OPTIMAL ANALYSIS OF NON-REGULAR GRAPHS USING THE RESULTS OF REGULAR MODELS VIA AN ITERATIVE METHOD

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ABSTRACT

In this paper an efficient method is developed for the analysis of non-regular graphs which contain regular submodels. A model is called *regular* if it can be expressed as the product of two or three subgraphs. Efficient decomposition methods are available in the literature for the analysis of some classes of regular models.

In the present method, for a non-regular model, first the nodes of the non-regular part of such model are ordered followed by ordering the nodes of the regular part. With this ordering the graph matrices will be separated into two blocks. The eigensolution of the non-regular part can be performed by an iterative method, and those of the regular part can easily be calculated using decomposition approaches studied in our previous articles. Some numerical examples are included to illustrate the efficiency of the new method.

1. INTRODUCTION

In our previous papers methods are already presented for the eigensolution and finding the

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inverse of block matrices. These matrices may arise in the adjacency or Laplacian matrices of the graphs [1]. The developed theorems can also be employed for structural models provided the necessary requirements mentioned in the theorems are fulfilled [2]. As an example, for some graph product and structural models these conditions are satisfied. Now we assume that for a given graph product or model with specified eigenvalues or inverse, some nodes or members are added. The main aim of the paper is the investigation of such problems. A special case of this model is when two graphs are connected to each other. For this purpose we first briefly study all those models for which the eigensolutions are available.

Some product graphs and closed form solutions for determining their eigenvalues are introduced in the next section. There are other product graphs for which no closed form solution are available, however it is possible to calculate their eigenvalues transforming the problem into solvable forms. Similar to the Laplacian matrices of graphs, one comes to the stiffness matrices of some structures for which the eigensolution or inverse can easily be obtained.

Now we add some node or member to such a model. For solving this problem one can use iterative methods. One of these methods is the condensation approach in which some of the degrees of freedom, denoted by s, are omitted and some others, denoted by m, are remained. The method of Guyan is one of the most important approaches of this class. This method is a static condensation in which the terms corresponding to the omitted degrees of freedom (DOFs) are ignored. Then to reduce the errors involved in static condensation, some modifications are performed in the solution of dynamic problems. One of such methods is introduced by Paz and in fact it is a generalized static condensation. Other methods are also developed in some of which the expansion of the Taylor series is employed for inverting the stiffness matrix. As more recent methods one can refer to those developed in [3-9].

Non-regular structures which can be transformed into regular ones are studied in [10]. Here the matrices are decomposed such that the inverse of the regular part can easily be obtained by the inverse of its blocks. For this purpose only a suitable ordering is needed. First the numbering of the nodes of the regular part is performed followed by the remaining nodes. This can be performed in two ways. Either some nodes and elements are added to restore the regularity or some nodes and members are deleted.

In Ref. [11] first the static and dynamic analysis of repeated part of the model is performed. In these structures we may have two or more kinds of repetitions. Similar to the previous case the ordering is performed such that the stiffness and mass matrices are decomposed into block forms and then the inversion is carried out by inverting the regular matrices with a small amount of computational effort.

The content of this paper is organized as follows: In section 2, graph products and the corresponding theorems are briefly presented. Then the proposed method is described in section 3. Numerical examples are presented in section 4. In some of these examples the eigenvalues and in some others the inverse of the corresponding problem are investigated. Section 5 concludes the paper.

2. PRELIMINARIES

Before we introduce some graph products we refer to two important matrices associated

with graphs.

The adjacency matrix A of a graph: This is square matrix with dimension as the number of nodes of the graph. The entries of this matrix are 0 unless the two nodes corresponding to ith row and the jth column are connected, in which case the entry will be 1.

The Laplacian matrix L of a graph: This matrix is defined as $L = D - A$ with D is a diagonal matrix, where each diagonal entry is the degree of the corresponding node (the number of member connected to that node).

When a sequence of n nodes of a graph are connected to each other, we will have a path denoted by P_n. If the first node and the last node of such a path coincide, then we will have a cycle, C_n.

The Kronecker product of two matrices A and B, denoted by $S = A \otimes B$, is a matrix the ijth entry of which is $a_{ij}B$. Now we introduce three graph products.

1. Cartesian Product: The Cartesian product of two graphs K and H denoted by $S = K \times H$ is defined as follows: For a pair of nodes $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $N(K) \times N(H)$ a member uv will exist in $M(S)$ if one of the following conditions hold:

$$u_1 = v_1 \quad and \quad u_2 v_2 \in M(H)$$
$$or \qquad\qquad\qquad\qquad (1)$$
$$u_2 = v_2 \quad and \quad u_1 v_1 \in M(K)$$

This means two nodes will be connected in Cartesian product if their first entries are identical the second entries will be connected to each other and vice versa.

2. Direct Product: The direct product of two graphs K and H denoted by $S = K * H$ is defined as follows: For a pair of nodes $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $N(K) \times N(H)$ a member uv will exist in $M(S)$ if one of the following conditions hold:

$$u_1 v_1 \in M(K) \quad and \quad u_2 v_2 \in M(H) \qquad (2)$$

This means two nodes will be connected in direct product if their first entries in the first graph and their second entries in the second graph are identical.

3. Strong Cartesian Product: The direct product of two graphs K and H, denoted by $S = K \boxtimes H$ is defined as follows: For a pair of nodes $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $N(K) \times N(H)$ a member uv will exist in $M(S)$ if one of the following conditions hold:

$$u_1 = v_1 \quad and \quad u_2 v_2 \in M(\,H\,)$$
$$or$$
$$u_2 = v_2 \quad and \quad u_1 v_1 \in M(\,K\,) \tag{3}$$
$$or$$
$$u_1 v_1 \in M(\,K\,) \quad and \quad u_2 v_2 \in M(\,H\,)$$

This means that the strong Cartesian product is the combination of the previously described products. Having these definitions, now the theorems corresponding to the eigenvalues of the Laplacian matrices of these products are described:

Let us assume M to be as the sum of some Kronecker products, as

$$M = \sum_{i=1}^{k} (\,A_i \otimes B_i\,) \tag{4}$$

Now if the matrix P diagonalizes all the A_is simultaneously, then it is previously shown that $U = P \otimes I$ can also block diagonalize the matrix M. The necessary and sufficient condition for P to exists is that all pairs of A_is commute, i.e.

$$A_i A_j = A_j A_i \tag{5}$$

Then

$$\lambda_M = \bigcup_{i=1}^{n} eig(\,M_i\,) ; M_i = \sum_{j=1}^{k} (\,\lambda_i(\,A_j\,) B_j\,) \tag{6}$$

In this relation, the dimension of A_i is equal to n, and that of B_i is equal to m.

One can perform similar calculations if the matrix P simultaneously diagonalizes the B_i matrices, since

$$\sum_{i=1}^{k} (\,A_i \otimes B_i\,) = \sum_{i=1}^{k} (\,B_i \otimes A_i\,) \tag{7}$$

If A_is and B_is posses the commutativity property, then the calculations will be easier, because

$$eig(\,\sum_{i=1}^{k} A_i \otimes B_i\,) = \sum_{i=1}^{k} eig(\,A_i \otimes B_i\,) \tag{8}$$

Another special case is when a matrix is block circulant. For this type of matrices, the eigenvalues can easily be obtained as described in [12].

Now consider the following set of equations:

$$Mx = B \qquad (9)$$

It is obvious that after calculating the eigenvalues and eigenvectors of M, the set of equations (9) can be solved [13]. If λ_i s and $\{\varphi\}_i$ s are the eigenvalues and eigenvectors of M, the introducing $B_j = \{\varphi\}_j^t B$ we will have:

$$y_j = \frac{B_j}{\lambda_j} \Rightarrow \{x\}_n = \sum_{i=1}^{n} \{\varphi\}_i y_i = \sum_{i=1}^{n} \{\varphi\}_i \frac{B_i}{\lambda_i} = \sum_{i=1}^{n} \frac{\{\varphi\}_i \{\varphi\}_i^t}{\lambda_i} B \qquad (10)$$

If A_i s and B_i s have do not posses the commutativity property with respect to multiplication, then for the case $k = 2$ one can use QZ transformation for the solution of Eq. (9). This transformation is introduced in Ref. [14]. Here and also for Eq. (10), one does not need to calculate the stiffness matrix.

However, one can easily find M^{-1} having the eigenvalues and eigenvectors of M. Having the matrix V of eigenvectors, and the matrix D of eigenvalues in its diagonal, then $M = VDV^t$. Since the eigenvalues of M^{-1} are the inverse of those of M with identical eigenvectors, therefore

$$M^{-1} = VD^{-1}V^t = V \begin{bmatrix} 1/\lambda_1 & & & & 0 \\ & 1/\lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 1/\lambda_{m*n} \end{bmatrix} V^t \qquad (11)$$

In this relation D^{-1} can easily be obtained by finding the inverse of the diagonal entries of D. The eigenvector of such a matrix will be $u \otimes v$ in which u is a vector that diagonalizes the two matrices A_1 and A_2 simultaneously, and v is the eigenvector of $M_i = \sum_{j=1}^{k} (\lambda_i (A_j) B_j)$.

3. THE PROPOSED METHOD

In this method with transforming the matrices of graphs into block form, and employing dynamic condensation the eigenvalues are calculated. For simplicity a graph for which the eigenvalues can be calculated using the above mentioned method (or any other method) is denoted by S. The dimension of the matrices of this graph is denoted by s which corresponds to the DOFs which is supposed to be omitted. We also assume that the primary matrices have dimension equal to n.

In general we aim at solving the following eigenvalue problem:

$$(A - \lambda_i B)\Phi_i = 0 \qquad ; \qquad i = 1:n \tag{12}$$

This matrix can be decomposed into two parts havening dimensions of m and s, where $n = m + s$. It is obvious s corresponds to the part for which the inverse can be found using the previously developed methods.

$$\left(\begin{bmatrix} A_{mm} & A_{ms} \\ A_{ms}^t & A_{ss} \end{bmatrix} - \lambda \begin{bmatrix} B_{mm} & B_{ms} \\ B_{ms}^t & B_{ss} \end{bmatrix} \right) \begin{pmatrix} \Phi_m \\ \Phi_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

The calculations corresponding to the eigenvalues and eigenvectors can be performed using the iterative approach of [5]. Summary of these calculations and application can be found in Ref. [11].

It should be noted that in structural problems the smallest eigenvalues [11] and in the graph Laplacian matrices the second eigenvalues are of importance.

It can be recognized that in the present method ordering plays and import role. According to the above explanations, first a subgraph S of the graph is selected. This part is selected such that the inverse of the corresponding matrix can be calculated. The remaining part of the model is denoted as M. For nodal ordering, first the nodes of M and then the nodes of S are numbered.

Here applications of this method in graph are described. One of these applications is the calculation of the eigenvalues of the Laplacian matrices of direct and strong Cartesian graph products. For these products the Laplacian matrices can be expressed as the sum of two Kronecker products. Defining

$$F_n(A_m, B_m, C_m, D_m) = \begin{bmatrix} A_m & B_m & D_m \\ B_m & C_m & B_m & D_m \\ D_m & B_m & C_m & B_m \\ & D_m & . & . & . \\ & & . & . & . \\ & & & . & . & . & D_m \\ & & & & B_m & C_m & B_m & D_m \\ & & & & D_m & B_m & C_m & B_m \\ & & & & & D_m & B_m & A_m \end{bmatrix}_n \tag{14}$$

It should be mentioned that when this matrix is used with 3 arguments, then we will have $D_m = 0$.

In this way for the direct product we will have

$$M_{ef} = F_f(0,1,0) \otimes F_e(0,-1,0) + F_f(1,0,2) \otimes F_e(1,0,2) \tag{15}$$

and for strong Cartesian product we have

$$M_{ef} = F_f(1,1,1) \otimes F_e(-1,-1,-1) + F_f(2,0,3) \otimes F_e(2,0,3)$$ (16)

Here both terms are in the form $A_1 \otimes B_1 + A_2 \otimes B_2$. However, in both terms $A_1 A_2 \neq A_2 A_1$. Thus one can not block diagonalize using the given form.

In Ref. [15] members were added to the four edges of the graph to perform the calculations. This changes A_2 and B_2 in the direct product to $2I$ and in strong Cartesian product it changes to $3I$, to provide the decomposability condition. However, in Ref. [1] it is shown that one does not need to alter B_2 and changes on A_2 is sufficient for block diagonalization.

Since the addition of members to all four edges of the graph is unnecessary and two opposite edges are sufficient (e.g. upper and lower edges), thus M_{ef} can be written as

$$M_{ef} = I_f \otimes (A+B)_e + F_f(1,-1,2) \otimes (-B)_e$$ (17)

Since in this case $I_f T_f = T_f I_f$, therefore M_{ef} can be diagonalized and we have

$$eig(M_{ef}) = \bigcup_{i=1}^{f} \{ eig[A_e - (1 + 2\cos\frac{i\pi}{f})B_e] \}$$ (18)

Due to the change on the primary graph, the magnitude of λ_2 will have some approximation. In order to improve this approximation one can use Rayleigh's method. In this method there is no need to add member to the edges of the model and one can use the relationships which are described in Ref. [11] for finding the eigenvalues. Details of this approach will be illustrated in Example 1.

For some repeated graphs when we use this method to find the eigenvalues, the inverse of a special block matrix is involved in the following form, and for simplicity we denote it by R:

$$R_n(A,B,B^t) = \begin{bmatrix} A & B & 0 & 0 & . & . & 0 \\ B^t & A & B & 0 & & & \\ 0 & B^t & A & . & & & \\ 0 & 0 & . & & . & & \\ . & & & . & . & B & \\ . & & & & B^t & A & B \\ 0 & & & & & B^t & A \end{bmatrix}_n$$ (19)

In this matrix, the submatrices A, B, B^t are $l \times l$ matrices.

If R contains two blocks B and B^t in its corners, then we will have a circulant matrix and one can then easily obtain the eigensolution and inverse of the matrix [12]. Thus we rewrite R in the following form:

$$R = L' + L'' \Rightarrow \begin{bmatrix} A & B & 0 & & & \\ B^t & A & B & & & \\ 0 & B^t & A & B & & \\ & & & . & & \\ & & & . & B \\ & & & & B^t & A \end{bmatrix}_{n \times n} = \begin{bmatrix} A & B & 0 & & & B^t \\ B^t & A & B & & & \\ 0 & B^t & A & B & & \\ & & & . & & \\ & & & . & B \\ B & & & & B^t & A \end{bmatrix}_{n \times n} + \begin{bmatrix} 0 & 0 & 0 & & & -B^t \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ & & & . & & \\ & & & . & 0 \\ -B & & & & 0 & 0 \end{bmatrix}_{n \times n} \quad (20)$$

Showing the inverse of L' by E, then we will have

$$R^{-1} = \left[L' + L'' \right]^{-1} = \left\{ L' \left[I + L'^{-1} L'' \right] \right\}^{-1} = \left[I + E L'' \right]^{-1} E \quad (21)$$

Exchanging the row and columns of block n with those of block 2 in matrices L'' and E, these are transformed into two block matrices:

$$\overline{L''} = \begin{bmatrix} 0 & -B^t & & & & 0 \\ -B & 0 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & 0 & 0 \\ 0 & & & & 0 & 0 \end{bmatrix}_{n \times n} = \begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (22)$$

$$L_1 = \begin{bmatrix} 0 & -B^t \\ -B & 0 \end{bmatrix} = -R_2(0, B^t, B)$$

Where $\overline{L''}$ is the matrix as L'' after the exchange of the row and column. Similar to the method presented in [16] and using Eq. (21), we obtain:

$$\left[I + E L'' \right]^{-1} E = \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix} \right\}^{-1} E$$
$$= \begin{bmatrix} I + E_{11} L_1 & 0 \\ E_{21} L_1 & I \end{bmatrix}^{-1} E = \begin{bmatrix} (I + E_{11} L_1)^{-1} & 0 \\ -E_{21} L_1 (I + E_{11} L_1)^{-1} & I \end{bmatrix} E \quad (23)$$

The inverse of the matrix L' can be obtained using its eigenvalues, since

$$L' = I \otimes A + H \otimes B + H^t \otimes B^t = \sum_{i=1}^{3} A_i \otimes B_i \qquad (24)$$

where H is a rotation matrix in the form:

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ & & & & . & . & \\ 0 & & & & & . & 1 \\ 1 & 0 & & & & & 0 \end{bmatrix}_{n \times n} \qquad (25)$$

Here H is an orthogonal matrix and the matrices H and H^t are commutative with respect to multiplication, i.e.,

$$HH^t = H^t H = I \qquad (26)$$

Therefore considering Eq. (6) and Eq. (24), the eigenvalues of L' is obtained as follows:

$$eig(L') = \bigcup_{i=1}^{n} eig(L_i') \ ; \ L_i' = \lambda_i(A_1)B_1 + \lambda_i(A_2)B_2 + \lambda_i(A_3)B_3 \qquad (27)$$

After calculating the eigenvalues, using Eq. (11) the inverse of L' is obtained.
The inverse of R can also be obtained with the following simpler approach:

$$R(A,B,B^t) = \begin{bmatrix} A & B & 0 & 0 & . & . & B^t \\ B^t & A & B & 0 & & & \\ 0 & B^t & A & B & & & \\ 0 & 0 & B^t & A & & & \\ . & & & & . & & \\ . & & & & & . & B \\ B & & & & B^t & A \end{bmatrix} - \begin{bmatrix} B^t & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ . & . \\ . & . \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & . & . & I \\ I & 0 & 0 & 0 & . & . & 0 \end{bmatrix} = L' - UV \qquad (28)$$

$$\Rightarrow R^{-1}(A,B,B^t) = (L' - UV)^{-1} = L'^{-1} + L'^{-1}U(I - VL'^{-1}U)^{-1}VL'^{-1}$$

From the above formula it can be seen that this kind of formulation for obtaining the inverse of R requires the inversion of two matrices. One is $I - VL'^{-1}U$ which is the same as $I + E_{11}L_1$, and the second one, L', which is the sum of three Kronecker products and its eigenvalues and hence its inverse can be obtained from n eigenvalues of the matrix.

Here the repeated matrices which do not satisfy Eq. (5) and thus are not decomposable

are studied. Using some matrix operations, the inverse of these systems is related to the inverse of some decomposable block matrices. In this way simple analytical relations are obtained for finding the inverse of repeating graphs. In Example 2 some of such graphs are introduced and investigated.

Also there are some graphs which do not satisfy Eq. (5), however, with a proper nodal ordering one can identify a subgraph for the condition of decomposiblity holds, as shown in Example 3.

Another application consists of connecting two graphs which is studied in Example 4. Another application of the present approach is studying those graphs which become a product graph by the addition of some members, as illustrated in Example 5.

4. NUMERICAL EXAMPLES

Example 1: In this example the aim is to find the eigenvalues of the Laplacian matrix of a graph in the form of strong Cartesian product. The strong Cartesian product $P_4 \boxtimes P_5$ is shown in Figure 1. The main problem is that the upper and lower nodes are not identical to the intermediate nodes and if only the intermediate nodes are considered then one can find the eigenvalues followed by the inverse of the corresponding matrix. Therefore for nodal ordering first the upper and lower nodes are labeled followed by the numbering of the intermediate nodes.
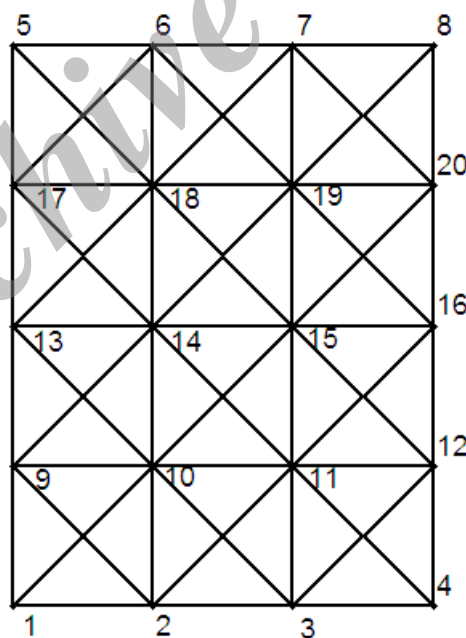

Figure 1. The Cartesian product $P_5 \boxtimes P_4$ and its nodal numbering

Considering the above nodal ordering the Laplacian matrix is decomposed into two blocks as follows:

$$L = \begin{bmatrix} L_{mm} & L_{ms} \\ L_{ms}^t & L_{ss} \end{bmatrix}$$

Here $m = 8$ and $s = 12$. Therefore instead of calculating the eigenvalues of a matrix of dimension 20, only a matrix of dimension 12 is inversed together with an eigensolution of dimension 8. The L_{ss} matrix is in the following form:

$$L_{ss} = \begin{bmatrix} A & B & 0 \\ B & A & B \\ 0 & B & A \end{bmatrix} = F_3(A,B,A) = I_3 \otimes F_4(5,-1,8) + F_3(0,1,0) \otimes F_4(-1,-1,-1)$$

The important point is that for inverting L_{ss} there is no need for direct calculations since in this matrix $A_1A_2 = A_2A_1$ and eigenvalues are obtained using Eq. (6) and the inversion is performed employing Eq. (11). Thus we have

$$\lambda_L = \bigcup_{i=1}^{3} \left[\lambda_i(I_3)F_4(5,-1,8) + \lambda_i(F_3(0,1,0)F_4(-1,-1,-1) \right]$$

In this way, instead of solving an eigenvalue problem of dimension 20, the eigenvalues of three matrices of dimension 4 and one of dimension 8 should be calculated.

Example 2: Suppose we want to find the second eigenvalue of the Laplacian matrix of the graph shown in Figure 2. This graph has a repeated form and we use the dynamic condensation approach. As it can be seen from the figure, first the nodes of the right-hand side and left-hand side (non-repeated parts) are numbered followed by the nodal numbering of the internal nodes.
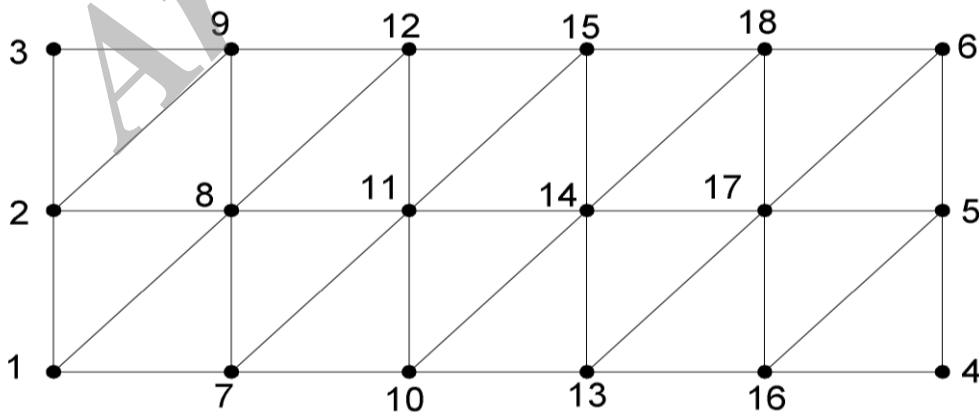

Figure 2. A repeated graph and a suitable nodal numbering for the formation of the Laplacian matrix in a 2-block form

With the above numbering the Laplacian matrix will have the following form:

$$L = \begin{bmatrix} C & 0 & B & 0 & 0 & 0 \\ 0 & C' & 0 & 0 & 0 & B^t \\ B^t & 0 & A & B & 0 & 0 \\ 0 & 0 & B^t & A & B & 0 \\ 0 & 0 & 0 & B^t & A & B \\ 0 & B & 0 & 0 & B^t & A \end{bmatrix} = \begin{bmatrix} L_{mm} & L_{ms} \\ L_{ms}^t & L_{ss} \end{bmatrix}$$

Where

$$L_{ss} = R_4(A, B, B^t), L_{ms}^t = \begin{bmatrix} B^t & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B \end{bmatrix}, \quad L_{ms} = \begin{bmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & B^t \end{bmatrix}, L_{mm} = \begin{bmatrix} C & 0 \\ 0 & C' \end{bmatrix}$$

In these relations the matrices A, B, C and C^t are 3×3 matrices. The nodes 1 to 6 are shown with subscript m and the nodes 7 to 18 are denoted by subscript s. Similar to the previous section, in using the dynamic condensation we will need the inversion of a matrix of dimension L_{ss} (which is equal to 12 in this example) and an eigensolution problem of a matrix of dimension L_{mm} (that is equal to 6 in this example).

Therefore we will need to find the inverse of the matrix R. For calculating this inverse one can use either Eqs. (20 to 27) or Eq. (28). In this example, $n = 4$ and $l = 3$.

As it can be seen from Eqs. (20-27), in this method instead of finding the inverse of R (of dimension 12), the inverse of two regular matrices $I + E_{11}L_1$ (Eq. (23)) and L' (Eq. (24) are needed. Where the first one is of dimension 6, and the second needs the eigensolution of 4 matrices of dimension 3.

If we use Eq. (28), then for finding the inverse of R we need to find the inverse of two matrices. One matrix is $I - VL'^{-1}U$ which is the same as $I - VL'^{-1}U$ and in this example it is a 2-block matrix each block being of dimension 3. The other one is L' which is the sum of three Kronecker products, and the eigenvalues and hence inversion is obtained from the eigenvalues of 4 matrices of dimension 3.

Similar calculations are applicable to other repeated graphs. Examples of such cases are strong Cartesian products and direct products, where Eq. (5) is not applicable. Some of such graphs are provided in Figure 3. For the first two graphs, the nodal numbers are also shown.
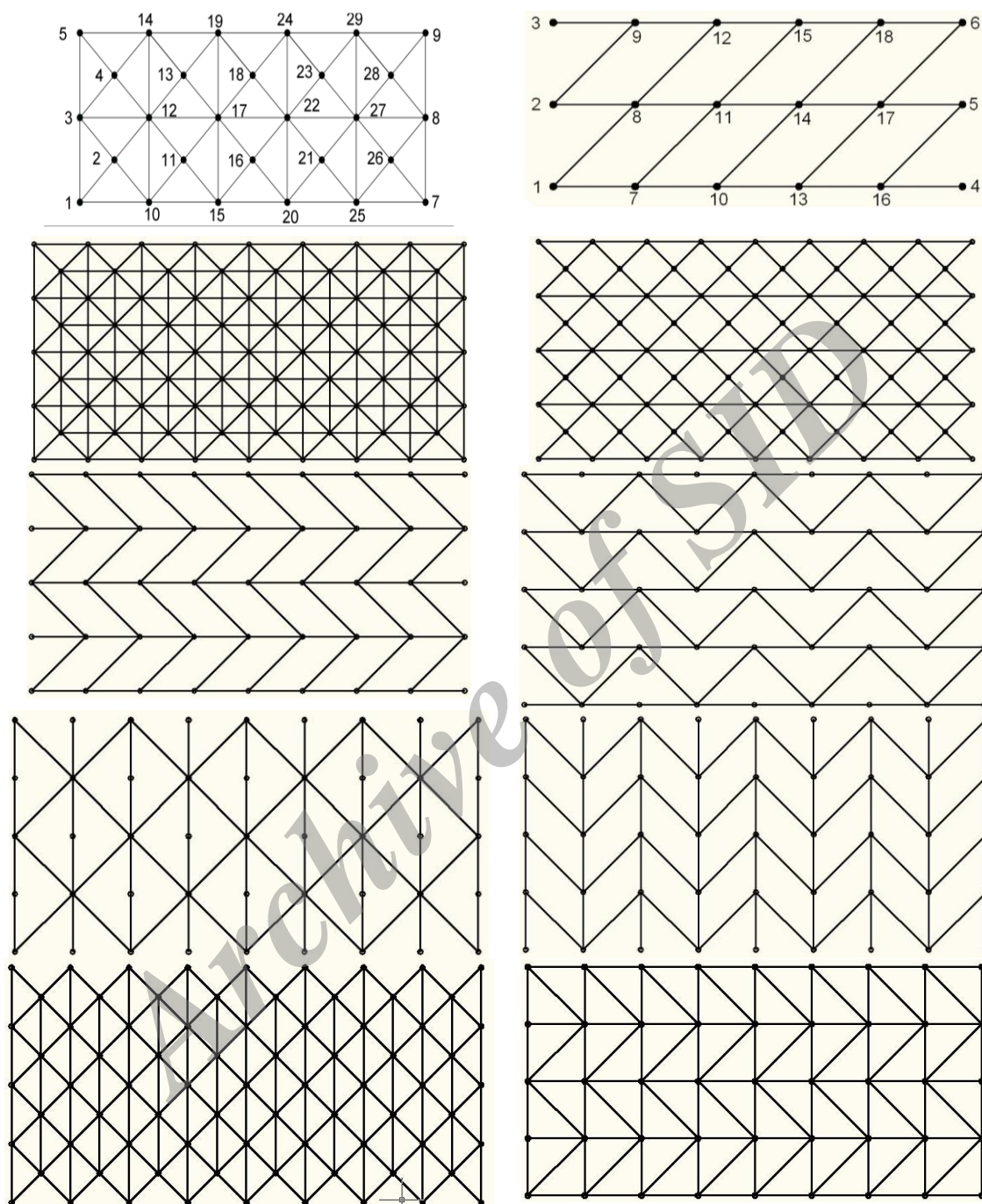
Figure 3. Different forms of repeated graphs and suitable nodal numbering of two models

Example 3: In Figure 4(a) a graph is shown which is similar to strong Cartesian product, and the only difference is that the crossing point are considered as nodes belonging the subgraph S. The remaining nodes are contained in the subgraph M. Using the aforementioned numbering, the matrix L_{ss} will be equal to $4I_6$, where I is a unit matrix.

Therefore $L_{ss}^{-1} = \dfrac{1}{4}I_6$ and it is sufficient to calculate the eigenvlaues of a matrix of dimension $m = 12$. It should be mentioned that for calculating the eigenvalues of this matrix similar to the previous example, one can decompose the graph into two subgraphs with $m = 6$ and $s = 6$. For this example the second eigenvalue of the Laplacian matrix is obtained as $\lambda_2 = 0.7226$.

As an application, the eigenvector of this eigenvalue, known as the Fiedler vector, is obtained for Figure 2(a), and the nodal ordering is performed for reducing the profile of the stiffness matrix. The Fiedler vector is as

$$v_2 = \{\,0.3156, 0.3074, 0.3156, 0.1375, 0.1365, 0.1375, -0.1375, -0.1365, -0.1375, -0.1356, -0.3074,$$
$$-0.3156, 0.2737, 0.2737, 0, 0, -0.2737, -0.2737\,\}$$

Ordering the nodes according to this vector the new nodal numbers is obtained as illustrated in Figure 4(b).
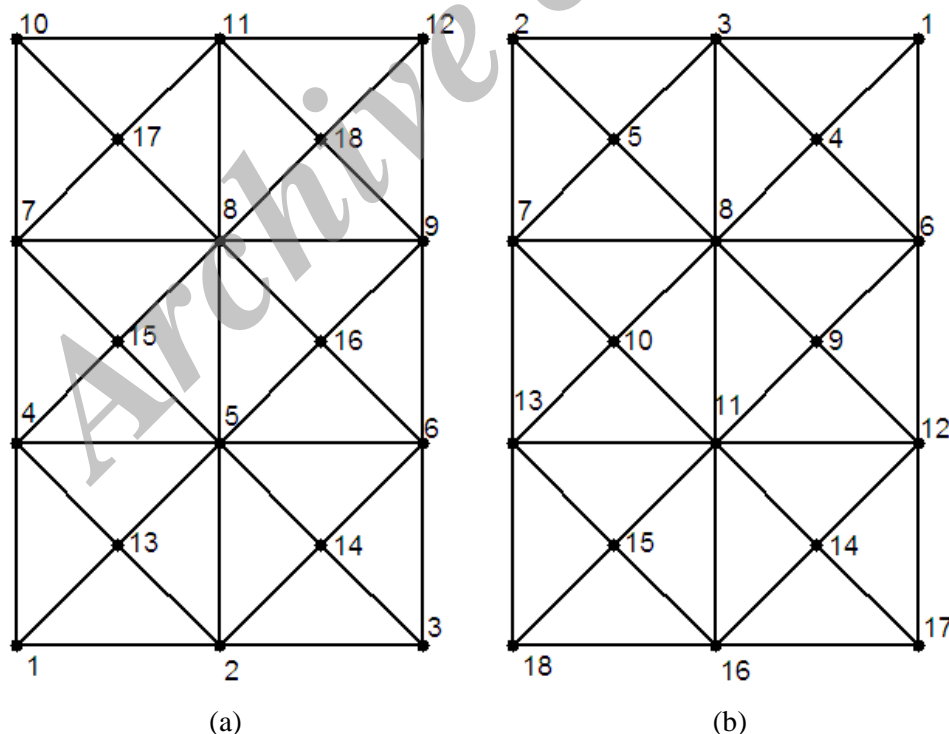

(a)   (b)

Figure 4. (a) The graph introduced in Example 3, (b) Nodal ordering using Fiedler's vector

Example 4: In this example we want to calculate the eigenvalues of graph obtained by connecting two subgraphs. An example of such a graph is illustrated in Figure 5(a). One subgraph is the product graph $P_7 \boxtimes P_4$ and the other one is the product graph $P_3 \times P_3$. In nodal numbering one should pay special attention to the block which should be inverted and its eigenvalues should be calculable. Similar to the previous example, the intermediate part of the strong Cartesian product is considered and numbered, and the remaining nodes are number next.
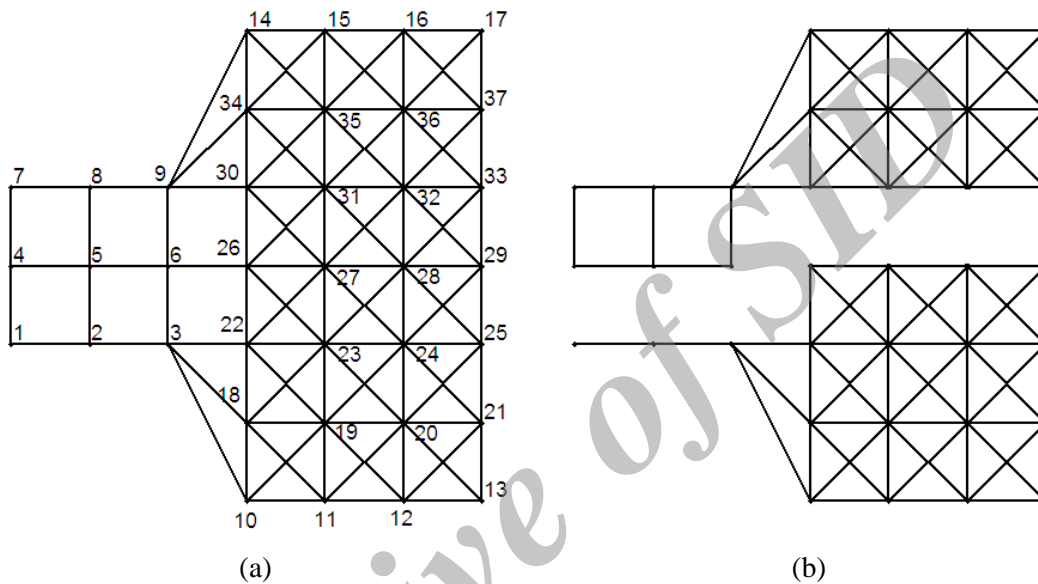

Figure 5. (a) Two graphs connected to each other and the nodal numbering (b) Partitioning of the graph

Therefore for this graph we have

$$L = \begin{bmatrix} L_{mm} & L_{ms} \\ L_{ms}^t & L_{ss} \end{bmatrix}$$

Here we have $m = 17$ and $s = 20$. With this numbering and inverting a matrix of dimension 20, we will face an eigenvalue problem of dimension 17. The matrix L_{ss} we have the following form:

$$L_{ss} = F_5(A, B, A) = I_5 \otimes \begin{bmatrix} 6 & -1 & 0 & 0 \\ -1 & 8 & -1 & 0 \\ & -1 & 8 & -1 \\ 0 & 0 & -1 & 5 \end{bmatrix} + F_5(0,1,0) \otimes F_4(-1,-1,-1)$$

In this way instead of inverting a matrix of dimension 20, one should find the eigesolution of 5 matrices of dimension 4 to be employed in Eq. (11). In this problem, the

relationships for calculating of eigenvalues in [5] are used twice and $\lambda_2 = 0.3078$ is obtained, while the exact answer is $\lambda_2 = 0.3101$. As an application this eigenvalue is used to find the Fiedler's vector. Ordering the entries of this vector, the model can be bisected such that the two obtained subgraphs have identical number of members and the number of members connecting the two subgraphs is minimum. Figure 5(b) shows these two subgraphs. This approach is extensively employed in parallel computing.

Example 5: We want to study a graph obtained by addition of two members to the product graph $P_5 \boxtimes C_4$. The Laplacian matrix of this graph will have the following form, and its eigenvalues can be calculated as follows:

$$L_{mn} = F_n(A_m, B_m, C_m)$$
$$A_m = G_m(5, -1, 5) \quad , \quad B_m = G_m(-1, -1, -1) \text{ and } \quad C_m = G_m(8, -1, 8)$$

Since

$$A_m = 6I_m + B_m, \quad C_m = 9I_m + B_m$$

Therefore

$$L_{mn} = F(6I_m + B_m, B_m, 9I_m + B_m) = 3F_n(2,0,3) \otimes I_m + F_n(1,1,1) \otimes B_m$$

We know that

$$eig(B_m) = -(1 + 2 \cos \frac{2k\pi}{m}) \quad k = 1 : m$$

Using Eq. (7) we have

$$eig(\sum(A_i \otimes B_i)) = eig(\sum(B_i \otimes A_i))$$

Therefore

$$eig(L_{mn}) = eig\{ I_m \otimes 3F_n(2,0,3) + B_m \otimes F_n(1,1,1)\}$$

Since both I_m and B_m commute in the multiplication, thus we have

$$eig(L_{mn}) = \bigcup_{k=1}^{m}[eig\{ 3F_n(2,0,3) - (1 + 2 \cos \frac{2k\pi}{m})F_n(1,1,1)\}]$$

As it can be observed from Figure 6, we have $m = 12$ and $s = 8$. Here the matrix L_{ss} has the following form:

$$L_{ss} = G_4(A, B, A) \quad ; \quad A = \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix} \quad ; \quad B = -I_4$$

where

$$G_n(A_m, B_m, C_m) = \begin{bmatrix} A_m & B_m & & & & & & B_m \\ B_m & C_m & B_m & & & & & \\ & B_m & C_m & B_m & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & B_m & C_m & B_m & \\ & & & & & B_m & C_m & B_m \\ B_m & & & & & & B_m & A_m \end{bmatrix}_n$$

For this matrix $A_1 A_2 = A_2 A_1$ and therefore the calculation of the eigenvalues and the inverse of the matrix becomes feasible. In this example utilizing the relationships for calculating of eigenvalues (see [5]) one obtains $\lambda_2 = 1.1001$, while the exact answer is $\lambda_2 = 1.1027$.
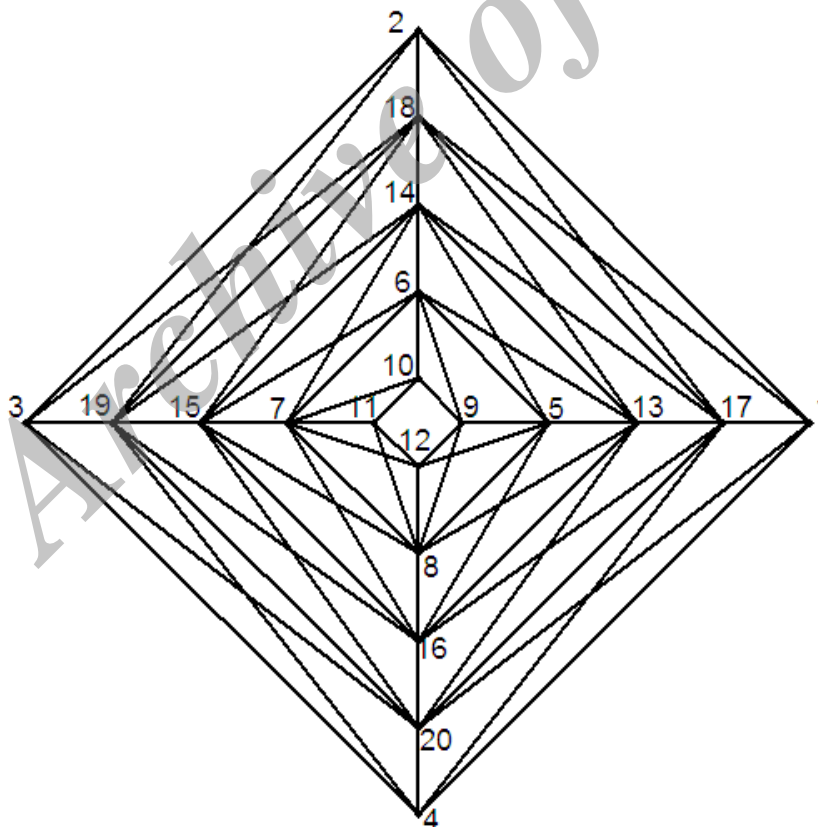

Figure 6. The product graph $P_5 \boxtimes C_4$ with two deleted members

5. CONCLUDING REMARKS

In this paper the analysis of those graphs are studied from which a regular model can be extracted, and inversion is performed using the previously developed methods.

The method developed in this paper is an iterative approach, and its important feature is that it performs the inversion and eigensolution on matrices smaller dimension than the matrix of the original model only once. This means that in the iterations of this method, only we have only numerical calculations and not matrix calculation. It is seen that after decomposition the dimensions of the matrices are reduced [17] .

In this way when a graph is irregular graph, has some a regular submodels, first the nodes corresponding to the irregular submodel and we show the corresponding Laplacian matrix by subscript m . According to dynamic condensation approach, the dimension of this matrix specifies the number of eigensolution problems which should be solved. Also the number of nodes in the regular part specifies the number of inversion that should be performed. In the process of inversion we face matrices which are in the form $R_n(A, B, B^t)$, and if this matrix contains n blocks each block being of dimension l, then for inversion n times the eigenvalues of dimension l should be calculated, and also the inverse of the matrix $I + E_{11}L_1$ or $I - VL^{r-1}U$ of dimesion $2l$ will be needed.

Different types of graphs are studied in this paper. The examples consist of product graphs which either do not satisfy Eq. (5), or can be transformed into solvable forms by adding or removing some members. In another example, two graphs are connected to each other. Also some new product graphs together with the method of calculating their eigenvalues are provided.

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