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ROMAN GAME DOMINATION SUBDIVISION NUMBER OF A GRAPH

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Abstract. A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of a Roman dominating function is the value $w(f) = \sum_{v\in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G. The Roman game domination subdivision number of a graph G is defined by the following game. Two players $\mathcal{D}$ and $\mathcal{A}$, $\mathcal{D}$ playing first, alternately mark or subdivide an edge of G which is not yet marked nor subdivided. The game ends when all the edges of G are marked or subdivided and results in a new graph G'. The purpose of $\mathcal{D}$ is to minimize the Roman domination number $\gamma_R(G')$ of G' while $\mathcal{A}$ tries to maximize it. If both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies, $\gamma_R(G')$ is well defined. We call this number the *Roman game domination subdivision number* of G and denote it by $\gamma_{Rgs}(G)$. In this paper we initiate the study of the Roman game domination subdivision number of a graph and present sharp bounds on the Roman game domination subdivision number of a tree.

1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). The number of vertices of a graph G is its *order* $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. For a vertex v in a rooted tree T, let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$.

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The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. For terminology and notation on graph theory not defined here, the reader is referred to [7].

A vertex $v \in V$ is said to *dominate* all the vertices in its closed neighborhood $N[v]$. A subset D of V is a *dominating set* of G if D dominates every vertex of $V \setminus D$ at least once. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets of G. Similarly, a subset D of V is a 2-*dominating set* of G if D dominates every vertex of $V \setminus D$ at least twice. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality among all 2-dominating sets of G.

The game domination subdivision number of graph G, introduced by Favaron et al. in [5], is defined by the following game. Two players $\mathcal{A}$ and $\mathcal{D}$ alternately play on a given graph G, $\mathcal{D}$ playing first, by marking or subdividing an edge of G. An edge which is neither marked nor subdivided is said to be *free*. At the beginning of the game, all the edges of G are free. At each turn, $\mathcal{D}$ marks a free edge of G and $\mathcal{A}$ subdivides a free edge of G by a new vertex. The game ends when all the edges of G are marked or subdivided and results in a new graph G'. The purpose of $\mathcal{D}$ is to minimize the domination number $\gamma(G')$ of G' while $\mathcal{A}$ tries to maximize it. If both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies, $\gamma(G')$ is well defined. We call this number the *game domination subdivision number* of G and denote it by $\gamma_{gs}(G)$.

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [10, 11] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF is the value $w(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, the weight of f is $w(f) = |V_1| + 2|V_2|$. The Roman domination number has been studied by several authors (see for example [2, 3, 6, 8, 9]).

We propose here a similar game based on the Roman domination number. Two players $\mathcal{A}$ and $\mathcal{D}$ alternately play on a given graph G, $\mathcal{D}$ playing first, by marking or subdividing an edge of G. At the beginning of the game, all the edges of G are free. At each turn, $\mathcal{D}$ marks a free edge of G and $\mathcal{A}$ subdivides a free edge of G by a new vertex. The game ends when all the edges of G are marked or subdivided and results in a new graph G'. The purpose of $\mathcal{D}$ is to minimize the Roman domination number $\gamma_R(G')$ of G' while $\mathcal{A}$ tries to maximize it. If both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies, $\gamma_R(G')$ is well defined. We call this number the *Roman game domination subdivision number* of G and denote it by $\gamma_{Rgs}(G)$. As the Roman domination number of any graph obtained by subdividing some of its edges is at least as large as the Roman domination number of the graph itself, we clearly have $\gamma_R(G) \le \gamma_{Rgs}(G)$.

Our purpose in this paper is to initiate the study of the Roman game domination subdivision number of a graph. We first determine $\gamma_{Rgs}(G)$ for several classes of graphs, and then we establish some bounds on it when G is a tree.

2. Preliminary Results

Observation 2.1. Consider the variant of the game defined by the same rule with the exception that in one turn of the game, $\mathcal{D}$ is allowed to mark two free edges instead of one. For this variant, the Roman game domination subdivision number γ'_{Rgs} satisfies $\gamma'_{Rgs}(G) \le \gamma_{Rgs}(G)$.

Proposition 2.2. *Let G be a connected graph, D a dominating set of G such that $V \setminus D$ is independent, A the set of the vertices of $V \setminus D$ having at least two neighbors in D, and $B = V \setminus (D \cup A)$. Then*

$$\gamma_{Rgs}(G) \le 2|D| + \left\lfloor \frac{|B|}{2} \right\rfloor.$$

Proof. First Player $\mathcal{D}$ marks an edge between D and B when $B \neq \emptyset$, otherwise any free edge, and continues as follows. When $\mathcal{A}$ subdivides an edge between D and B then $\mathcal{D}$ marks a free edge between D and B if any, otherwise any free edge. When $\mathcal{A}$ subdivides an edge wu between D and A with $w \in D$ and $u \in A$ then $\mathcal{D}$ marks a free edge $w'u$ with $w' \in D$ if any, otherwise any free edge. In the end of the game, the vertices of A and those of a subset B' of B are extremity of a marked edge and the function $f : V(G') \to \{0, 1, 2\}$ defined by $f(v) = 2$ for $v \in D$, $f(v) = 1$ for $v \in B \setminus B'$ and $f(v) = 0$ otherwise, is a Roman dominating function of the resulting graph G'. Since $\mathcal{D}$ began, $|B'| \ge \lceil \frac{|B|}{2} \rceil$. Hence $\gamma_{Rgs}(G) = \gamma_R(G') \le 2|D| + \left\lfloor \frac{|B|}{2} \right\rfloor$. □

The next result is an immediate consequence of Proposition 2.2.

Proposition 2.3. If X is an independent set of G such that $V \setminus X$ is a 2-dominating set, then $\gamma_{Rgs}(G) \le 2(n - |X|)$. In particular, if $\delta(G) \ge 2$ then $\gamma_{Rgs}(G) \le 2(n - \alpha(G))$.

Proposition 2.4. ([12]) If G is a bipartite graph, then $\alpha(G) = n(G)/2$ if and only if $n(G)$ is even and G has a perfect matching.

Corollary 2.5. If G is a bipartite graph with $\delta(G) \ge 2$, then $\gamma_{Rgs}(G) \le n(G)$, with equality only if $n(G)$ is even and G has a perfect matching.

Proof. If G is a bipartite graph, then $\alpha(G) \ge n(G)/2$. Using Proposition 2.3, we arrive at $\gamma_{Rgs}(G) \le 2(n(G) - \alpha(G)) \le n(G)$. If $n(G)$ is odd or G has no perfect matching, then Proposition 2.4 implies that $\alpha(G) < n(G)/2$ and it follows from Proposition 2.3 that $\gamma_{Rgs}(G) \le n(G) - 1$. □

Proposition 2.6. For $n \ge 2$, $\gamma_{Rgs}(K_{1,n-1}) = \lceil \frac{n+2}{2} \rceil$.

Proof. $\mathcal{A}$ subdivides $\lfloor \frac{n-1}{2} \rfloor$ edges and thus $\gamma_{Rgs}(K_{1,n-1}) = \lfloor \frac{n-1}{2} \rfloor + 2 = \lceil \frac{n+2}{2} \rceil$. □

The next proposition can be found in [3].

Proposition 2.7. For the classes of paths P_n and cycles C_n,

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proposition 2.8. For $n \ge 2$ for a path and $n \ge 3$ for a cycle, $\gamma_{Rgs}(P_n) = \gamma_{Rgs}(C_n) = n$.

Proof. In the game on a path or a cycle, all the strategies of $\mathcal{D}$ and $\mathcal{A}$ are equivalent since subdividing any edge of a path or cycle results a new path or cycle with one more vertex. If $G = P_n$, then $\mathcal{A}$ subdivides $\lfloor \frac{n-1}{2} \rfloor$ edges and $G' = P_{n'}$ with $n' = n + \lfloor \frac{n-1}{2} \rfloor$. If $G = C_n$, then $\mathcal{A}$ subdivides $\lfloor \frac{n}{2} \rfloor$ edges and $G' = C_{n'}$ with $n' = n + \lfloor \frac{n}{2} \rfloor$. Applying Proposition 2.7, we have $\gamma_R(P_{n'}) = \gamma_R(C_{n'}) = \lceil \frac{2n'}{3} \rceil$ and therefore

$$\gamma_{Rgs}(P_n) = \gamma_{Rgs}(C_n) = \left\lceil \frac{2(n + \lfloor \frac{n-1}{2} \rfloor)}{3} \right\rceil = \left\lceil \frac{2(n + \lfloor \frac{n}{2} \rfloor)}{3} \right\rceil = n.$$

□

If G is an even cycle C_n, then Proposition 2.8 shows that equality in Corollary 2.5 is possible. The following lower bound for the Roman domination number of any graph is proved in [4].

Proposition 2.9. For any graph G of order n and maximum degree $\Delta \geq 1$,

$$\gamma_R(G) \geq \frac{2n}{\Delta + 1}.$$

Proposition 2.10. Let G be an r-regular graph of order n with $r \geq 2$. Then

$$\gamma_{Rgs}(G) \geq \left\lceil \frac{2(n + \lfloor (rn)/4 \rfloor)}{r+1} \right\rceil.$$

Proof. The graph G has $(rn)/2$ edges. Therefore player $\mathcal{A}$ subdivides $\lfloor (rn)/4 \rfloor$ edges. It follows that the resulting graph G' has maximum degree r and $n + \lfloor (rn)/4 \rfloor$ vertices. Using Proposition 2.9, we deduce that

$$\gamma_{Rgs}(G) = \gamma_R(G') \geq \left\lceil \frac{2(n + \lfloor (rn)/4 \rfloor)}{r+1} \right\rceil.$$

□

If C_n is a cycle of order n, then Proposition 2.8 shows that

$$\gamma_{Rgs}(C_n) = n = \left\lceil \frac{2(n + \lfloor (2n)/4 \rfloor)}{3} \right\rceil.$$

Therefore Proposition 2.10 is sharp, at least for $r = 2$.

The Dutch-windmill graph, $K_3^{(m)}$, is a graph which consists of m copies of K_3 with a vertex in common.

Proposition 2.11. For every positive integer m, $\gamma_{Rgs}(K_3^{(m)}) = 2 + \lceil \frac{m}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor$.

Proof. By Proposition 2.8, we may assume that $m \geq 2$. Let v, u_i, w_i are the vertices of the i-th copy of K_3 in $K_3^{(m)}$ (v is the common vertex). In the graph $K_3^{(m)'}$ obtained at the end of the game, let p and q be the numbers of cycles whose one edge respectively, two edges are subdivided. Then clearly $\gamma_R(K_3^{(m)'}) = 2 + p + 2q$.

The strategy of $\mathcal{A}$ is as follows. When some edge remains free after $\mathcal{D}$ has played, $\mathcal{A}$ subdivides a free edge in a cycle whose one edge is marked and one edge is subdivided if possible, otherwise a free edge in a cycle with two marked edges if possible, otherwise a free edge of cycle that all its edges are free if possible, otherwise a free edge in the cycle still having free edges. On this way, the number of cycles with at least two subdivided edges is $\lfloor \frac{m}{2} \rfloor$ and the number of cycles with one subdivided edge is $\lceil \frac{m}{2} \rceil$. Hence $\gamma_R(K_3^{(m)'}) = 2 + \lceil \frac{m}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor$. $\square$

For two positive integers p and q, we call a *double star* $DS_{p,q}$ the graph obtained from two stars $K_{1,p}$ of center u and $K_{1,q}$ of center v by adding the edge uv.

Proposition 2.12. *For the double star $DS_{1,q}$ of order $n = q + 3$,*

$$\gamma_{Rgs}(DS_{1,q}) = 2 + \left\lceil \frac{n}{2} \right\rceil.$$

Proof. If $q = 1$, the result follows from Proposition 2.8. Let $q \geq 2$. Then, Player $\mathcal{D}$ cannot prevent $\mathcal{A}$ to subdivide some edge of the star $K_{1,q}$. If $q = 2$, then clearly $\gamma_{Rgs}(DS_{1,2}) = 5 = 2 + \lceil \frac{n}{2} \rceil$. Assume henceforth $q \geq 3$. Player $\mathcal{A}$ subdivides $\lfloor \frac{q+2}{2} \rfloor$ edges that among them q' are edges of the star $K_{1,q}$ with $0 < q' \leq \lfloor \frac{q+2}{2} \rfloor < q$. Thus the resulting graph $DS'_{1,q}$ has Roman domination number $q' + 3$ if $q' = \lfloor \frac{q+2}{2} \rfloor$ and $q' + 4$ when $q' \leq \lfloor \frac{q}{2} \rfloor$. Hence $\mathcal{D}$ tries to mark and $\mathcal{A}$ to subdivide the largest possible number of edges of the star $K_{1,q}$. At the end of the game, as $\mathcal{D}$ began, $\lfloor \frac{q}{2} \rfloor$ edges of the star are subdivided and $\gamma_R(DS'_{1,q}) = \lfloor \frac{q}{2} \rfloor + 4 = \lfloor \frac{n+1}{2} \rfloor + 2 = \lceil \frac{n}{2} \rceil + 2$. $\square$

Proposition 2.13. *For the double star $DS_{p,q}$ of order $n = p + q + 2$ with $2 \leq p \leq q$,*

$$\gamma_{Rgs}(DS_{p,q}) = \begin{cases} \frac{n+1}{2} + 2 & \text{if } n \text{ is odd} \\ \frac{n}{2} + 3 & \text{if } n \text{ is even} \end{cases} = \left\lceil \frac{n+1}{2} \right\rceil + 2.$$

Proof. In the graph $DS'_{p,q}$ obtained at the end of the game, let p' and q' be the numbers of edges which have been subdivided in the stars $K_{1,p}$ and $K_{1,q}$ respectively. Moreover, let $\eta = 1$ if uv is subdivided, $\eta = 0$ otherwise. Clearly $p' + q' + \eta = \lfloor \frac{n-1}{2} \rfloor$ and $p' + q' \leq \lfloor \frac{n-1}{2} \rfloor < n - 2 = p + q$. Then

$$\gamma_R(DS'_{p,q}) = p' + q' + 4 = \left\lfloor \frac{n-1}{2} \right\rfloor - \eta + 4.$$

The strategy of $\mathcal{A}$ is as follows. When some edge remains free after $\mathcal{D}$ has played, $\mathcal{A}$ subdivides a free edge in a star already containing marked edges if possible, otherwise a free edge of the star still having the maximum number of free edges if possible, otherwise the edge uv. On this way, $\mathcal{A}$ never simultaneously subdivides uv and all the edges of a star. Hence $\gamma_R(DS'_{p,q}) \geq \lfloor \frac{n-1}{2} \rfloor + 4$. Moreover if n is even, then $\mathcal{A}$ does not subdivide uv, $p' < p$, $q' < q$, $p' + q' = \frac{p+q}{2}$, and $\gamma_R(DS'_{p,q}) = p' + q' + 4 = \frac{p+q}{2} + 4 = \frac{n-2}{2} + 4 = \frac{n}{2} + 3$. If n is odd, the total number of edges is even and if $\mathcal{D}$ never marks uv, $\mathcal{A}$ is obliged to subdivide it. Hence $\eta = 1$ and $\gamma_R(DS'_{p,q}) = \lfloor \frac{n-1}{2} \rfloor + 3 = \frac{n+1}{2} + 2$. $\square$

Theorem 2.14. If p and q are two integers with $2 \leq p \leq q$, then

$$p + 2 \leq \left\lceil \frac{2(p + q + \lfloor (pq)/2 \rfloor)}{q+1} \right\rceil \leq \gamma_{Rgs}(K_{p,q}) \leq 2p.$$

In particular, $\gamma_{Rgs}(K_{p,q}) = 2p$ for $2 \leq p \leq 4$,

Proof. Applying Proposition 2.3, we obtain $\gamma_{Rgs}(K_{p,q}) \leq 2(p + q - q) = 2p$.

The graph $G = K_{p,q}$ has pq edges. Therefore player $\mathcal{A}$ subdivides $\lfloor (pq)/2 \rfloor$ edges. It follows that the resulting graph G' has maximum degree q and $p + q + \lfloor (pq)/2 \rfloor$ vertices. Using Proposition 2.9, we deduce that

$$\gamma_{Rgs}(G) = \gamma_R(G') \geq \left\lceil \frac{2(p + q + \lfloor (pq)/2 \rfloor)}{q+1} \right\rceil.$$

Now it is straightforward to verify that

$$\left\lceil \frac{2(p + q + \lfloor (pq)/2 \rfloor)}{q+1} \right\rceil \geq p + 2,$$

and thus the desired inequality chain is proved. It follows that $\gamma_{Rgs}(K_{2,q}) = 4 = 2p$ for $p = 2$.

For $p = 3$, we deduce that $5 \leq \gamma_{Rgs}(K_{3,q}) \leq 6$. Suppose that there exists a $\gamma_R(G')$-function $f = (V_0, V_1, V_2)$ with weight $\gamma_R(G') = 5$. Since $\gamma_R(G') = 5 = |V_1| + 2|V_2|$, we see that $|V_1| \geq 1$ and $|V_2| \leq 2$. Thus $V_1 \cup V_2$ dominates at most $1 + 2(q+1) = 2q + 3$ vertices of G'. However, G' contains $q + 3 + \lfloor (3q)/2 \rfloor$ vertices. This leads to the contradiction $q + 3 + \lfloor (3q)/2 \rfloor > 2q + 3$, and so $\gamma_{Rgs}(K_{3,q}) = 6$.

For $p = 4$, we deduce that

$$7 = \left\lceil \frac{2(4 + q + 2q)}{q+1} \right\rceil \leq \gamma_R(G') = \gamma_{Rgs}(K_{4,q}) \leq 2p = 8.$$

Suppose that there exists a $\gamma_R(G')$-function $f = (V_0, V_1, V_2)$ with weight $\gamma_R(G') = 7$. Since $\gamma_R(G') = 7 = |V_1| + 2|V_2|$, we see that $|V_1| \geq 1$ and $|V_2| \leq 3$. Thus $V_1 \cup V_2$ dominates at most $1 + 3(q+1) = 3q + 4$ vertices of G'. Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be the partite sets of $K_{4,q}$, and let $V_2 = \{u_1, u_2, u_3\}$ and $V_1 = \{v\}$. Since G' contains exactly $3q + 4$ vertices, we observe that $|N_{G'}[u_i]| = q + 1$ and $N_{G'}[u_i] \cap N_{G'}[u_j] = \emptyset$ for $1 \leq i \neq j \leq 3$. Therefore $\{u_1, u_2, u_3\} \subset X$ and $v \in X$. We note that it is no problem for player $\mathcal{A}$ to subdivide an edge adjacent to u_1, u_2, u_3 and v. If player $\mathcal{A}$ has subdivided the edge vw by vz and zw, then the vertex z is not dominated by $V_1 \cup V_2$. This contradiction shows that $\gamma_R(G') = 7$ is not possible and so $\gamma_R(G') = \gamma_{Rgs}(K_{4,q}) = 2p = 8$. □

Conjecture 1. *If p and q are two integers with $2 \leq p \leq q$, then $\gamma_{Rgs}(K_{p,q}) = 2p$.*
Theorem 2.14 shows that Conjecture 1 is valid for $2 \leq p \leq 4$.

3. Trees

In this section we present lower and upper bounds on the Roman game domination subdivision number of a tree.

3.1. An upper bound in terms of 2-domination number.

First we present an upper bound on the Roman game domination subdivision number of trees in terms of 2-domination number and then we characterize all extremal graphs. A vertex of degree one is a *leaf* and a *support vertex* is a vertex that is adjacent to at least one leaf.

Theorem 3.1. For any tree T of order $n \geq 2$,

$$\gamma_{Rgs}(T) \leq 2\gamma_2(T) - 1.$$

Proof. The proof is by induction on n. Obviously, the statement is true for $n \le 3$. Assume the statement is true for all trees of order less than n, where $n \ge 4$. Let T be a tree of order n. If T is a star $K_{1,n-1}$, then $\gamma_2(T) = n-1$ and, by Proposition 2.6, $\gamma_{Rgs}(T) = \lceil \frac{n+2}{2} \rceil$. This implies that $\gamma_{Rgs}(T) < 2\gamma_2(T) - 1$. If T is a double star $DS_{p,q}$ then by Propositions 2.12 and 2.13, $\gamma_{Rgs}(T) < 2\gamma_2(T) - 1$. Hence, we assume that T is not a star or double star. Then $\mathrm{diam}(T) \ge 4$. Assume that $P = v_1v_2\dots v_k$, $k \ge 5$, is a longest path of T. We denote $d_T(v_{k-1}) = t$. Let D be a minimum 2-dominating set of T not containing v_{k-1}. It is easy to see that such a minimum 2-dominating set exists. We consider two cases. In each of them, we define a subtree T_1 of order at least two of T and a strategy for $\mathcal{D}$. We denote by T' and T_1' the trees obtained from T and T_1 at the end of the game.

Case 1. Assume that $t \ge 3$.
Let $u_1, u_2, \dots, u_{t-2}, v_k$ be the leaves adjacent to v_{k-1}. Then $\{u_1, u_2, \dots, u_{t-2}, v_k\} \subseteq D$. Clearly, the set $D \setminus \{u_1, u_2, \dots, u_{t-2}, v_k\}$ is a 2-dominating set of the tree $T_1 = T - T_{v_{k-1}}$ and thus

$$\gamma_2(T_1) \le \gamma_2(T) - (t-1).$$

Player $\mathcal{D}$ plays the game according to an optimal strategy on T_1 as long as $\mathcal{A}$ subdivides an edge of T_1. If $\mathcal{A}$ subdivides an edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k, v_{k-1}u_1, \dots, v_{k-1}u_{t-2}\}$, then $\mathcal{D}$ marks a free edge in $\{v_{k-1}v_k, v_{k-1}u_1, \dots, v_{k-1}u_{t-2}\}$ provided that is any. Otherwise, $\mathcal{D}$ marks an arbitrary free edge in T_1. It follows from Observation 2.1 that $\gamma_R(T_1') \le \gamma_{Rgs}(T_1)$. It is easy to see that $\gamma_R(T') \le \gamma_R(T_1') + 2 + \lfloor \frac{t}{2} \rfloor$. Hence

$$\gamma_{Rgs}(T) \le \gamma_R(T') \le \gamma_R(T_1') + 2 + \lfloor \frac{t}{2} \rfloor \le \gamma_{Rgs}(T_1) + 2 + \lfloor \frac{t}{2} \rfloor.$$

It follows from the induction hypothesis and the fact $t \ge 3$ that

$$\gamma_{Rgs}(T) \le \gamma_{Rgs}(T_1) + 2 + \lfloor \frac{t}{2} \rfloor \le (2\gamma_2(T_1) - 1) + 2 + \lfloor \frac{t}{2} \rfloor \le 2\gamma_2(T) - 2t + 3 + \lfloor \frac{t}{2} \rfloor \le 2\gamma_2(T) - 2. \tag{3.1}$$

Case 2. Assume that $t = 2$.
Since $v_{k-1} \notin D$, $\{v_k, v_{k-2}\} \subseteq D$ and $D \setminus \{v_k\}$ is a 2-dominating set of the tree $T_1 = T - \{v_k, v_{k-1}\}$. Thus

$$\gamma_2(T_1) \le \gamma_2(T) - 1.$$

Player $\mathcal{D}$ plays the game according to an optimal strategy on T_1 as long as $\mathcal{A}$ subdivides an edge of T_1 and when $\mathcal{A}$ subdivides one edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{D}$ marks the second edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$. We may assume, without loss of generality, that $\mathcal{A}$ subdivides the edge $v_{k-1}v_k$ by a new vertex z. We can extend each $\gamma_R(T_1')$-function, f, to a Roman dominating function of T' by assigning 2 to z. Hence

$$\gamma_{Rgs}(T) \le \gamma_{Rgs}(T_1) + 2.$$

By the induction hypothesis we have

$$\gamma_{Rgs}(T) \le \gamma_{Rgs}(T_1) + 2 \le 2\gamma_2(T_1) - 1 + 2 \le 2\gamma_2(T) - 1. \tag{3.2}$$

This completes the proof. □

In what follows, we provide a constructive characterization of all trees T for which $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$. To do this, we describe a procedure to build a family $\mathcal{F}$ of labeled trees that attains the bound in Theorem 3.1. First we define the following operation on labeled trees. The label of a vertex is also called its *status* and denoted $sta(v)$. Let $\mathcal{F}$ be the family of labeled trees that:

(1) contains P_3 where the two leaves have status A, and the central vertex has status B, and
(2) is closed under the operation $\mathfrak{T}$, which extend the tree T by attaching a tree to the vertex $y \in V(T)$, called the *attacher*.

Operation $\mathfrak{T}$. If $sta(y) = A$, then $\mathfrak{T}$ adds a path yxw to T with $sta(x) = B$ and $sta(w) = A$.

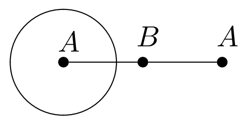
$\mathfrak{T}$:


FIGURE 1. The operation

If $T \in \mathcal{F}$, we let $A(T)$ and $B(T)$ be the set of vertices of status A and B, respectively, in T.

Observation 3.2. Let $T \in \mathcal{F}$ and $v \in V(T)$.

(1) If v is a leaf, then $sta(v) = A$ and if v is a support vertex, then $sta(v) = B$.
(2) If $sta(v) = B$, then $\deg(v) = 2$ and the neighbors of v have status A.
(3) If $sta(v) = A$, then all neighbors of v have status B.
(4) $A(T)$ is an independent set.
(5) $B(T)$ is an independent set.

Proposition 3.3. If $T \in \mathcal{F}$, then $\gamma_2(T) = |A(T)|$.

Proof. We show that A is the unique minimum 2-dominating set of T. By Observation 3.2, the set A is a 2-dominating set of T, but not B which contains no leaf. Let $A_1 \cup B_1$ with $A_1 \subset A$ and $B_1 \subseteq B$ be another 2-dominating set. Every vertex of $A \setminus A_1$ is 2-dominated by B_1. Since T is a tree, the number $e(B_1, A \setminus A_1)$ of edges between B_1 and $A \setminus A_1$ satisfies

$$2(|A| - |A_1|) \le e(B_1, A \setminus A_1) \le |B_1| + |A| - |A_1| - 1.$$

Hence $|B_1| + |A_1| \ge |A| + 1$, which shows that A is the unique minimum 2-dominating set of T and $\gamma_2(T) = |A|$. □

Proposition A. (Atapour et al. [1]) Let T' be a tree of order at least 3 and $y \in V(T')$. Let T be a tree obtained from T' by attaching a path uvw to T' with an edge yu. Then $\gamma_R(T) = \gamma_R(T') + 2$.

Theorem 3.4. If $T \in \mathcal{F}$, then $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$.

Proof. Let $T \in \mathcal{F}$ be obtained from a path P_3 by successive operations $\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i = \mathfrak{T}$ if $m \ge 1$ and $T = P_3$ if $m = 0$. The proof is by induction on m. If $m = 0$, then clearly the statement is true. Let $m \ge 1$ and assume that the statement holds for all trees which are obtained from P_3

by applying at most $m-1$ operations. Let T_{m-1} be the tree obtained from P_3 by the first $m-1$ operations $\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^{m-1}$. Then $\mathfrak{T}^m$ adds a path xx_1x_2 in which $x \in A(T_{m-1})$.

The strategy of $\mathcal{A}$ is that he plays the game according to an optimal strategy on $T_{m-1} = T_m - \{x_1, x_2\}$, as long as $\mathcal{D}$ marks edges of T_{m-1}, and when $\mathcal{D}$ marks an edge in $F = \{xx_1, x_1x_2\}$ then $\mathcal{A}$ subdivides the other edge in F with a new vertex z. Suppose T'_m is the tree obtained at the end of the game. Then $T'_m - \{x_1, x_2, z\}$ is the tree T'_{m-1} obtained from T_{m-1} at the end of the game and $\gamma_{Rgs}(T_{m-1}) = \gamma_R(T'_{m-1})$. It follows from Proposition A that

$$\gamma_{Rgs}(T_m) = \gamma_R(T'_m) = \gamma_R(T'_{m-1}) + 2 = \gamma_{Rgs}(T_{m-1}) + 2.$$

By the inductive hypothesis and Proposition 3.3, we have

$$\begin{aligned}\gamma_{Rgs}(T_m) &= \gamma_{Rgs}(T_{m-1}) + 2 = 2\gamma_2(T_{m-1}) + 1 = 2|A(T_{m-1})| + 1 \\ &= 2(|A(T_{m-1})| + 1) - 1 = 2|A(T_m)| - 1 = 2\gamma_2(T_m) - 1.\end{aligned}$$

This completes the proof. □

Theorem 3.5. Let T be a tree of order $n \geq 2$. Then $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$ if and only if $T \in \mathcal{F}$.

Proof. By Theorem 3.4, we only need to prove that every tree T with $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$ is in $\mathcal{F}$. We prove this by induction on n. Since $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$, we have $n \geq 3$. If $n = 3$, then the only tree T of order 3 and $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$ is $P_3 \in \mathcal{F}$. Let $n \geq 4$ and assume that the statement holds for every tree of order less than n with $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$. Let T be a tree of order n and $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$. Assume that $P = v_1v_2\ldots v_r$ is a longest path in T. Obviously, $\deg(v_1) = \deg(v_r) = 1$ and $\deg(v_2) = \deg(v_{r-1}) = 2$ by the proof of Theorem 3.1. Since $\gamma_2(P_4) = 3$ and $\gamma_{Rgs}(P_4) = 4$, we have $n \geq 5$. Suppose that T is rooted at v_1.

Let $T_1 = T - \{v_r, v_{r-1}\}$. Since $\gamma_{Rgs}(T) = 2\gamma_2(T) - 1$, the inequalities occurring in (3) become equalities. Hence $\gamma_{Rgs}(T_1) = 2\gamma_2(T_1) - 1$. By the inductive hypothesis, $T_1 \in \mathcal{F}$. We claim that $sta_{T_1}(v_{r-2}) = A$. In that case, T can be obtained from T_1 by the operation $\mathfrak{T}$. Suppose to the contrary that $sta_{T_1}(v_{r-2}) = B$. Suppose that D is a minimum 2-dominating set of T not containing v_{r-1}. Then $v_{r-2} \in D$ and $D - \{v_r\}$ is a 2-dominating set of T_1 containing a vertex with status B. It follows from the proof of Proposition 3.3 that $|D| - 1 \geq \gamma_2(T_1) + 1$ and hence $\gamma_2(T) = \gamma_2(T_1) + 2$. Then

$$\gamma_{Rgs}(T) \leq \gamma_{Rgs}(T_1) + 2 = 2\gamma_2(T_1) - 1 + 2 = 2(\gamma_2(T) - 2) + 1 \leq 2\gamma_2(T) - 3,$$

which is a contradiction. This completes the proof. □

3.2. **Bounds in terms of order.** A support vertex is said to be an *end-support vertex* if all its neighbors except one of them are leaves.

Theorem 3.6. For any tree T of order $n \geq 2$,

$$\gamma_{Rgs}(T) \geq \left\lceil \frac{n+2}{2} \right\rceil,$$

with equality if and only if T is a star.

Proof. The proof is by induction on n. Clearly, the statement is true for $n = 2, 3$. Assume that the statement is true for all trees of order less than n, where $n \geq 4$. Let T be a tree of order n. If T is a star, then the result is immediate by Proposition 2.6. If T is a double star, then it follows from Propositions 2.12 and 2.13 that $\gamma_{gs}(T) > \lceil \frac{n+2}{2} \rceil$. Suppose that $\mathrm{diam}(T) \geq 4$. Let $P = v_1v_2 \ldots v_k$ be a diametral path in T and let $d = d_T(v_{k-1})$ and $t = d_T(v_2)$. Assume that $u_1, u_2, \ldots, u_{d-2}, v_k$ are the leaves adjacent to v_{k-1} if $d \geq 3$ and $u'_1, u'_1, \ldots, u'_{t-2}, v_1$ are the leaves adjacent to v_2 when $t \geq 3$.

First let d be even. The strategy of $\mathcal{A}$ is that he plays the game according to an optimal strategy on $T_1 = T - T_{v_{k-1}}$, where T is rooted in v_1, as long as $\mathcal{D}$ marks edges of T_1, and when $\mathcal{D}$ marks an edge in $F = \{v_{k-2}v_{k-1}, u_1v_{k-1}, \ldots, u_{d-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides a free edge in F. Suppose that T' is the tree obtained at the end of the game. Clearly $\mathcal{A}$ subdivides exactly $\frac{d}{2}$ edges in F. Let $\{z_1, z_2, \ldots, z_{\frac{d}{2}}\}$ be the subdivision vertices used to subdivide the edges in F. Then $T' - T'_{v_{k-1}}$ is the tree T'_1 obtained from T_1 at the end of the game and $\gamma_{Rgs}(T_1) = \gamma_R(T'_1)$. It is easy to see that $\gamma_R(T') \geq \gamma_R(T'_1) + \frac{d}{2} + 1$. This implies that $\gamma_{Rgs}(T) \geq \gamma_{Rgs}(T - \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}\}) + \frac{d}{2} + 1$. By the induction hypothesis we have

$$\begin{aligned}\gamma_{Rgs}(T) &\geq \gamma_{Rgs}(T - T_{v_{k-1}}) + \tfrac{d}{2} + 1 \\ &\geq \lceil \tfrac{n-d+2}{2} \rceil + \tfrac{d}{2} + 1 \\ &\geq \lceil \tfrac{n+4}{2} \rceil > \lceil \tfrac{n+2}{2} \rceil.\end{aligned}$$

Similarly, if t is even, then $\gamma_{Rgs}(T) > \lceil \frac{n+2}{2} \rceil$. Thus we assume that t and d are odd.

If $\mathrm{diam}(T) = 4$ and $\deg(v_3) = 2$, then it is easy to see that $\gamma_{Rgs}(T) > \lceil \frac{n+2}{2} \rceil$. Thus we assume that $\deg(v_3) \geq 3$ or $\mathrm{diam}(T) > 4$. Player $\mathcal{A}$ plays according to an optimal strategy on $T_1 = T - \{u_1, \ldots, u_{d-2}, v_{k-1}, v_k, u'_1, \ldots, u'_{t-2}, v_1, v_2\}$ as long as $\mathcal{D}$ marks edges of T_1, and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-1}, v_2v_3\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-1}, v_2v_3\}$ with vertex z, when $\mathcal{D}$ marks an edge in $F_1 = \{v_kv_{k-1}, u_1v_{k-1}, \ldots, u_{d-2}v_{k-1}\}$ then $\mathcal{A}$ subdivides a free edge in $\{v_kv_{k-1}, u_1v_{k-1}, \ldots, u_{d-2}v_{k-1}\}$ and when $\mathcal{D}$ marks an edge in $F_2 = \{v_1v_2, u'_1v_2, \ldots, u'_{t-2}v_2\}$ then $\mathcal{A}$ subdivides a free edge in $\{v_1v_2, u'_1v_2, \ldots, u'_{t-2}v_2\}$. Suppose that T' is the tree obtained at the end of the game. Clearly $\mathcal{A}$ subdivides one edge in $\{v_{k-2}v_{k-1}, v_2v_3\}$, exactly $\frac{d-1}{2}$ edges in F_1 and exactly $\frac{t-1}{2}$ edges in F_2. Let $\{z_1, z_2, \ldots, z_{\frac{d-1}{2}}\}$ be the subdivision vertices used to subdivide the edges in F_1 and $\{z'_1, z'_2, \ldots, z'_{\frac{t-1}{2}}\}$ be the subdivision vertices used to subdivide the edges in F_2. Then $T' - \{z, u_1, \ldots, u_{d-2}, v_{k-1}, v_k, z_1, \ldots, z_{\frac{d-1}{2}}, v_1, v_2, z'_1, \ldots, z'_{\frac{t-1}{2}}\}$ is the tree T'_1 obtained from T_1 at the end of the game and $\gamma_{Rgs}(T_1) = \gamma_R(T'_1)$. It is easy to see that $\gamma_R(T') \geq \gamma_R(T'_1) + 2 + \frac{d-1}{2} + \frac{t-1}{2}$. It follows from the induction hypothesis that $\gamma_{Rgs}(T) \geq \lceil \frac{n-d-t+2}{2} \rceil + 2 + \frac{d-1}{2} + \frac{t-1}{2} > \lceil \frac{n+2}{2} \rceil$. This completes the proof. $\square$

Theorem 3.7. For any tree T of order $n \geq 2$,

$$\gamma_{Rgs}(T) \leq n.$$

Moreover, $\gamma_{Rgs}(T) = n$ only when $n = 2, 3$, $T \in \{DS_{1,1}, DS_{1,2}, DS_{2,2}\}$ or $\mathrm{diam}(T) \geq 4$ and all the end-support vertices of T have degree at most 4 and at most one of them has degree 4.

Proof. The proof is by induction on n. If $n = 2, 3$, then clearly $\gamma_{Rgs}(T) = n$. Let $n \geq 4$. Assume that the result is true for any non-trivial tree of order less than n, and let T be a tree of order n. If T is a star, then $\gamma_{Rgs}(T) < n$ by Proposition 2.6 and $n \geq 4$. Assume that T is not a star. If T is a double star, then it follows from Propositions 2.12 and 2.13 that $\gamma_{Rgs}(T) \leq n$ with equality if and only if $T \in \{DS_{1,1}, DS_{1,2}, DS_{2,2}\}$.

Now let $\mathrm{diam}(T) \geq 4$, and let x be an end-support vertex of degree $d_T(x) = t$ of T, $y^1, y^2, \ldots, y^{t-1}$ the leaves attached at x, and z the neighbor of x of degree at least 2. The tree $T_1 = T - \{x, y^1, y^2, \ldots, y^{t-1}\}$ has order at least two. In the following three cases, we define a strategy for $\mathcal{D}$ and denote by T' and T_1' the trees obtained from T and T_1 at the end of the game.

Case 1. The tree T has an end-support vertex of degree at least 5.
Player $\mathcal{D}$ plays its best strategy on T_1 as long as $\mathcal{A}$ subdivides edges of T_1. When $\mathcal{A}$ subdivides an edge of $\{xz, xy^1, \ldots, xy^{t-1}\}$, then $\mathcal{D}$ marks a free edge in $\{xy^1, \ldots, xy^{t-1}\}$. It is easy to see that $\gamma_R(T') \leq \gamma_R(T_1') + \lfloor \frac{t}{2} \rfloor + 2$. Hence, by the induction hypothesis and $t \geq 5$,

$$\gamma_{Rgs}(T) = \gamma_R(T') \leq \gamma_R(T_1') + \left\lfloor \frac{t}{2} \right\rfloor + 2 = \gamma_{Rgs}(T_1) + \left\lfloor \frac{t}{2} \right\rfloor + 2 \leq n - t + \left\lfloor \frac{t}{2} \right\rfloor + 2 < n.$$

Case 2. T admits two end-support vertices x, x' of degree 4.
Assume that y'^1, y'^2, y'^3 are the leaves attached at x', and z' the neighbor of x' of degree at least 2. Suppose that $T_2 = T - \{x, y^1, y^2, y^3, x', y'^1, y'^2, y'^3\}$. If $T_2 = K_1$, then $z = z'$ and it is easy to see that $\gamma_{Rgs}(T) < n$. Let T_2 have order at least two. The strategy of $\mathcal{D}$ is that he plays its best strategy on T_2 as long as $\mathcal{A}$ subdivides edges of T_2. When $\mathcal{A}$ subdivides an edge of $\{xy^1, xy^2, xy^3, x'y'^1, x'y'^2, x'y'^3\}$, $\mathcal{D}$ marks a free edge of $\{xy^1, xy^2, xy^3, x'y'^1, x'y'^2, x'y'^3\}$ and when $\mathcal{A}$ subdivides an edge in $\{xz, x'z'\}$, $\mathcal{D}$ marks the other edge in $\{xz, x'z'\}$. Clearly $\gamma_R(T') \leq \gamma_R(T_2') + 7$. By the inductive hypothesis, we have

$$\gamma_{Rgs}(T) = \gamma_R(T') \leq \gamma_R(T_2') + 7 = \gamma_{Rgs}(T_2) + 7 \leq n - 8 + 7 < n.$$

Case 3. All the end-support vertices of T have degree at most 4 and at most one of them has degree 4.
Let x be an end-support vertex of degree t. Player $\mathcal{D}$ plays its best strategy on T_1 as long as $\mathcal{A}$ subdivides edges of T_1. When $\mathcal{A}$ subdivides an edge of $\{xy^1, xy^2, \ldots, xy^{t-1}, xz\}$, $\mathcal{D}$ marks an edge of $\{xy^1, xy^2, \ldots, xy^{t-1}\}$ if possible, otherwise the edge xz if still free, otherwise any other free edge of T_1. At the end of the game, at most $\lfloor \frac{t}{2} \rfloor$ edges of $xy^1, xy^2, \ldots, xy^{t-1}$ are subdivided. It is easy to see that $\gamma_R(T') \leq \gamma_R(T_1') + 2$ when $t = 2$ and $\gamma_R(T') \leq \gamma_R(T_1') + 2 + \lfloor \frac{t}{2} \rfloor$ when $t = 3, 4$. It follows from the induction hypothesis that $\gamma_{Rgs}(T) \leq \gamma_R(T') \leq n$ and the proof is complete. □

We conclude this paper with the following conjecture.

Conjecture 2. For any graph G of order n, $\gamma_{Rgs}(G) \leq n$.

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