)}{2} + \frac{n\,r}{r+1} + r(r+2)\,Kf(G) \right]$$

$$+ \frac{n^2\,r^{2k+2}\,(r-2)(r+1) + n\,r^{k+3}}{r+2}$$

$$= \quad r^{k+2}\,(r+2)^k \left[n^2\,(r-2)(r+1) \left(\frac{1}{2} + \frac{r^k}{(r+2)^{k+1}} \right) \right.$$

$$+ \quad \frac{n\,r}{r+1} + r(r+2)\,Kf(G) \bigg] + \frac{n\,r^{k+3}}{r+2} \qquad \text{for } k \to \infty \ .$$

Also, from [21], we have

$$Kf(C^k(G)) \sim r^k(r+2)^k \left(\frac{n^2}{2} + Kf(G) \right) \qquad \text{for } k \to \infty \ , \ r \to \infty \ .$$

Hence, for $k \to \infty$ and $r \to \infty$, we have:

$$Kf^*(C^{k+1}(G)) \quad \sim \quad r^{k+3}\,(r+2)^{k+1} \left(\frac{n^2}{2} + Kf(G) \right) + n^2\,r^{2k+3} + n\,r^{k+2}$$

$$= \quad n\,r^{k+2} \left[(r+2)^{k+1} \left(\frac{n}{2} + \frac{Kf(G)}{n} \right) + n\,r^{k+1} + 1 \right].$$

$\square$

3. **Applications**

In this section, we apply our results given the Section 2 and gives formulas for the degree Kirchhoff index of some classes of graphs.

Example 3.1.   In this example, degree Kirchhoff index of some graphs constructed from complete graph K_n, $n \geq 2$, are computed. From [2], we have that $KG_{1,n} \cong K_n$, $n \geq 3$. Thus, $Kf^*(K_n) = Kf^*(KG_{1,n}) = (n-1)^3$. Note that K_n is $(n-1)$-regular with n vertices and $Kf(K_n) = n-1$. In [3], a triangular graph Δ_n, is defined as a line graph of the complete graph K_n and in [2], it is proved that $\overline{KG_{2,n}} \cong L(K_n)$, $n \geq 5$. Now by Lemma 2.7, we have

$$Kf^*(\overline{KG_{2,n}}) = Kf^*(L(K_n)) = \frac{n^4}{2}(n-3) - 4(n^3 - 5\,n^2 + 6n - 2)\,.$$

By Lemma 2.9,

$$Kf^*(C(K_n)) = \frac{n}{n+1}(n^5 - 5n^4 + 8n^3 - 6n^2 + 3n - 1) + n^5 - 3n^4 + 2n^3 + 2n^2 - 3n + 1\,.$$

Since Δ_n, is an $(2n-4)$-regular graph with $\frac{n(n-1)}{2}$ vertices and $Kf(\Delta_n) = \frac{1}{8}(n-2)(n^2 + 3n - 2)$, by Lemma 2.7

$$Kf^*(L(\Delta_n)) = n^7 - 8\,n^6 + \frac{1}{4}(105\,n^5 - 267\,n^4 + 745\,n^3 - 1439\,n^2) + 330\,n - 100\,.$$

Note that, $Kf^*(L(\Delta_n)) = Kf^*(L(\overline{KG_{2,n}})) = Kf^*(L^2(K_n))$. By Lemma 2.8,

$$\begin{aligned}
Kf^*(C(\Delta_n)) &= \frac{n}{n-1}\big(2\,n^7 - 21\,n^6 + 89\,n^5 - 193\,n^4 + 219\,n^3 - 108\,n^2 - 4n + 16\big) \\
&+ 2\big(n^7 - 6\,n^6 + 3\,n^5 + 58\,n^4 - 184\,n^3 + 240\,n^2 - 144\,n + 32\big)\,.
\end{aligned}$$

Example 3.2.   Consider the cycle graph C_n. It is well know that $Kf(C_n) = \frac{n}{12}(n^2 - 1)$. Since C_n is 2-regular with n vertices, and $Kf^*(C_n) = Kf^*(L(C_n))$, by Lemma 2.7, $Kf^*(L(C_n)) = \frac{n}{3}(n^2 - 1)$. Apply Lemma 2.9, we have $Kf^*(C(C_n)) = \frac{2n}{3}(4n^2 - 1)$.

Example 3.3.   Consider the complete bipartite graph $K_{n,n}$, for which $Kf = 4n - 3$. Recall that the normalized Laplacian spectrum of $K_{n,n}$ is $\{2, \underbrace{1, \ldots, 1}_{2n-2}, 0\}$ and $K_{n,n}$ is n-regular with $2n$ vertices and n^2 edges. Then $Kf^*(K_{n,n}) = n^2(4n - 3)$. The line graph of $K_{n,n}$, $n \geq 2$, is known as the lattice graph $L_2(n)$. By Lemma 2.7, we have $Kf^*(L(K_{n,n})) = 2n(n^4 - 6n^2 + 8n - 3)$. On the other hand, by Lemma 2.9,

$$Kf^*(C(K_{n,n})) = n^3(4\,n^2 + 5n - 6) + \frac{2n^4}{n+2}(2\,n^2 - 2n - 3)\,.$$

By continuing this reasoning, since $K_{n,n,n}$ is $2n$-regular of order $3n$ and $Kf(K_{n,n,n}) = \frac{1}{2}(9n - 5)$, $Kf^*(L(K_{n,n,n})) = n(36\,n^4 - 67\,n^2 + 49\,n - 10)$, it follows that

$$Kf^*(C(K_{n,n,n})) = \frac{12\,n^4}{n+1}(6\,n^2 - 3n - 2) + 8\,n^3(9n^2 + 4n - 5)\,.$$

Example 3.4. Cocktail–party graph CP_n. Notice that CP_n is $(2n-2)$-regular with $2n$ vertices and $2n(n-1)$ edges. As well-known, $Kf(CP_n) = \frac{2n^2-2n+1}{n-1}$ and $Kf^*(CP_n) = 4(2n^3 - 4n^2 + 3n - 1)$. By Lemma 2.7,

$$Kf^*(L(CP_n)) = 4(4\,n^5 - 12\,n^4 + n^3 + 28\,n^2 - 30\,n + 9)$$

whereas by Lemma 2.9,

$$Kf^*(C(CP_n)) = 8(8\,n^5 - 22\,n^4 + 23\,n^3 - 13\,n^2 + 5\,n^5 - 1)\ .$$

Acknowledgments

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