Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 2 No. 4 (2013), pp. 25-36.

www.combinatorics.ir
www.ui.ac.ir

GRAPH THEORETICAL METHODS TO STUDY CONTROLLABILITY AND LEADER SELECTION FOR DEAD-TIME SYSTEMS

M. NAJAFI* AND F. SHEIKHOLESLAM

Communicated by Alireza Abdollahi

Abstract. In this article a graph theoretical approach is employed to study some specifications of dynamic systems with time delay in the inputs and states, such as structural controllability and observability. First, the zero and non-zero parameters of a proposed system have been determined, next the general structure of the system is presented by a graph which is constructed by non-zero parameters. The structural controllability and observability of the system is investigated using the corresponding graph. Our results are expressed for multi-agents systems with dead-time. As an application we find a minimum set of leaders to control a given multi-agent system.

1. Introduction

Recently, graph theory has wide applications for modeling of large scale systems, distributed systems and multi-agent systems. Graph is an interested tool to visual some specification of dynamic systems, especially for multi agent systems [1], like as complex networks [2, 3, 4, 5], swarm group of unmanned air vehicles (UAV) or autonomous underwater vehicles (AUV) [6, 7, 8, 9, 10] and physiological systems and gene network [11].

One of the important application of graph theory is to study the controllability and observability of systems [1, 12, 13, 14, 15]. For this goal, a graph is associated to a proposed system where the vertices are corresponded to states, inputs or outputs and the edges are introduced the relations between them. The associated graph may be assumed directed or undirected [1, 15] also weighted or unweighted [14, 16]. The system specification can also be studied using the topology [13], the adjacency matrix [14] or the laplacian matrix [1] of a graph. The adjacency or the laplacian matrix

MSC(2010): Primary: 93C75; Secondary: 94C15; 05C50.
Keywords: Graph methods, Dead-time systems, Multi-agent systems, structural controllability.
Received: 13 September 2013, Accepted: 30 September 2013.
*Corresponding author.

are obtained using defined parameters of a given system, while graph topology is constructed due to the location of zero and non-zero parameters of the system. The latter case generally discusses about structural controllability and observability of every system having a graph topology.

Moreover, some researchers claim that a delay must be considered between relations of distinct states in real multi-agent systems such as complex networks [2, 3, 4, 5] and swarm UAVs or AUVs [6, 9]. So the delay parameter must be assumed in the model and accordingly in the associated graph. Ji and et al. [14] discuss about controllability of multi-agent systems with a time-delay. In [14], the controllability of proposed system are studied by investigating the eigenvalues of adjacency matrix of the associated graph. However in [14] the authors do not consider any delay in the leader states, while usually there is no difference between leaders and followers to have delay in their communications. Moreover, usually it is considered that the leaders are preassumed.

One of the interested and newest research aria in multi-agent systems is to present a method for leaders selection [16, 17, 18, 19]. Finding the minimum possible leaders given a controllable multi-agent system is the main challenge in this topic. Increasing the number of leaders significantly growths up the complexity of any control algorithm; because leaders must lead the group of followers coordinately, while they are conceptually independent. In other words, a follower may control by different leaders which declare conflicting commands and cause a fault in the system. Therefore we are interested on presenting a method to find a minimum possible leaders. Some authors [16, 18] consider the followers and theirs relation are identified and it is possible to add a number of leaders with arbitrary influence. However usually in multi-agent systems as swarm group of UAVs or complex network, the leaders must be chosen among the existing states (with defined relation) and remaining states are appropriated followers [18].

In this paper, the structural controllability and observability of dead-time systems are studied by using an associated graph. The graph is made based zero and non-zero parameters of the proposed system. In other words, the structure of relation between the inputs, states and output are presented by the graph. As mentioned in [13], it is possible to study the controllability of a system by investigation the paths topology in the associated graph. We use the latter result for dead-time systems and extend it for multi-agent systems with delay in state and input. Consequently, the result of the structural controllability is employed to find minimum number of leaders such that the proposed multi agent system potentially controlled. Here we assume that the defined state must be divided to leaders and followers group.

2. **Preliminaries**

In this section, some definitions and basic theorems of control and graph theory are presented. After that, the method of associating graph to a system is described.

2.1. **Control Theory.** The goal in the control theory is to design a suitable input for a proposed system such that the output converge to the desired value. For this target, the model of the system must be defined such that the relations between inputs and outputs are determined. On the other hand, all systems have dynamic behavior and must be described by a differential equation. State space is the famous method for modeling of the system in the control engineering. Generally, the state space is obtain as following for linear time invariant (LTI) systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\dot{x}(t)$ is the derivation of $x(t)$, $u(t) \in \mathbb{R}^m$ is controllable input vector, $y(t) \in \mathbb{R}^p$ is the output vector and A, B, C and D are constant matrices. The constant matrices play a fundamental role in control concepts such as controllability, observability, stabilizeability or detectability. In this article we focus to the controllability and observability of systems. Generally, the controllability definition and its basic theorem are presented as following.

Definition 2.1. *System* (2.1) *is said to be controllable, if for any initial state* $x(0) = \varphi$ *and any terminal state* x_f*, there exist a real number* $T > 0$ *and input* $u(t)$ *(for* $t \in [0, T]$*) such that* $x(T) = x_f$*.*

Theorem 2.2. [20] *System* (2.1) *is controllable if and only if the rank of the* $(n \times n.m)$ *matrix* $M_c = [B, AB, A^2B, \ldots, A^{n-1}B]$ *is* n*.*

Sometimes, states are affected by other states and inputs with a dead-time because of the distance between state, the communication time, processing lag or actuators performance. In this case, a delay is introduced in the model as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t-d) \\ y(t) = Cx(t) + Du(t-d) \end{cases}, \tag{2.2}$$

where $d \in \mathbb{R}^+$ is the delay time. Distributed systems, multi agent systems, complex networks and so on can be modeled by dead-time systems. Controllability concept of dead-time systems has some differences compared with non-delayed systems. For example, the controllability definition and its basic theorem are modified as follows.

Definition 2.3. *System* (2.2) *is said to be controllable, if for any initial state* $x(\tau) = \varphi(\tau)$*, any initial input* $u(\tau) = \psi(\tau)$ *(for* $\tau \in [-d, 0]$*) and any terminal state* x_f*, there exist a real number* $T > d$ *and input* $u(t)$ *(for* $t \in [0, T]$*) such that* $x(T) = x_f$*.*

Theorem 2.4. [14] *System* (2.2) *is controllable if and only if the rank of* $(n \times n(n+1)m/2)$ *matrix* $Q = [Q_1^1, Q_1^2, Q_2^2, Q_1^3, \ldots, Q_n^n]$ *is* n*, where* $Q_1^1 = B$*,* $Q_j^{k+1} = AQ_j^k + A_d Q_{j-1}^k$ *for* $j = 1, 2, \ldots, k$ *and* $Q_j^k = 0$ *for* $j > k$*.*

Note that the presence or absence of a delay in the input causes no change in the Theorem 2.4 while Definition 2.3 are changed if there is no delay in the input.

One of the other important features of a system is identified by observability. Based on Duality proposition [20], the observability investigation of a system are similar to controllability investigation, if the input and output are replaced with together. The observability of dead-time systems are introduced as follows.

Definition 2.5. *System* (2.2) *is said to be observable, if for any unknown initial state* $x(0) = \varphi$, *there exists a real number* $T > 0$ *such that the knowledge of* $u(\tau)$ *(for* $\tau \in [-d, T - d]$*) and* $y(t)$ *(for* $t \in [0, T]$*) are sufficient to determine uniquely initial state* $x(0)$.

In some kinds of systems (e.g. multi agent systems), the system topology or location of agent causes relationships between some states while some other states may be independent. Also the coefficients of related states can be arbitrary. Due to states dependability, the constant matrices A, A_d, B, C and D have some zero and non-zero elements, for example:

$$A(\lambda) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 \\ \lambda_4 & 0 & 0 \end{bmatrix}, A_d(\lambda) = \begin{bmatrix} 0 & \lambda_5 & 0 \\ 0 & \lambda_6 & 0 \\ 0 & \lambda_7 & 0 \end{bmatrix}, B(\lambda) = \begin{bmatrix} \lambda_8 & 0 \\ 0 & 0 \\ 0 & \lambda_9 \end{bmatrix}, C(\lambda) = \begin{bmatrix} 0 & \lambda_{10} & \lambda_{11} \\ 0 & 0 & \lambda_{12} \end{bmatrix} \tag{2.3}$$

A zero entry in ith row and jth column of A or A_d means the ith state are not affected by jth state. Also it is possible to set the non-zero entries of the constant matrices arbitrarily. In this case, the controllability of a system depends on locations of zero and non-zero elements; we call these locations the structure of the system. So the controllability and observability must be re-defined for structure of the system.

Definition 2.6. *System* (2.1) *is structurally controllable if at least there exists a suitable set of* λ_i*s such that system* (2.1) *is controllable.*

Definition 2.7. *System* (2.1) *is structurally observable if at least there exists a suitable set of* λ_i*s such that system* (2.1) *is observable.*

2.2. **Graph terminology.** A graph G contains a number of vertices and edges which is presented by $G(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, $E = \{(v_i, vj), \ldots, (v_k, v_f)\}$ is a set of ordered pairs of elements of V, and (v_i, vj) denotes a directed edge from v_i to v_j. The graph is called directed graph or digraph where its edges are directed. A graph $H = (V', E')$ is called a subgraph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

In this article all edges are unweighted and we have no multi edges. The adjacency matrix $\mathcal{A}$ of the graph G is the matrix $\mathcal{A} = [a_{ij}]$, where $a_{ij} = 1$ if there exists an edge from v_j to v_i and $a_{ij} = 0$ otherwise. So unlike to undirected graph, $\mathcal{A}$ is not symmetric . Here self loop can be considered, thus the diagonal entries of $\mathcal{A}$ can be non-zero.

We say that a path from v_0 to v_l exists if there is a subset $\{v_0, v_1, \ldots, v_{l-1}, v_l\} \subseteq V$ such that all ordered pairs $(v_{i-1}, v_i) \in E$. It is said that the path consists of $l + 1$ vertices with l-link. Also v_0 is called the begin vertex and v_l is called the end vertex. As there exists a 1-link (an edge) from v_j to v_i if the ijth entry of $\mathcal{A}$ is non-zero; it is well known that the ijth entry of l power of $\mathcal{A}$ ($\mathcal{A}^l$) are non-zero, if there exists an l-link path from v_j to v_i. A path is called simple where every vertex on the path occurs in only once. Two paths are said disjoint if the set of their vertices are disjoint. Also a closed path is a path such that $v_0 = v_l$.

Consider U, X and Y are three nonempty subsets of V. There exists a path from U to X if there exists a path from $u_i \in U$ to $x_j \in X$; i.e. the begin vertex of a path is in U and the end vertex of it is in X. A path is called a U-rooted path if the begin vertex of path is in U. Accordingly, a U-rooted path family consists of a number of mutually disjoint U-rooted paths. Similarly, Y-topped path is a path such that the end vertex is in Y and Y-topped path family consists of a number of mutually disjoint Y-topped paths. Also, a cycle is a simple and closed path in X, of the form $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), \ldots, (x_t, x_i)$ with unique start and end vertex. A number of mutually disjoint cycle is called cycle family. An union of U-rooted path family, Y-topped Path family and cycle family is disjoint if their path mutually have no vertices in common. Moreover, X is reachable by U if each vertex in X are the end vertex of at least a U-rooted path.

2.3. **State space description by graph method.** Consider system (2.2) where $x = [x_1, \ldots, x_n]^T$, $u = [u_1, \ldots, u_m]^T$ and $y = [y_1, \ldots, y_p]^T$. The structure of a dynamic system is represented by an associated graph $G(V, E)$, where $V = U \cup X \cup Y$, $U = \{u_1, \ldots, u_m\}$, $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_p\}$, $E = E_B \cup E_A \cup E_{A_d} \cup E_C \cup E_D$, $E_B = \{(u_j, x_i) | (B)_{ij} \neq 0\}$, $E_A = \{(x_j, x_i) | (A)_{ij} \neq 0\}$, $E_{A_d} = \{(x_j, x_i) | (A_d)_{ij} \neq 0\}$, $E_C = \{(x_j, y_i) | (C)_{ij} \neq 0\}$ and $E_D = \{(u_j, y_i) | (D)_{ij} \neq 0\}$. Similarly, the unweighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{(n+m+p)\times(n+m+p)}$ can be obtained by A, A_d, B, C and D as following:

$$\mathcal{A}(A, A_d, B, C, D) \triangleq \begin{bmatrix} 0 & 0 & 0 \\ B(1) & (A + A_d)(1) & 0 \\ D(1) & C(1) & 0 \end{bmatrix}, \tag{2.4}$$

where $(A + A_d)(1), B(1), C(1)$ and $D(1)$ mean that the λ_i's and all combinations are replaced by 1. For example, if an entry of $(A + A_d)$ is (λ_k) or $(\lambda_h + \lambda_p)$, it is replaced by 1. Note that A and A_d can cause similar edges between elements of X. Also inputs (U) are not affected by any vertices and also outputs (Y) do not affect on any vertices, so the first row and the last column of $\mathcal{A}$ are blocking zero. For example, the associated graph to the system (2.3) is shown in Figure 1.

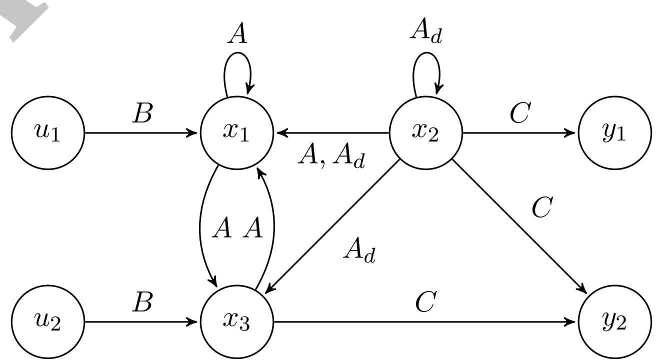

Figure 1. Associated graph to the system (2.3).

3. Structural controllability of dead-time system

As we mentioned in Section 2, a graph can represent the structure of a dead-time system. This section is mainly interested on studying the structural specification of dead-time systems based on the associated graph. The structural controllability of dead-time systems are described in Theorem 3.1.

Theorem 3.1. *System* (2.2) *with a graph* $G(\{U,X,Y\},\{E_B,E_A,E_{A_d},E_C,E_D\})$*, is structurally controllable if one of the following statements holds:*

- *(a) In one of sub-graphs* $G(\{U,X,Y\},\{E_B,E_A,\emptyset,E_C,E_D\})$ *or* $G(\{U,X,Y\},\{E_B,\emptyset,E_{A_d},E_C,E_D\})$*, every state is the end vertex of an* U*-rooted path and there exist a disjoint union of an* U*-rooted path family and a cycle family such that covers all state vertices.*
- *(b) Every state in graph* $G(\{U,X,Y\},\{E_B,E_A,E_{A_d},E_C,E_D\})$ *is the end vertex of an* U*-rooted path and there exist a disjoint union of an* U*-rooted path family and a cycle family such that covers all state vertices.*

Also system (2.2) *can be structurally controllable only if every state in graph* $G(\{U,X,Y\},\{E_B,E_A,E_{A_d},E_C,E_D\})$ *is the end vertex of an* U*-rooted path.*

Proof. (a) Based on Theorem 2.2, System (2.2) is controllable if $Rank(Q)=n$, where:

$$Q=[Q_1^1,Q_1^2,Q_2^2,Q_1^3,\ldots,Q_n^n].$$

One can calculate Q_j^k of Q as follows:

$$Q_1^1=B,Q_2^2=AB,Q_3^3=A^2B,\ldots,Q_n^n=A^{n-1}B,$$
$$Q_1^1=B,Q_1^2=A_dB,Q_1^3=A_d^2B,\ldots,Q_1^n=A_d^{n-1}B.$$

Consider two linear non-delayed systems as:

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{3.1}$$

$$\dot{x}(t)=A_dx(t)+Bu(t) \tag{3.2}$$

The controllable matrices of system (3.1) and (3.2) are equal to $\bar{Q}_1=[Q_1^1,Q_2^2,\ldots,Q_n^n]$ and $\bar{Q}_2=[Q_1^1,Q_1^2,\ldots,Q_1^n]$ respectively. On the other hand, $Rank(Q)=n$ if $Rank(\bar{Q}_1)=n$ or $Rank(\bar{Q}_2)=n$ because Q contains $\bar{Q}_1$ and $\bar{Q}_2$ elements. So system (2.2) is structurally controllable if system (3.1) or (3.2) is structurally controllable. Theorem 1 of [13] shows that a system as (3.1) or (3.2) is structurally controllable if condition (a) of Theorem 3.1 holds.

(b) Now consider a linear non-delayed system as follows:

$$\dot{x}(t)=(A+A_d)x(t)+Bu(t)\ . \tag{3.3}$$

The controllability matrix of system (3.3) is equal to $\bar{Q}_3=[B,(A+A_d)B,\ldots,(A+A_d)^{n-1}B]$. On the other hand we have:

$$Q_1^2+Q_2^2=(A+A_d)B,Q_1^3+Q_2^3+Q_3^3=(A+A_d)^2B,\ldots,Q_1^n+\cdots+Q_n^n=(A+A_d)^{n-1}B.$$

One can see that $\bar{Q}_3$ consists of linear combinations of partitions of Q (i.e. Q_j^k). So if $Rank(\bar{Q}_3) = n$ then $Rank(Q) = n$. The rest of the proof can be followed similar to the part (a).

Now consider in graph $G(\{U, X, Y\}, \{E_B, E_A, E_{A_d}, E_C, E_D\})$, $X_{\bar{u}} \subseteq X$ is the set of states which are not the end of any U-rooted path. Let assume that $X_{\bar{u}} = \{x_i | i \in I\}$ for some $I \subseteq \{1, \ldots, n\}$. On the other hand, consider $X_u \subseteq X$ is the set of states such that there exists a direct edge from U to each them. Let also assume that $X_u = \{x_j | j \in J\}$ for some $J \subseteq \{1, \ldots, n\}$. The jth ($j \in J$) rows of B are non-zero and other rows (included ith ($i \in I$) rows) of B is completely zero. As there is no path from X_u to $X_{\bar{u}}$, the ijth entries of $(A + A_d)^l$ are zero (for $i \in I$, $j \in J$ and $l = 0, 1, \ldots, n-1$). So the ith rows of $(A+A_d)^l B$ is completely zero (because in these rows, for non-zero rows of B, the corresponded columns of $(A + A_d)^l$ is zero). One can see that $\sum_{i=1}^{l} Q_l^i = (A + A_d)^{l-1} B$. So the ith rows of $\sum_{i=1}^{l} Q_l^i$ are zero. In other words, the ith rows of all Q_l^i are zero (for $j = 1, 2, \ldots, l$ and $l = 1, 2, \ldots, n$), because the entries of Q_l^i are parametric and the summation of them are zero when all of the rows are zero. In this case, the rank of Q is less than n and the system (2.2) is not controllable. Therefore system (2.2) can be structurally controllable only if every state in graph $G(\{U, X, Y\}, \{E_B, E_A, E_{A_d}, E_C, E_D\})$ is the end vertex of a U-rooted path. □

Theorem 3.1 can be very useful to design the structure of multi agent systems. For swarm group of UAVs in example, the position of UAVs must be first selected, next the possibility of communications between each ordered pairs of UAVs must be determined. Finally the controllability of the structure of group must be investigated. Based on Theorem 3.1, the associated graph in Figure 1 is not controllable because x_2 is not the end vertex of an U-rooted path.

Usually observability can be defined very similar to controllability where the output concept is replaced by input one. While we are especially interested on controllability but it is possible to present a corollary about structural observability based on previous result.

Corollary 3.2. *System (2.2) with a graph $G(\{U, X, Y\}, \{E_B, E_A, E_{A_d}, E_C, E_D\})$ is structurally observable if one of the following statements holds:*

- *(a) In one of sub-graphs $G(\{U, X, Y\}, \{E_B, E_A, \emptyset, E_C, E_D\})$ or $G(\{U, X, Y\}, \{E_B, \emptyset, E_{A_d}, E_C, E_D\})$, every state is the begin vertex of a Y-topped path and there exists a disjoint union of a Y-topped path family and a cycle family that covers all state vertices.*
- *(b) Every state in graph $G(\{U, X, Y\}, \{E_B, E_A, E_{A_d}, E_C, E_D\})$ is the begin vertex of a Y-topped path and there exists a disjoint union of a Y-topped path family and a cycle family that covers all state vertices.*

Also system (2.2) can be structurally observable only if every state in graph $G(\{U, X, Y\}, \{E_B, E_A, E_{A_d}, E_C, E_D\})$ is the begin vertex of a Y-topped path.

Proof. Based on Duality proposition of Linear system [20] and similar to [13], the proof of corollary 3.2 can be followed as Theorem 3.1. □

4. Structural controllability of dead-time multi agent system

One of the important advantages of using graph method is for controllability studying and leader selection of multi agent systems such as complex networks and swarm vehicles, UAVs or AUVs systems. In these cases usually some states (called followers) must be controlled by some other states (called leaders) [1]. The main challenge in leader follower systems is the possibility of followers controllability versus to leaders. Therefore the relation between leaders and followers must be chosen such that the system is structurally controllable.

In the real applications as shown in [2, 3, 4, 5, 6, 9], there exists a delay between states because of the communication delay, process lag, non-ideal actuator or computation times. In this case, the model of system can be considered as follows:

$$\dot{x}(t) = Ax(t) + A_d x(t-d). \tag{4.1}$$

Usually there exists no delay for affecting a state in itself [14], so it is considered that A is diagonal and the diagonal entries of A_d is zero.

As we noted before, some states in multi agent systems are separated to leaders and followers. Therefore system (4.1) is divided into two parts; followers states and leader states, i.e.:

$$x^T = \begin{bmatrix} \bar{x}_1^T, & \dots, & \bar{x}_{n-m}^T, & z_1^T, & \dots, & z_m^T \end{bmatrix}, A = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}, A_d = \begin{bmatrix} A_d^{11} & A_d^{12} \\ A_d^{21} & A_d^{22} \end{bmatrix}, \tag{4.2}$$

$$\bar{x}^T = \begin{bmatrix} \bar{x}_1^T, & \dots, & \bar{x}_{n-m}^T \end{bmatrix}, z^T = \begin{bmatrix} z_1^T, & \dots, & z_m^T \end{bmatrix}.$$

Matrices A and A_d are partitioned to four blocks corresponded to $\bar{x}$ and z. As A is diagonal, $A^{12} = 0$ and $A^{21} = 0$. Moreover, $A_d^{21} = 0$ because the leaders are independent versus to followers. Consequently, the model of follower states is obtained as:

$$\dot{\bar{x}}(t) = A^{11}\bar{x}(t) + A_d^{11}\bar{x}(t-d) + A_d^{12}z(t-d). \tag{4.3}$$

Ji and et al. [14] do not consider the delay in the leaders while usually there exists no difference between leaders and followers in the presence of delay in relationship. Hence the model of multi-agent systems with dead-time is considered as (4.3) in this paper. Figures 2 and 3 present two examples of the associated graphs to multi-agent systems. In those figures all loops are caused by A and other edges are corresponded to entries A_d which having a delay.

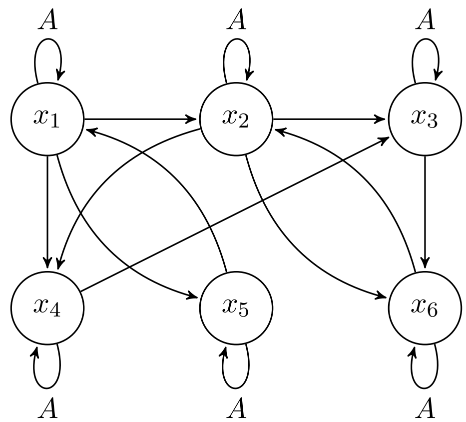

FIGURE 2. An example for associating a graph to multi-agent with a dead-time.

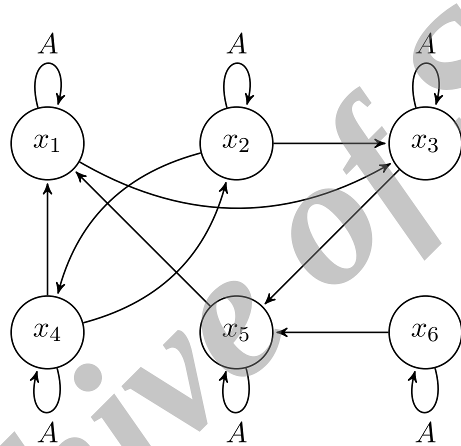

FIGURE 3. An example for associating a graph to multi-agent with a dead-time.

The structural controllability of a multi-agent system can be studied using Theorem 3.1, if the leaders and structure of the proposed systems are identified. However, sometimes leaders must be selected based on the given structure. Moreover a system must be ideally controlled by a leader or at least by a minimum number of leaders. Increasing the number of leaders causes more complexity of group control because each leaders must control the followers coordinately while leaders are independent. Next section will address the minimum leader selection for multi-agent systems with dead-time using the associated graphs.

5. Leader selection for dead-time multi agent system

Consider a dead-time multi agent system as (4.1). A graph can be assigned to this systems as follows:

$$G(\{X\},\{E_A,E_{A_d}\}) = G(\{\bar{X},Z\},\{E_{A^{11}},E_{A^{22}},E_{A_d^{11}},E_{A_d^{12}},E_{A_d^{22}}\}),$$

where $X = \{\bar{x}_1, \ldots, \bar{x}_{n-m}, z_1, \ldots, z_m\}$, $\bar{X} = \{\bar{x}_1, \ldots, \bar{x}_{n-m}\}$, $Z = \{z_1, \ldots, z_m\}$, $E_{A^{11}} = \{(\bar{x}_j, \bar{x}_i)| (A^{11})_{ij} \neq 0\}$, $E_{A^{22}} = \{(z_j, z_i)| (A^{22})_{ij} \neq 0\}$, $E_{A_d^{11}} = \{(\bar{x}_j, \bar{x}_i)|(A_d^{11})_{ij} \neq 0\}$, $E_{A_d^{12}} = \{(z_j, \bar{x}_i)|(A_d^{12})_{ij} \neq 0\}$, $E_{A_d^{22}} = \{(z_j, z_i)|(A_d^{22})_{ij} \neq 0\}$. In this section we try to find a leader subset $Z \subseteq X$ of the minimum size such that the corresponding systems is structural controllable. First Theorem 5.1 describes which subset of X has the capability to be a leader set.

Theorem 5.1. *Consider multi-agent system* (4.1) *and its associated graph. A subset of states $Z \subset X$ can be selected as leaders such that the multi-agent* (4.3) *is structural controllable, if $\bar{X}$ are reachable by Z.*

Proof. Matrix A is assumed be diagonal for dead-time multi agent systems [14], so it is possible to find an union of a disjoint cycle family covered all states. Therefore if $\bar{X}$ are reachable by leader set Z, it is possible to find a Z-rooted path for each followers such that all followers are the end vertices of a Z-rooted path. Hence the condition (b) of Theorem 3.1 is satisfied and (4.3) is structural controllable. □

Note that the leaders conceptually must be independent from follower states, so there must be no path from the followers to the leaders. Although this issue is not considered in Theorem 5.1, but it is possible to eliminate all edges from followers to leaders as λ_is are arbitrary.

Usually it is not easy to search all subsets of X satisfying the condition of Theorem 5.1, next find a leader set of the minimum size. For this proposed, Theorem 5.2 can offer a method to find the leader set of the minimum size using adjacency matrix of the associated graph.

Theorem 5.2. *Consider multi-agent system* (4.1) *and its associated adjacency matrix $\mathcal{A}$. Minimum number of controllable leaders for system* (4.3) *are minimum value of m such that at least the summation of m columns of $\sum_{k=1}^{n-1} \mathcal{A}^k$ has no zero entry. Moreover, the corresponded states of these columns can be a candidate for leaders.*

Proof. If ijth entry of $\sum_{k=1}^{n-m} \mathcal{A}^k$ is non-zero, it means that there is a path from x_j to x_i which can be 1-link, 2-link, ... or $(n-m)$-link. Therefore, there exists at least a path form x_j to all states if all entries of jth column of $\sum_{k=1}^{n-m} \mathcal{A}^k$ are non-zero. Consequently, there is at least a path from m states to other remaining states if the summation of corresponding columns of these m state in $\sum_{k=1}^{n-m} \mathcal{A}^k$ is non-zero.

Note that the computing of $\sum_{k=1}^{n-1} \mathcal{A}^k$ is enough to find minimum leaders, because if an entry of $\sum_{k=1}^{n-m} \mathcal{A}^k$ is non-zero, it is certainly non-zero in $\sum_{k=1}^{n-1} \mathcal{A}^k$. Also $\sum_{k=1}^{s} \mathcal{A}^k$ (for $s > n-1$) gives us no additional information because the graph has no simple path with more than $(n-1)$-link. Moreover iith entries of $\sum_{k=1}^{n-m} \mathcal{A}^k$ are non-zero since A is diagonally non-zero, (i.e. Z are reachable by itself). □

It is possible to use Theorem 5.2 to select leader set with minimum size for a multi-agent system such as shown in Figure 2 or 3. One can obtain $\sum_{k=1}^{n-1}\mathcal{A}^k$ for the graph of Figure 2 as follows:

$$\sum_{k=1}^{6-1}\mathcal{A}^k = \begin{bmatrix} 31 & 0 & 0 & 0 & 31 & 0 \\ 61 & 54 & 27 & 16 & 34 & 38 \\ 77 & 65 & 27 & 22 & 39 & 43 \\ 65 & 38 & 16 & 11 & 43 & 27 \\ 31 & 0 & 0 & 0 & 31 & 0 \\ 73 & 81 & 38 & 27 & 30 & 54 \end{bmatrix}. \tag{5.1}$$

Columns 1 and 5 in (5.1) are completely non-zero. Therefore subset $\{x_1\}$ or $\{x_5\}$ can be selected for a leader set of the graph of Figure 2. On the other hand, we can calculate $\sum_{k=1}^{n-1}\mathcal{A}^k$ of Figure 3 as following:

$$\sum_{k=1}^{6-1}\mathcal{A}^k = \begin{bmatrix} 27 & 50 & 43 & 45 & 65 & 43 \\ 0 & 31 & 0 & 31 & 0 & 0 \\ 22 & 54 & 27 & 54 & 43 & 22 \\ 0 & 31 & 0 & 31 & 0 & 0 \\ 43 & 57 & 65 & 46 & 92 & 65 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}. \tag{5.2}$$

It can be seen that there exists no column such that all its entries are non-zero. However it is possible to get $\{x_2, x_6\}$ or $\{x_4, x_6\}$ as leaders.

6. Problems

As shown in this paper, it is possible to investigate the structural controllability of dead-time system based on an associated graph using sufficient condition of Theorem 3.1. It seems the condition (b) of Theorem 3.1 can be necessary and sufficient condition. However it must be proven dependently.

Moreover, the method of finding minimum leaders is described in Section 5. Sometimes there exist multiple choices for leader set with the minimum size, while all of them have no equivalent connectivity. For example in Figure 2, both states x_1 and x_5 can be leader. Nevertheless, if the edge between x_5 and x_1 is eliminated, there exists no path from x_5 to other states. On the other hand, there are three dependent paths from x_1 to other states. It can be studied by investigating the weight of columns of $\sum_{k=1}^{n-1}\mathcal{A}^k$. For example in (5.1), the entries of first column are greater than or equal to fifth column.

The degree of leaders connectivity can be equivalent to reliability concept in control theory. Therefore a relation between the weight of columns of $\sum_{k=1}^{n-1}\mathcal{A}^k$ and leaders reliability can be interested problems in future works.

References

[1] A. Rahmani, M. Ji, M. Mesbahi and M. Egerstedt, Controllability of multi-agent systems from a graph-theoretic perspective, *SIAM J. Control Optim.*, **48** no. 1 (2009) 162–186.

[2] , R. Olfati-Saber and R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Automat. Control*, **49** no. 9 (2004) 1520–1533.

[3] Y. G. Sun, L. Wang and G. Xie, Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays, *Systems Control Lett.*, **57** no. 2 (2008) 175–183.

[4] F. Xiao and L. Wang, Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays, *IEEE Trans. Automat. Control*, **53** no. 8 (2008) 1804–1816.

[5] J. Qin, H. Gao and W. X. Zheng, Second-order consensus for multi-agent systems with switching topology and communication delay, *Systems Control Lett.*, **60** no. 6 (2011) 390–397.

[6] R. Ghabcheloo, A. P. Aguiar, A. Pascoal, C. Silvestre, I. Kaminer and J. Hespanha, Coordinated path-following in the presence of communication losses and time delays, *SIAM J. Control Optim.*, **48** no. 1 (2009) 234–265.

[7] J. Guo, Z. Lin, M. Cao and H. Yan, Adaptive control schemes for mobile robot formations with triangularised structures, *Control Theory & Applications, IET*, **4** no. 9 (2010) 1817–1827.

[8] B. Yun, B. M. Chen, K. Y. Lum and T. H. Lee, *A leader-follower formation flight control scheme for UAV helicopters*, Proceeding of Automation and Logistics, ICAL 2008. IEEE International Conference on, 39–44.

[9] S. Khosravi, M. Jahangir and H. Afkhami, Adaptive fuzzy SMC-based formation design for swarm of unknown time-delayed robots, *Nonlinear Dynam.*, **69** no. 4 (2012) 1825–1835.

[10] M. Mesbahi and F. Y. Hadaegh, Formation flying control of multiple spacecraft via graphs, matrix inequalities, and switching, *Journal of Guidance, Control, and Dynamics*, **24** no. 2 (2001) 369–377.

[11] C. P. Fall, *Computational cell biology*, Springer-Verlag, New York, **20** 2002.

[12] Y. Y. Liu, J. J. Slotine and A. L. Barabási, Controllability of complex networks, *Nature*, **473** no. 7346 (2011) 167–173.

[13] J. M. Dion, C. Commault, and J. Van Der Woude, Generic properties and control of linear structured systems: a survey, *Automatica J. IFAC*, **39** no. 7 (2003) 1125–1144.

[14] Z. Ji, Zi. Wang, H. Lin, and Zh. Wang, Controllability of multi-agent systems with time-delay in state and switching topology, *Internat. J. Control*, **83** no. 2 (2010) 371–386.

[15] D. Cvetkovic, P. Rowlinson, Z. Stanic and M. G. Yoon, Controllable graphs, *Bull. Cl. Sci. Math. Nat. Sci. Math.*, **143** no. 36 (2011) 81–88.

[16] S. Jafari, A. Ajorlou and A. G. Aghdam, *Leader selection in multi-agent systems subject to partial failure*, Proceeding of American Control Conference (ACC), 2011 5330–5335.

[17] A. Clark, L. Bushnell and R. Poovendran, *On leader selection for performance and controllability in multi-agent systems*, Proceeding of Decision and Control (CDC), 2012 IEEE 51st Annual Conference on, 86–93.

[18] M. A. Rahimian and A. G. Aghdam, *Structural controllability of multi-agent networks: Robustness against simultaneous failures*, Automatica, In press, (2013).

[19] M. Ji, A. Muhammad and M. Egerstedt, *Leader-based multi-agent coordination: Controllability and optimal control*, Proceeding of American Control Conference, 2006 6–12.

[20] C. T. Chen, *Linear system theory and design*, Oxford University Press, Inc., 1998.

Majdeddin Najafi

Avionics Research Institute, Isfahan University of Technology (IUT) Isfahan, Iran

Email: majd.najafi@ec.iut.ac.ir

Farid Shaikholeslam

Department of Electrical and Computer Engineering, Isfahan University of Technology (IUT) Isfahan, Iran

Email: shaikh@cc.iut.ac.ir