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GRAPHS COSPECTRAL WITH A FRIENDSHIP GRAPH OR ITS COMPLEMENT

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Abstract. Let n be any positive integer and F_n be the friendship (or Dutch windmill) graph with $2n+1$ vertices and $3n$ edges. Here we study graphs with the same adjacency spectrum as F_n. Two graphs are called cospectral if the eigenvalues multiset of their adjacency matrices are the same. Let G be a graph cospectral with F_n. Here we prove that if G has no cycle of length 4 or 5, then $G \cong F_n$. Moreover if G is connected and planar then $G \cong F_n$. All but one of connected components of G are isomorphic to K_2. The complement $\overline{F_n}$ of the friendship graph is determined by its adjacency eigenvalues, that is, if $\overline{F_n}$ is cospectral with a graph H, then $H \cong \overline{F_n}$.

1. Introduction

All graphs in this paper are simple of finite orders, i.e., graphs are undirected with no loops or parallel edges and with finite number of vertices. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. Also, $A(G)$ denotes the $(0,1)$-adjacency matrix of graph G. The characteristic polynomial of G is $\det(\lambda I - A(G))$, and we denote it by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjacency eigenvalues of G and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. If G has n vertices, then it has n eigenvalues and we denote its eigenvalues in descending order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, \ldots, m_s$, respectively. The multiset $\mathrm{Spec}(G) = \{(\lambda_1)^{m_1}, (\lambda_2)^{m_2}, \ldots, (\lambda_s)^{m_s}\}$ of eigenvalues of $A(G)$ is called the adjacency spectrum of G.

For two graphs G and H, if $\mathrm{Spec}(G) = \mathrm{Spec}(H)$, we say G and H are cospectral with respect to adjacency matrix. A graph G is said to be determined by its spectrum or DS for short, if $\mathrm{Spec}(G) =$

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Spec(H) for some graph H, then $G \cong H$. So far numerous examples of cospectral but non-isomorphic graphs are constructed by interesting techniques such as Seidel switching, Godsil-McKay switching, Sunada or Schwenk method. For more information, one may see [1, 2, 3] and the references cited in them. Only a few graphs with very special structures have been reported to be determined by their spectra (see [1, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references cited in them). Recently Wei Wang and Cheng-Xian Xu have developed a new method in [12] to show that many graphs are determined by their spectrum and the spectrum of their complement.

The friendship (or Dutch windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. By construction, the friendship graph F_n is isomorphic to the windmill graph $Wd(3,n)$ [13]. The Friendship Theorem of Paul Erdös, Alfred Rényi and Vera T. Sós [14] states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs.
Figure 1 shows some examples of friendship graphs.

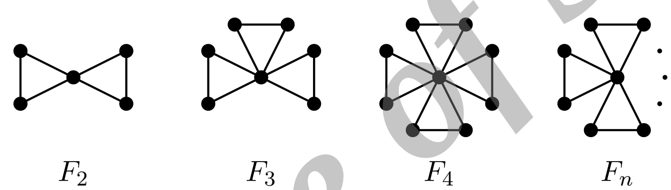

FIGURE 1. Friendship graphs F_2, F_3, F_4 and F_n

In [15] it is proved that the friendship graphs can be determined by the signless Laplacian spectrum and in [15, 16] the following conjecture has been proposed:

Conjecture 1. *The friendship graph is DS with respect to the adjacency matrix.*

Conjecture 1 has been recently studied in [17]. It is claimed as [17, Theorem 3.2] that Conjecture 1 is valid. We believe that there is a gap in the proof of [17, Theorem 3.2] where Interlacing Theorem has been applied for subgraphs of the graph which are not clear if they are induced or not. Therefore, we give our results independently.

The rest of this paper is organized as follows. In Section 2, we obtain some preliminary results about the cospectral mate of a friendship graph. In Section 2 we prove that if the cospectral mate of F_n is connected and planar then it is isomorphic to F_n. In Section 3, it is proved that, if the cospectral mate of F_n is connected and does not have C_5 as a subgraph, then it is isomorphic to F_n. Also, we prove that, if there are two adjacent vertices with degree 2 in a cospectral mate of F_n, then G is isomorphic to F_n and some variations of the latter result is studied. In Section 4, the complement of the cospectral mate is studied and we show that if this complement is disconnected, then the cospectral mate is isomorphic to F_n. Also, it is shown that the complement of the friendship graph F_n is DS.

2. Some Properties of Cospectral Mate of F_n

We first give some preliminary facts and theorems which are useful in the sequel. For the proof of these facts one may see [18].

Lemma 2.1. *Let G be a graph. For the adjacency matrix of G, the following information can be deduced from the spectrum:*

1. *The number of vertices*
2. *The number of edges*
3. *The number of closed walks of any length*
4. *Being regular or not and the degree of regularity*
5. *Being bipartite or not.*

Theorem 2.2 (Interlacing Theorem, Theorem 2.5.1 of [3]). *Let G be a graph of order n and H be an induced subgraph of G of order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Proposition 2.3. *Let F_n denote the friendship graph with $2n+1$ vertices. Then*

$$Spec(F_n) = \left\{ \left(\frac{1}{2} - \frac{1}{2}\sqrt{1+8n}\right)^1, (-1)^n, (1)^{n-1}, \left(\frac{1}{2} + \frac{1}{2}\sqrt{1+8n}\right)^1 \right\}.$$

Proof. The friendship graph F_n with $2n+1$ vertices is the cone of the disjoint union of n complete graphs K_2: $K_1 \nabla nK_2$. It follows from Theorem 2.1.8 of [18] that the characteristic polynomial of F_n is:

$$P_{F_n}(x) = (x+1)\left(x^2-1\right)^{n-1}\left(x^2-x-2n\right).$$

This completes the proof. □

Let H be any graph. A graph G is called H-free if it does not have an induced subgraph isomorphic to H. In the following, we examine the structure of G as a cospectral graph of F_n.

Proposition 2.4. *Let G be a graph cospectral with friendship graph F_n. Then*

1. *If H is a graph with $\lambda_2(H) > 1$, then G is H-free.*
2. *If H is a graph having at least two eigenvalues less than -1, then G is H-free.*

Proof. We know that $\lambda_2(F_n) = 1$ and F_n has only one eigenvalue less than -1. Now applying Interlacing Theorem for the induced subgraph H, it follows that G is H-free. □

Theorem 2.5. *Let G be a graph cospectral with friendship graph F_n. Then G is either connected or it is a disjoint union of some K_2 and a connected component.*

Proof. It is easy to see that $\lambda_2(K_3 \cup P_3)$ and $\lambda_2(K_3 \cup K_3)$ are both greater than 1. Thus by Proposition 2.4 all but one of the connected components of G do not contain K_3 or P_3 as an induced subgraph. So, if G is not connected, all but one connected components of G must be isomorphic to K_2, since G does not have any isolated vertices. □

Definition 2.6. [19] *A graph is triangulated if it has no chordless induced cycle with four or more vertices. It follows that the complement of a triangulated graph cannot contain a chordless cycle with five or more vertices.*

Proposition 2.7. *Let G be a connected, planar and cospectral graph with friendship graph F_n. Then G is triangulated.*

Proof. The graph G is planar and connected with n triangles, $2n+1$ vertices and $3n$ edges. Also, the number of faces is an invariant parameter between two cospectral connected planar graphs, since it only depends on the number of vertices and edges. Let $f(G)$ denote the number of faces of graph G. By Euler formula for connected planar graphs, $f(G) = 2 - |V(G)| + |E(G)|$, the number of faces of G is $n+1$ and each inner face of G must be an induced triangle. Therefore G has no chordless induced cycle with four or more vertices. □

In the following, we express an interesting corollary extracted from a theorem of Vladimir Nikiforov in [20], and we use it to prove some results.

Theorem 2.8. [20, Theorem 3] *Let G be a graph of order n with $\lambda_1(G) = \lambda$. If G has no 4-cycles, then*

$$\lambda^2 - \lambda \geq n - 1,$$

and equality holds if and only if every two vertices of G have exactly one common neighbour.

Proof. Apply Theorem 3 of [20] for $k = l = 1$. See also the abstract of [21]. □

Corollary 2.9. *Let G be connected, planar and cospectral with friendship graph F_n. Then G is isomorphic to F_n.*

Proof. By Proposition 2.7, the graph G is C_4-free and $\lambda_1^2(G) - \lambda_1(G) = 2n$. Therefore by Theorem 2.8, the graph G must be isomorphic to F_n. □

Suppose $\chi(G)$ and $\omega(G)$ denote chromatic number and clique number of a graph G, respectively. A graph G is called perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. It is proved that a graph G is perfect if and only if G is Berge, that is, it contains no odd hole or antihole, where odd hole and antihole are odd cycle, C_m for $m \geq 5$, and its complement, respectively. Also in 1972 Lovász proved that, a graph is perfect if and only if its complement is perfect [22].

Proposition 2.10. *Let G be a graph cospectral with F_n. Then both G and $\overline{G}$ are perfect.*

Proof. The spectrum of a hole, that is an n-cycle C_n for n odd and $n \geq 5$, is $\lambda_j = 2\cos(\frac{2\pi j}{n})$ for $j = 0, 1, \ldots, n-1$. It is easy to check that for n odd and $n \geq 5$, $\lambda_{n-1}(C_n)$ and $\lambda_n(C_n)$ is strictly less than -1. Therefore by Proposition 2.4, any hole cannot be an induced subgraph of G. Also, since the spectrum of an antihole that is the complement of a hole, are $n-3$ and $-1-\lambda_j$ $(j = 1, \ldots, n-1)$, it follows that any antihole has at least two eigenvalues less than -1. Now Proposition 2.4 shows that G cannot have any antihole. So, both G and $\overline{G}$ are perfect graphs. □

Theorem 2.11 (Theorem 6 of [23])**.** *A graph G is P_6-free if and only if each connected induced subgraph of G contains a dominating induced C_6 or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

Proposition 2.12. *Let G be a connected and cospectral graph with F_n. Then each connected induced subgraph of G contains a dominating (not necessarily induced) complete bipartite graph.*

Proof. By Theorem 2.11 and Proposition 2.4, we must only prove that G has no induced dominating C_6 as a subgraph. Suppose G has an induced dominating C_6 subgraph. By hand checking and Interlacing Theorem, it is not hard to see that a seventh vertex of G (except the vertices of the latter C_6) must join to three non adjacent vertices of C_6. Also, for each $2n+1-7=2n-6$ remaining vertices, we have at least two edges. So, the total number of edges in G is at least $2(2n-6)+3+6=4n-3$, that is contradiction. □

3. Structural Properties of Cospectral Mates of F_n

It can be seen by Theorem 2.8 that, if G is cospectral with F_n and does not have C_4 as a subgraph then G is isomorphic to F_n. In the following, we study the cospectral mate of F_n with respect to the C_5 subgraph. Also, we know that in the graph F_n, and also in its cospectral mate, the average number of triangles containing a given vertex is $\frac{6n}{4n+2}$, that is strictly greater than one. Using the latter property we obtain some results about the cospectral mate of F_n.

Firstly, we will prove that if G is a connected graph cospectral with F_n and does not have C_5 as a subgraph, then G is isomorphic to F_n. We need the following well-known result.

Lemma 3.1. [18] *Suppose that the graph G is connected and H is a proper subgraph of G. Then*

$$\lambda_{max}(H) < \lambda_{max}(G).$$

Lemma 3.2. *Let G be a connected graph cospectral with F_n and $\delta(G)$ be the minimum degree of G. Then, $\delta(G)=2$ and G has at least three vertices with this minimum degree.*

Proof. Suppose, for a contradiction, that G has at least one vertex with degree 1, say v. Suppose that v is adjacent to the vertex w. The graphs $G\setminus\{v\}$ and $G\setminus\{v,w\}$ are induced subgraphs of G. Let $\mu_1\geq\mu_2\geq\ldots\geq\mu_{2n-1}$ be the eigenvalues of graph $G\setminus\{v,w\}$. Then Interlacing Theorem implies that

$$\lambda_j\geq\mu_j\geq\lambda_{j+2},(j=1,2,\ldots,2n-1),$$

where λ_i, $(i=1,2,\ldots,2n+1)$ are the eigenvalues of G. Thus

i) $|\mu_1|\leq|\lambda_1|=\left|\frac{1+\sqrt{8n+1}}{2}\right|$,

ii) $|\mu_{2n-1}|\leq|\lambda_{2n+1}|=\left|\frac{1-\sqrt{1+8n}}{2}\right|$,

iii) $|\mu_j|\leq 1$ for $j=2,3,\ldots,2n-2$.

Now Theorem 2.2.1 of [18] implies that

$$P_G(x)=xP_{G\setminus\{v\}}(x)-P_{G\setminus\{v,w\}}(x).$$

It follows that

$$(-1)^n(2n) = P_G(0) = P_{G\backslash\{v,w\}}(0) = \prod_{j=1}^{2n-1} \mu_j.$$

Therefore, by $(i), (ii)$ and (iii) we obtain $\mu_1 = \lambda_1$, contradicting Lemma 3.1. So, G does not have any vertex of degree one.

Now, suppose G has t vertices of degree 2. Therefore

$$2t + 3(2n+1-t) \leq \sum_{i=1}^{2n+1} deg(v_i(G)) = 6n.$$

It follows that $t \geq 3$, $\delta(G) = 2$ and G has at least three vertices of degree two. □

Remark 3.3. *By similar arguments as in Lemma 3.2, one can show that if G is not connected, then the component of G that is not isomorphic to K_2 does not have any vertex with degree one. In this case, the minimum degree of G depends to the number of components isomorphic to K_2.*

Lemma 3.4. *Let G be a graph of order n. Then, the number of closed walks of length five in G is given by*

$$tr(A^5(G)) = \sum_{i=1}^{n} \lambda_i^5(G) = 30N_G(C_3) + 10N_G(C_5) + 10N_G(C_3^*),$$

where C_3^* is isomorphic to K_3 with one pendant.

Proof. It is easy to see that, the only subgraphs of G occur in counting of closed walks of length five are, C_3, C_5 and C_3^*. By summation the fifth power of the eigenvalues of these graphs, the coefficients of $N_G(C_3), N_G(C_5)$ and $N_G(C_3^*)$ must be 30, 10 and 40, respectively. Since C_3 is counted twice in G and C_3^*, 30 times for each C_3^*, and we have to subtract it. This completes the proof. □

Lemma 3.5. *Suppose $S : \mathbb{R}^n \to \mathbb{R}$ is a function defined as $S(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i^2$ and let $\sum_{i=1}^n x_i = M$. Then*

i) If $x_i \geq 0$ for $i = 1, 2, \ldots, n$, then the maximum of S is M^2 and this value only happens in $M\boldsymbol{e}_i$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is the standard orthogonal basis of $\mathbb{R}^n$.

ii) If $x_i \geq d$ $(i = 1, 2, \ldots, n)$, then the maximum of S is $(n-1)d^2 + (M-(n-1)d)^2$ and this value only happens in $(M-(n-1)d)\boldsymbol{e}_i + d\boldsymbol{j}$, where $\boldsymbol{j}$ denotes the all-1 vector of size $1 \times n$.

Proof. Let $T = 2\sum_{1 \leq i < j \leq n} x_i x_j$. To prove case (i), it suffices to note that $S = M^2 - T$, and for maximizing the function S, we must minimize the function T. But the minimum of T is zero and it happens only in $M\boldsymbol{e}_i$, since we have $\sum_{i=1}^n x_i = M$.

For proving case (ii), let $y_i = x_i - d$ $(i = 1, 2, \ldots, n)$ and $T(y_1, y_2, \ldots, y_n) = \sum_{i=1}^n y_i^2$. So, $y_i \geq 0$ and $\sum_{i=1}^n y_i = M - nd$. Now by using part (i), the maximum of T is $(M-nd)^2$ and it only happens in $(M-nd)\boldsymbol{e}_i$. Therefore, by backing the changed variables and the fact $S = T - nd^2 + 2Md$, this completes the proof. □

Lemma 3.6. *Suppose $S : \mathbb{R}^n \to \mathbb{R}$ is the function defined as $S(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} t_i x_i$, where t_i $(i = 1, 2, \ldots, n)$ are real numbers, $\sum_{i=1}^{n} x_i = M$ and $x_i \geq d$. If $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, then the maximum of function S is $d(t_1 + t_2 + \ldots + t_{n-1}) + t_n(M - (n-1)d)$.*

Proof. Since the real numbers t_i $(i = 1, 2, \ldots, n)$ are in increasing order, the result follows from Lemma 3.5. □

Theorem 3.7. *Suppose that G is connected graph cospectral with F_n. If G does not have C_5 as a subgraph, then G is isomorphic to F_n.*

Proof. Since G is cospectral with F_n, the number of vertices, edges, triangles and closed walks of length 5 are the same in both graphs G and F_n. By Lemma 3.4, we have $N_G(C_3^*) = N_{F_n}(C_3^*)$. Now, we calculate the number of $N_G(C_3^*)$ in two ways. Suppose $v_1, v_2, \ldots, v_{2n+1}$ are the vertices of G. Let t_i $(i = 1, 2, \ldots, 2n+1)$ denote the number of triangles having v_i as a vertex. So the total number of C_3^* having v_i as a vertex with degree three is $t_i(deg_G(v_i) - 2)$. Therefore,

$$N_G(C_3^*) = \sum_{i=1}^{2n+1} t_i(deg_G(v_i) - 2) = \sum_{i=1}^{2n+1} t_i deg_G(v_i) - 2\sum_{i=1}^{2n+1} t_i. \tag{3.1}$$

On the other hand

$$\sum_{i=1}^{2n+1} t_i = 3N_G(C_3) = 3n \Longrightarrow N_G(C_3^*) = \sum_{i=1}^{2n+1} t_i deg_G(v_i) - 6n. \tag{3.2}$$

Since $N_{F_n}(C_3^*) = 2n^2 + 4n - 6n$, by (3.1) and (3.2) we obtain

$$\sum_{i=1}^{2n+1} t_i deg_G(v_i) = 2n^2 + 4n \tag{3.3}$$

Now we prove that G is isomorphic to F_n. Suppose $x_i \geq 2, y_i \geq 0$ $(i = 1, 2, \ldots, 2n+1)$, $\sum_{i=1}^{2n+1} x_i = 3n$, $\sum_{i=1}^{2n+1} y_i = 6n$ and define the function F as follow

$$F(x_1, x_2, \ldots, x_{2n+1}, y_1, y_2, \ldots, y_{2n+1}) = \sum_{i=1}^{2n+1} x_i y_i.$$

We show that, if $(x_1, \ldots, x_{2n+1}) = (t_1, \ldots, t_{2n+1})$ and $(y_1, \ldots, y_{2n+1}) = (deg_G(v_1), \ldots, deg_G(v_{2n+1}))$, then the maximum of function F is happen for the graph F_n.

Let $\mathcal{A} = \{G_1, G_2, \ldots, G_k\}$ be the set of all connected graphs with $2n+1$ vertices, $3n$ edges, n triangles, minimum degree 2 and without any subgraph isomorphic to C_5. The vertices of G_i $(i = 1, 2, \ldots, k)$ can be labeled in such a way that, for each graph G_i we have $t_1 \leq t_2 \leq \ldots \leq t_{2n+1}$. It is easy to see that, F_n is a member of $\mathcal{A}$. Now, we want to find the maximum of $\sum_{i=1}^{2n+1} t_i deg_G(v_i)$ among the members of $\mathcal{A}$. We prove that, the maximum value of $\sum_{i=1}^{2n+1} t_i deg_G(v_i)$ is equal to $2n^2 + 4n$ and it only happens for the graph F_n.

For each graph $G \in \mathcal{A}$, let $X_G = (t_1, t_2, \ldots, t_{2n+1})$, $Y_G = (deg_G(v_1), deg_G(v_2), \ldots, deg_G(v_{2n+1}))$ and $F(G) = F(X_G, Y_G) = X_G \cdot Y_G$. It is clear that $F(F_n) = 2n^2 + 4n$. By Lemma 3.6, for each graph $G \in \mathcal{A}$ we have

$$F(G) = t_1 deg_G(v_1) + \ldots + t_{2n+1} deg_G(v_{2n+1}) \leq 2t_1 + \ldots + 2nt_{2n+1}.$$

The latter inequality implies that, for each graph $G \in \mathcal{A}$

$$F(G) \leq F(X_G, Y_0),$$

where $Y_0 = (2, 2, \ldots, 2n)$. Among the members of $\mathcal{A}$, the only graph having Y_0 as a degree sequence is F_n. Therefore, the graph G is isomorphic to F_n, since by (3.3) we have $\sum_{i=1}^{2n+1} t_i deg_G(v_i) = F(F_n)$. □

In the next proposition, we show that the cospectral mates of a friendship graph have'many' vertices of degree two.

Proposition 3.8. *Suppose that G is a graph cospectral F_n and let $d_2(G)$ and $\triangle(G)$ be the number of vertices of degree two and maximum degree of G, respectively. Then*

i) If G is disconnected, $G = mK_2 \cup G_1$, then

$$m \leq \frac{\lambda_{max}|V(G)| - 2|E(G)|}{-2\lambda_{min}}.$$

Moreover, if $d_2(G_1) \neq 0$, then $d_2(G_1) \geq \lambda_{max} - 4m$.

ii) If G is connected, then $d_2(G) \geq 1 + \lambda_{max}$.

Proof. It is easy to see that, if $G = mK_2 \cup G_1$ then the component G_1 has $3n - m$ edges, $2n + 1 - 2m$ vertices, $\lambda_{max} = \lambda_1(F_n)$ and $\lambda_{min} = \lambda_{2n+1}(F_n)$. Now it follows from Theorem 3.2.1 of [18] that

$$\frac{2(3n - m)}{2n + 1 - 2m} \leq \lambda_{max},$$

by simplification and using $\lambda_{max} - 1 = -\lambda_{min}$, we have proved the first part of (i). Again, using Theorem 3.2.1 of [18] and $\sum_{i=1}^{2n+1} deg_G(v_i) = 6n$, we obtain

$$2m + 2d_2(G_1) + \lambda_{max} + 3(2n + 1 - 2m - t - 1) \leq 6n,$$

so by simplification, the second part of (i) is proved.

To prove part (ii), notice that G is not regular. Thus Theorem 3.2.1 of [18] implies that $\triangle(G) \geq 1 + \lambda_{max}$. Therefore

$$2d_2(G) + 1 + \lambda_{max} + 3(2n + 1 - d_2(G) - 1) \leq 6n.$$

By simplification, we obtain the requested result. □

In the following, we obtain some structural properties of cospectral mates of a friendship graph. Actually, these results are some good evidences to show that the friendship graph is DS.

Definition 3.9. *Suppose that G is a graph and H is a subgraph of G. If x is a vertex of H with degree r in G, we denote it by $d_G(x) = r$.*

Lemma 3.10. *Suppose that G is a graph cospectral with F_n and G has a subgraph H isomorphic to K_3 having two vertices of degree 2 in G. Then G is isomorphic to F_n.*

Proof. Suppose that H has vertices $\{x, y, z\}$, where $d_G(x) = d_G(y) = 2$. We prove that an arbitrary triangle of G must share a common vertex with H at vertex z. Let $\{u, v, w\}$ be the vertices of an arbitrary triangle in G. At least one vertex of this triangle is joined to the vertex z, since G is $2K_3$-free. Therefore, the all cases that can happen are shown in Figure 2. The graph G is $\{A_2, A_3, A_4\}$-free, since $\lambda_2(A_2) = 1.73205$, $\lambda_2(A_3) = 1.50694$ and $\lambda_2(A_4) = 1.33988$. Now, we prove that, there is no edge between other $n-1$ triangles in G. Suppose that there are two triangles in G with some edges between them. Since these two triangles have a common vertex in z and all triangles in G must also have all possible cases showed in Figure 3. On the other hand G is $\{B_1, B_2\}$-free, since $\lambda_2(B_1) = 1.19799$ and $\lambda_2(B_2) = 1.28917$. This is a contradiction and so there are no edges between other $n-1$ triangles in G. This completes the proof. □

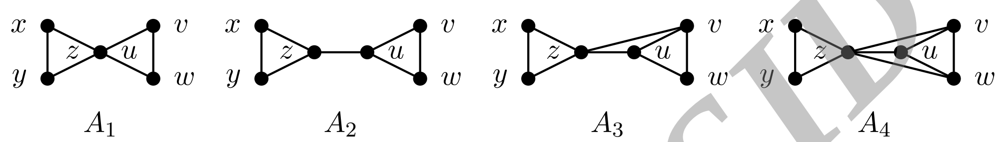

FIGURE 2. All possible cases between H and another triangle in G

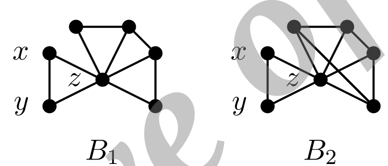

FIGURE 3. All possible cases between H and two other triangles in G

Theorem 3.11. *Suppose that G is a graph cospectral with F_n and G has two adjacent vertices of degree 2. Then G is isomorphic to F_n.*

Proof. Suppose $\{x, y\}$ are two adjacent vertices of degree 2 in G. If these two vertices are adjacent to a vertex z in G, then we have a triangle in G with vertices $\{x, y, z\}$. So, by Lemma 3.10, the result is clear. We show that, the latter is the only possible case. Suppose, for a contradiction, that the vertices x and y are adjacent to vertices a and b, respectively. Thus we have a P_4 with vertices $\{a, x, y, b\}$ as a subgraph of G. Therefore, at least one of the two cases in Figure 4 must be happen. First, we examine the graph C of Figure 4. For an arbitrary K_3 (or triangle) in G, we have $\lambda_2(C \cup K_3) = 2$. All possible cases that can be happen by C and K_3, are shown in Figure 5. Except of graphs C_1 and C_5 that have two eigenvalues less than -1, for all other graphs, $C_i (i = 2, \dots, 26)(i \neq 5)$, $\lambda_2(C_i) > 1$. Therefore the case C cannot happen in G.

Now, we examine the graph D of Figure 4. In this case, the graph D is an induced P_4 in G. For an arbitrary K_3 (or triangle) in G, we have $\lambda_2(D \cup K_3) = 1.61803$ and so, the all possible cases that D and K_3 can construct, are shown in Figure 6. Except the graphs D_3 that has two eigenvalues less than -1, for all other graphs, $D_i (i = 1, \dots, 20)(i \neq 3)$, we have $\lambda_2(D_i) > 1$. Therefore, the case D can not happen in G.

It follows that, if there are two adjacent vertices of degree 2 in the graph G, then they are adjacent to a common vertex in G, and this completes the proof.

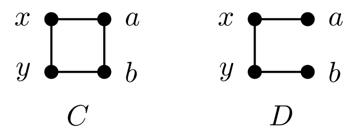

FIGURE 4. All possible cases for P_4 with vertices $\{a, x, y, b\}$ in G

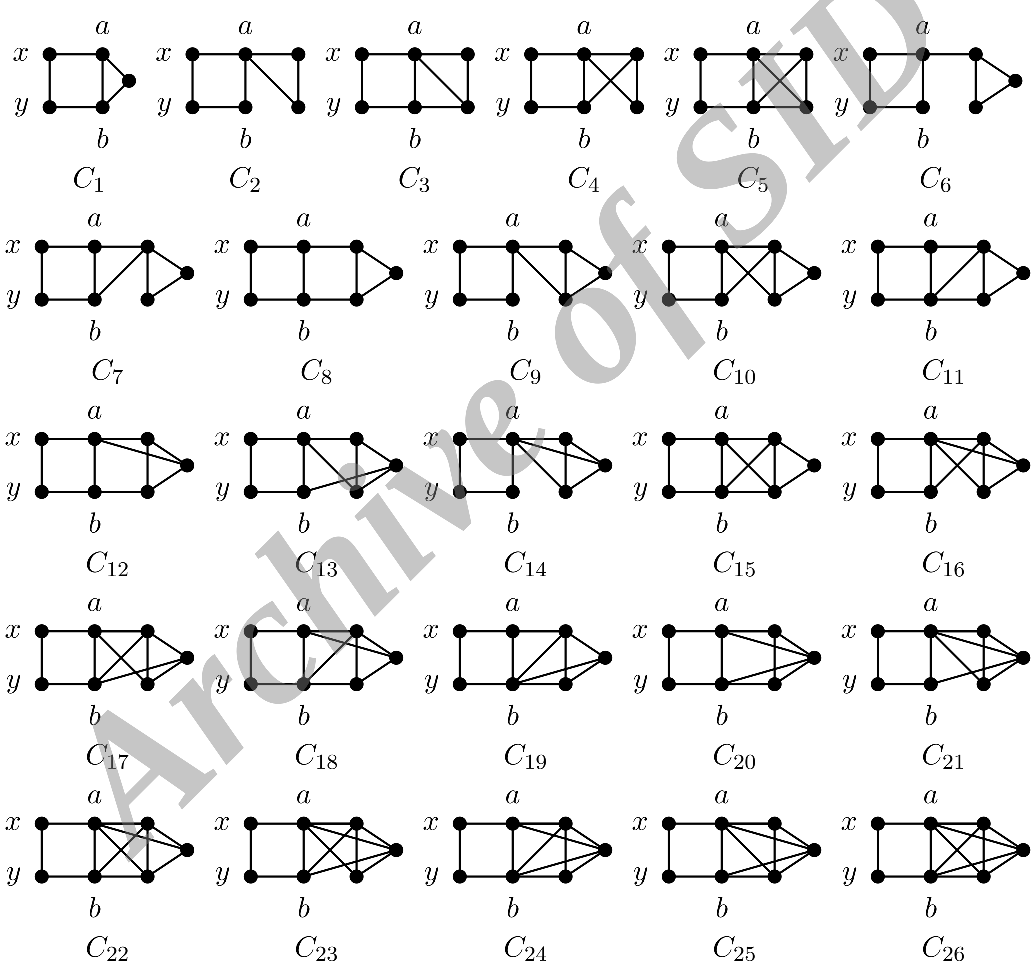

FIGURE 5. All possible cases between C and a triangle in G

□

Now suppose that G is a graph cospectral with F_n. We study the case in which two vertices of degree 2 in G are not adjacent. In this case, with one more condition we can prove that G is isomorphic to F_n.

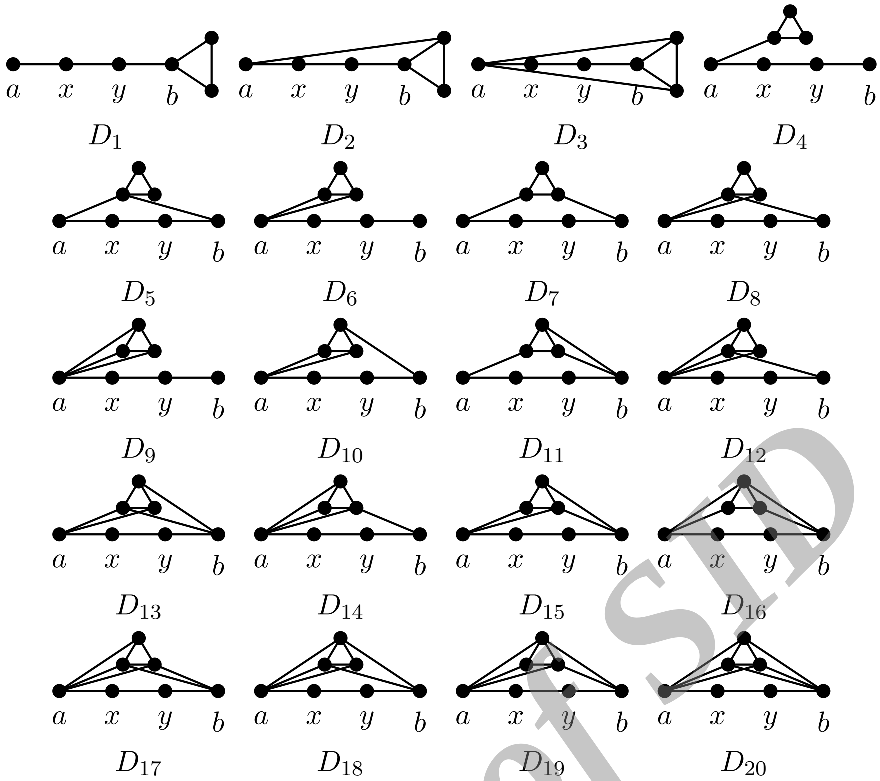

Figure 6. All possible cases between D and a triangle in G

Lemma 3.12. *Suppose that G is a graph cospectral with F_n. Let $\{x, y\}$ be two vertices of degree 2 in graph G, where these vertices are not adjacent. Then x and y does not have two common neighbors.*

Proof. Suppose, for contradiction, that x and y have common neighbors, say $\{a, b\}$. Thus one of the graphs in Figure 7 as a subgraph of G can occur. Suppose the adjacency matrix of G is $A(G)$ and, the first, second, third and fourth rows and columns of $A(G)$ are labeled by vertices x, y, a and b, respectively. The two first rows of $A(G)$ are identical, since $d_G(x) = d_G(y) = 2$ and they are not adjacent in G. Therefore, the dimension of the null space of $A(G)$ is greater than zero. Thus 0 is an eigenvalue of $A(G)$ and it is contradiction with cospectrality of G and F_n. This completes the proof. □

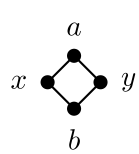

Figure 7. Both vertices x and y are adjacent to both a and b in G

It is known that the Kronecker product of paths P_2 and P_3, $P_2 \times P_3$, is two cycles C_4 that has a common edge.

Theorem 3.13. *Let $\{x, y\}$ be two non-adjacent vertices of degree 2 in G and G be $P_2 \times P_3$-free. If the vertices x and y have at least one common neighbour vertex, then G is isomorphic to F_n.*

Proof. Suppose the common neighbour vertex of two vertices x and y in G is z. Also, suppose x and y are adjacent to a and b, respectively. By Lemma 3.12, we can assume that $a \neq b$. So, we have the path P_5 with vertices $\{a, x, z, y, b\}$ as a subgraph of G. All possible induced subgraph that can be obtained from this P_5 are listed in Figure 8. The graphs E_4, E_5 and E_6 have two negative eigenvalues less than -1, so they can not happen in G. The vertex a in E_2 and E_3 must be join to an other vertex in G, say t. All possible cases for these two graphs with this new edge, are shown in Figure 9. All of them are forbidden subgraph of G. So, E_2 and E_3 can not happen in G. Therefore, the only case that can happen is E_1. Suppose the vertex a in E_1 is adjacent to vertex t of G, since the degree of a can not be 1. Now, the vertex t must be adjacent to some vertices of the set $\{z, b\}$, since G is P_6-free. It can not be adjacent to the both of z and b, since we do not have induced subgraph $P_2 \times P_3$. Also, t only is not adjacent to the vertex z, since its second largest eigenvalues are greater than 1. The only remaining case is that t be adjacent only to b. In this case we have an induced C_6 in G. If $d_G(b) = 2$, then by Lemma 3.10, G must be isomorphic to F_n and, nothing remain to prove. So, we must show that $d_G(b)$ can not be greater than 2. But, if $d_G(b) > 2$ and b is adjacent to the vertex f of G, by Interlacing Theorem, the vertex f must be adjacent to some vertices of the set $\{z, t, a\}$. But, all the resulted graphs are forbidden in G. This completes the proof. □

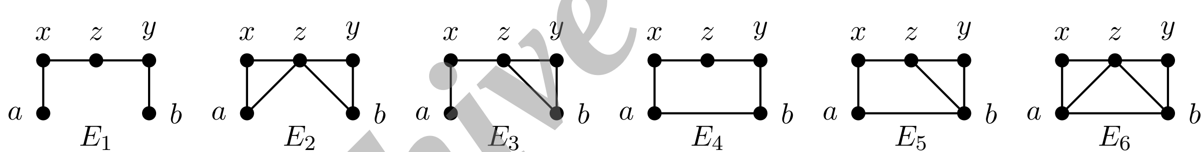

Figure 8. All induced subgraphs of P_5 of Lemma 3.13

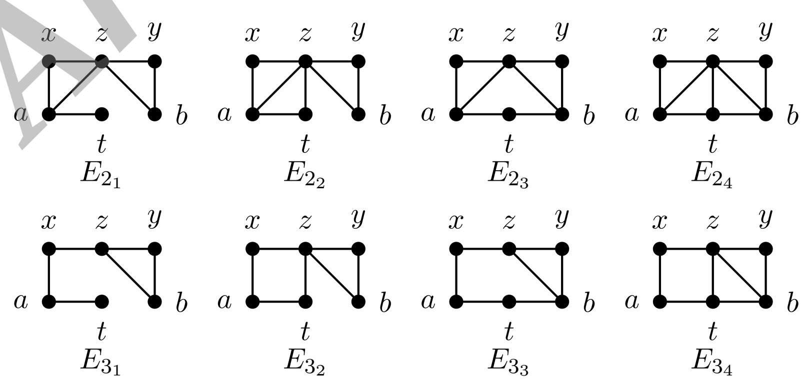

Figure 9. All induced subgraphs from E_2 and E_3 with one pendant at vertex a

Theorem 3.14. *The friendship graphs F_1, F_2 and F_3 are DS.*

Proof. The graph F_1 is isomorphic to graph K_3, and K_3 is *DS*. Suppose that G is graph cospectral with F_2, then by Theorem 2.5 and part (i) of Proposition 3.8, G must be connected. So G is connected and planar, since it does not have K_5 or $K_{3,3}$ as a subgraph. Therefore, by Corollary 2.9, G is isomorphic to F_2. Now we prove that F_3 is *DS*. Let G be a cospectral graph with F_3. By part (i) of Proposition 3.8, G must be connected. Also, G does not contain graphs K_5 or $K_{3,3}$ as a subgraph. So by Corollary 2.9, G is isomorphic to F_3 and this completes the proof. □

4. Complement of Cospectral Mate of Friendship Graph

In this section, we study the complement of graph G, where G is cospectral with friendship graph F_n. Also, we show that the complement of friendship graph is DS.

Lemma 4.1. *Let G be a cospectral graph with F_n for some $n > 2$. Then $\overline{G}$ is either connected or it is the disjoint union of K_1 and a connected graph.*

Proof. Since F_n has $3n$ edges, G has the same number of edges and so $\overline{G}$ has $n(2n+1) - 3n = n(2n-2)$ edges. If $\overline{G}$ were disconnected, then the vertex set of $\overline{G}$ is partitioned into two parts of sizes k_1 and k_2 such that there is no edges between any two vertices of these two parts. So K_{k_1,k_2} is a subgraph of G and it follows that $3n \geq k_1k_2$. Without loss of generality we may assume that $k_2 \geq k_1$. Since $n > 2$ and $k_1 + k_2 = 2n + 1$, it follows that $k_1 = 1$ and $k_2 = 2n$. This completes the proof. □

Theorem 4.2. *Let G be a cospectral graph with F_n for some $n > 2$. If $\overline{G}$ is disconnected, then G is isomorphic to F_n.*

Proof. By Lemma 4.1, $\overline{G}$ has two connected components $\overline{L}$ and $\overline{K}$, where $\overline{K}$ has only one vertex. It follows that the complement L of $\overline{L}$ has $2n$ vertices and n edges. If every vertex of L has degree 1 in L, then L is the disjoint union of n complete graphs K_2. In the latter case, G will be isomorphic to F_n, since G is the join of L and K which is the same graph F_n. Since $\overline{G}$ is disconnected, G is connected and so by Lemma 3.2 G has no vertices of degree 1. Thus L has no vertices with the property that it has no neighbors in L (i.e. L has no isolated vertices in itself). Since the number of edges of G and F_n are the same, L is a graph with $2n$ vertices and n edges without isolated vertices. Therefore L is the disjoint union of n copies of K_2. It follows that G is the cone of K_1 over L which means that G is isomorphic to F_n. □

Lemma 4.3. *Let G be a graph cospectral with F_n for some $n > 2$. Then the eigenvalues of the complement $\overline{G}$ of G are -2, 0 and the roots of the following polynomial*

$$x^4 + (4-2n)x^3 + (4-4n)x^2 + (4bn^2 + 4cn^2 - 2cn + 2bn - 2c)x + 8cn^2 - 4cn - 4c,$$

where b and c are non-negative real numbers such that $b + c \leq 1$.

Proof. By [18, Proposition 2.1.3],

$$P_{\overline{G}}(x) = (-1)^{2n+1} P_G(-x-1) \left(1 - (2n+1) \sum_{i=1}^{4} \frac{\beta_i^2}{x+1+\mu_i}\right), \tag{1}$$

where $\mu_1 = \frac{1+\sqrt{8n+1}}{2}$, $\mu_2 = 1$, $\mu_3 = -1$, $\mu_4 = \frac{1-\sqrt{8n+1}}{2}$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are the main angles of G, see [18, page 15]. We know

$$\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1, \tag{2}$$

see [18, page 15], and it follows from [18, Theorem 1.3.5] that

$$6n = (2n+1)\left(\mu_1\beta_1^2 + \beta_2^2 - \beta_3^2 + \mu_4\beta_4^2\right). \tag{3}$$

Now let $b := \beta_2^2$ and $c := \beta_3^2$. Using identities (2) and (3), one may simplify $P_{\overline{G}}(x)$ given in (1) as a product of the polynomial given in the statement of the lemma and some positive powers of polynomials x and $x+2$. This completes the proof. □

It is well known that the minimal non-isomorphic cospectral graphs are $G_1 = C_4 \cup K_1$ and $G_2 = K_{1,4}$, where $G_1 = \overline{F_2}$ and G_2 is complete bipartite graph. So, we can see that $\overline{F_2}$ is not DS. The natural question is: what happen for the complement of remaining friendship graphs? We answer to this question in the next theorem.

Theorem 4.4. *Let $\overline{F_n}$ denote the complement of friendship graph F_n. Then for $n \geq 3$, $\overline{F_n}$ is DS.*

Proof. It is easy to check that the complement of friendship graph F_n is $CP(n) \cup K_1$, where $CP(n)$ is cocktail party graph. The spectrum of $\overline{F_n}$ is as follows:

$$Spec(\overline{F_n}) = \left\{[-2]^{n-1}, [0]^{n+1}, [2n-2]^1\right\}.$$

Let G be cospectral with $\overline{F_n}$. Firstly, we prove that G can not be connected. Suppose G is connected. Because for $n \geq 3$, there are $\frac{1}{6}((2n-2)^3 - 8(n-1))$ triangles in G, G is not bipartite and specially is not complete bipartite graph. Also in graph G, $(2n+1)(2n-2)$ is not eqal to the $(2n-2)^2 + 4(n-1)$, so by Corollary 3.2.2 of [18], G is not regular and specially is not strongly regular graph. Now by Theorem 7 of [24], G must be one of these graphs; cone over Petersen graph, the graph derived from the complement of the Fano plane, the cone over the Shrikhande graph, the cone over the lattice graph $L_2(4)$, the graph on the points and planes of $AG(3,2)$, the graph related to the lattice graph $L_2(5)$, the cones over the Chang graphs, the cone over the triangular graph $T(8)$, and the graph obtained by switching in $T(9)$ with respect to an 8-clique. But these graphs have spectrum $\left\{[-2]^5, [1]^5, [5]^1\right\}$, $\left\{[-2]^7, [1]^6, [8]^1\right\}$, $\left\{[-2]^{10}, [2]^6, [8]^1\right\}$, $\left\{[-2]^{10}, [2]^6, [8]^1\right\}$, $\left\{[-2]^{14}, [2]^7, [14]^1\right\}$, $\left\{[-2]^{16}, [3]^7, [11]^1\right\}$, $\left\{[-2]^{21}, [4]^7, [14]^1\right\}$, $\left\{[-2]^{21}, [4]^7, [14]^1\right\}$, and $\left\{[-2]^{28}, [5]^7, [21]^1\right\}$, respectively. But, because of the spectrum of G, this is contradiction and G is not connected.

Now, suppose G is disconnected. By similar discussion in the proof of Theorem 2.5, G must be the disjoint union of a connected graph G_1 and some isolated vertices, so $G = G_1 \cup mK_1$, for some $m > 0$. If $m > 2$, then G_1 has $2n+1-m$ vertices and $\triangle(G_1) \leq 2n-3$, but by Theorem 3.2.1 of [18], this is contradiction, since the index of G_1 is $2n-2$. If $m = 2$, then G_1 has $2n-1$ vertices and $\triangle(G_1) \leq 2n-2$. But, the index of G_1 is $2n-2$ and again by Theorem 3.2.1 of [18] we must have $\triangle(G_1) = 2n-2$. In this case, G_1 is complete graph with $2n-1$ vertices, that is contradiction. So, by the first part of proof, we have $m = 1$ and $G = G_1 \cup k_1$. Therefore, the spectrum of G_1 is $\left\{[-2]^{n-1}, [0]^n, [2n-2]^1\right\}$. It is well known that $CP(n)$ is DS and $Spec(G_1) = Spec(CP(n))$. Therefore G_1 is isomorph to $CP(n)$ and it shows that $G = CP(n) \cup K_1 = \overline{F_n}$. So, we obtain that $\overline{F_n}$ is DS. □

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