ASIAN JOURNAL OF CIVIL ENGINEERING (BUILDING AND HOUSING) VOL. 6, NO. 4 (2005)
PAGES 211-246

CONSTITUTIVE MODELLING OF CONCRETE: AN OVERVIEW

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ABSTRACT

In the last three decades, the constitutive modelling of concrete evolved considerably. This paper describes various developments in this field based on different approaches such as elasticity, plasticity, continuum damage mechanics, plastic fracturing, endochronic theory, microplane models, etc. In this article the material is assumed to undergo small deformations. Only time independent constitutive models and the issues related to their implementation are discussed.

Keywords: constitutive modelling, plasticity, failure criteria, continuum damage mechanics, endochronic theory, microplane model

1. INTRODUCTION

Concrete is a heterogeneous, cohesive-frictional material and exhibits complex non-linear inelastic behaviour under multi-axial stress states. The increased use of concrete as primary structural material in building complex structures such as reactor vessels, dams, offshore structures, etc., necessitates the development of sophisticated material models for accurate prediction of the material response to a variety of loading situations. The new developments which are taking place in the area of concrete technology resulted in new generation of concretes, which are better in terms of performance, such as high strength concrete (HSC) (Khaloo and Ahmad [106], ACI state-of art report [2], Candappa et al. [36]), reactive powder concrete (RPC), high performance light weight concrete (HPLC) and self compacting concrete, etc. Kmita [109] and Aitcin [4] further stressed the need for new material models.

Concrete structures are often analyzed by means of the finite element method. Analysis of a structural engineering problem by finite element method is based on solution of a set of equilibrium equations and a kinematically admissible displacement field. These are supplemented by boundary and initial conditions of a particular problem. These statically and kinematically admissible sets are independent of each other, and to link them material

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constitutive relations are required [35]. In recent decades, considerable effort has been undertaken to achieve this goal has resulted in partial success. With the present state of development of computer programs related to finite element method, inadequate modelling of engineering materials in general and concrete in particular is often one of the major factors limiting the capability of structural analysis (Chen [54], Bouzaiene and Massicotte [32]).

Concrete contains a large number of micro-cracks, especially at the interface between aggregates and mortar, even before the application of the external load. Many theories proposed in the literature for the prediction of the concrete behaviour such as empirical models, linear elastic, nonlinear elastic, plasticity based models, models based on endochronic theory of inelasticity, fracturing models and continuum damage mechanics models, micromechanics models, etc., are discussed in the following sections.

2. EMPIRICAL MODELS

The material constitutive law is, in general gained through a series of experiments [52, 92, 10]. The experimental data is then used to propose functions, which describe the material behaviour, by curve fitting. Obtaining the experimental data is not so easy. Even for the uniaxial case, there is little information available on strain softening portion and the difficulties are much more in case of multiaxial stress situations. One reason for insufficient experimental information after peak is due to difficulties associated with the testing techniques of materials [159]. Many testing machines used for standard compression test apply increasing loads rather than deformation which results in uncontrolled sudden failure after peak load. Several investigators have developed techniques to overcome this difficulty but some of them are costly which require stiff testing equipment which is not available in a normal testing lab (Wang et al. [197], Shah [173]). In most laboratories, cylindrical specimens are used for triaxial testing but the type of loading is unfortunately not truly triaxial in nature. The loading may be. Sometimes these are called untrue triaxial test or false triaxial test. Several investigators tried to develop a true triaxial system where all the three principal stresses can be varied independently and also for obtaining homogeneous state of stress in specimens. Bangash [10] reported experimental results for triaxial compression (see Figure 3).

Another reason for the scarcity of test data is scatter of the test data associated with machine precession, testing technique and statistical variation of material properties from sample to sample. There were many attempts in the literature to overcome the above mentioned difficulties for specific loading situation such as uniaxial, biaxial, triaxial and cyclic loading, etc. (see Table 1)

Table 1. References for different loading situations

Loading	References
Uniaxial	Khaloo and Ahmad [106], Tsai [187], Neville [141], Domingo and Chu [63], Shah [173], Wang et al. [197], Kotsovos and Newman [110], Drawin and Pecknold [59], Newman [144], Desayi and Krishnan [61], Smith and Young [182]
Biaxial	Gerstle [80, 78], Taylor [186], Newman [144],Kupfer [118]
Triaxial	Grassl et al. [82],Li and Ansari [128, 7], Attard and Setunge [8], Imran [92], Khaloo and Ahmad [106], Ahmad and Shah [3], Bazant and Oh [25], Gerstle [79], Cedolin et al. [46], Mills and Zimmerman [137], Akroyd [5], Domingo et al. [64]
Cyclic	Bahn and Hsu [9], Karsan and Jirsa [105],Sinha et al. [180], Fafitis and Shah [70], Yankelevsky and Reinhardt [202]
Confined	Iyengar et al. [93], Mander [133, 132], Attard and Setunge [8]

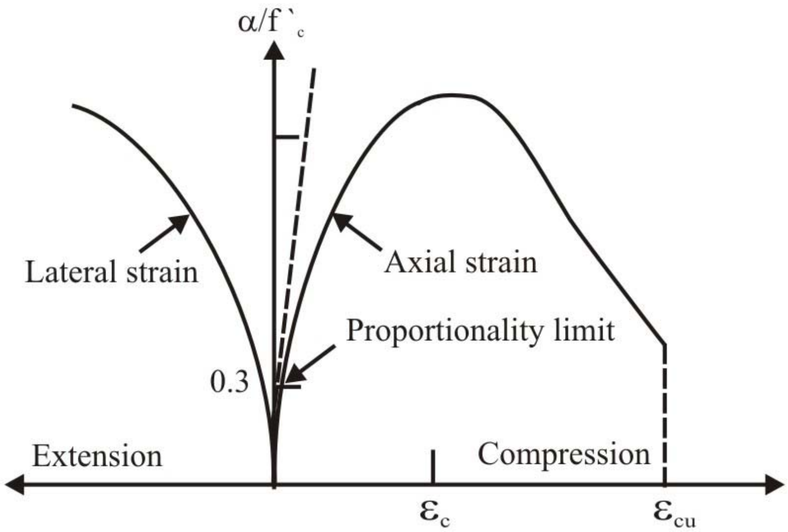

Figure 1. Uniaxial stress-strain curve [52]

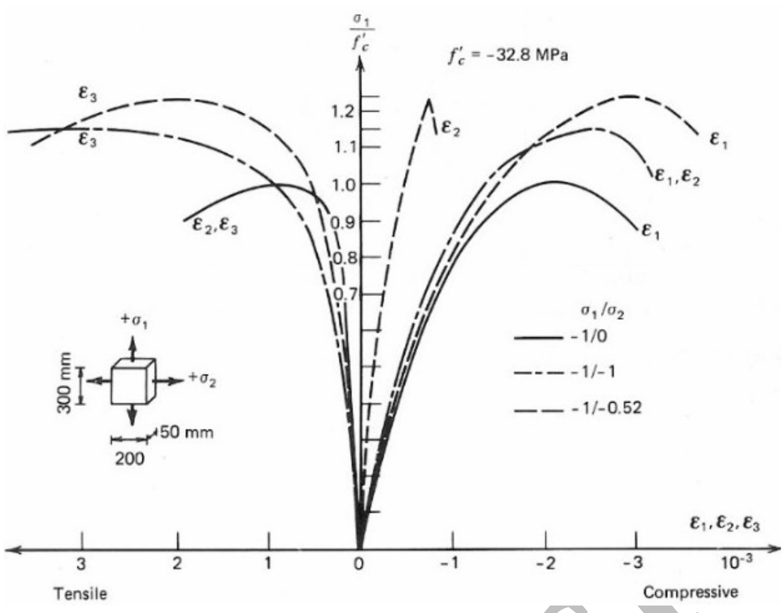

Figure 2. Biaxial stress-strain curve [118]

Many uniaxial and biaxial stress-strain relations are available in the literature. Figures 1, 2 shows a typical uniaxial compressive and biaxial stress-strain curves respectively. Some of the uniaxial stress-strain relations proposed by various researchers are given below:

Desayi and Krishan [61]

$$\sigma = \frac{E\varepsilon}{1+\left(\dfrac{\varepsilon}{\varepsilon_p}\right)^2} \tag{1}$$

where σ, ε are stress and strain tensors, E is Young's modulus, ε_p is strain at peak stress.

Saenz [171]

$$\sigma = \frac{E\varepsilon}{1+\left(\dfrac{E}{E_p}-2\right)\left(\dfrac{\varepsilon}{\varepsilon_p}\right)+\left(\dfrac{\varepsilon}{\varepsilon_p}\right)^2} \tag{2}$$

where E_p is Young's modulus at peak stress.

Smith and Young [182]

$$\sigma = E\varepsilon\varepsilon^{\frac{-\varepsilon}{\varepsilon_p}} \tag{3}$$

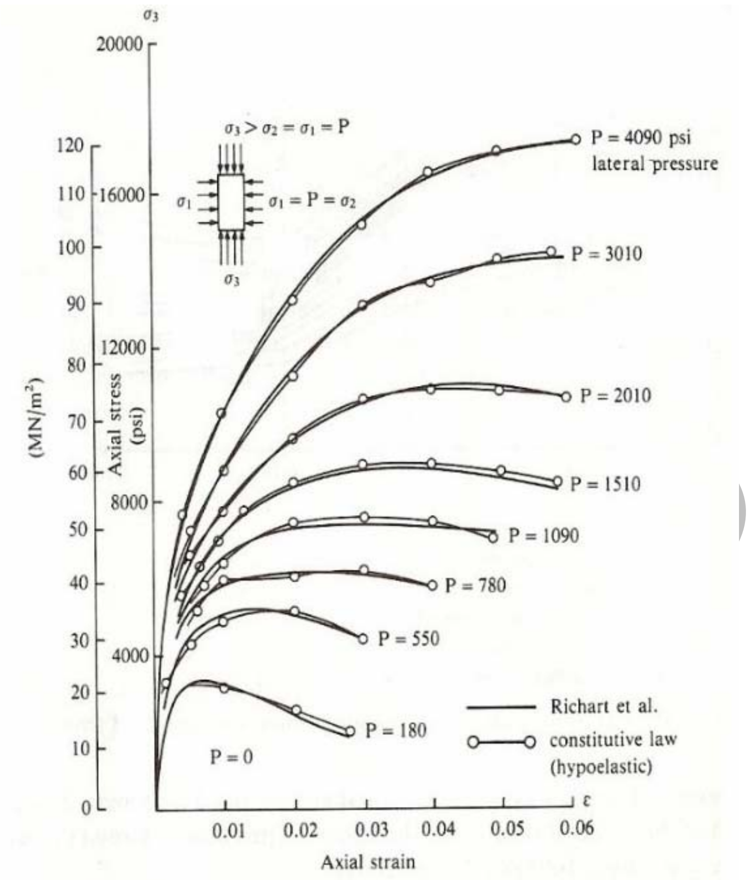

Figure 3. Triaxial stress-strain curve [10]

where, $e = \dfrac{E}{E_o}$, E_o initial tangent modulus.

The European Concrete Committee (CEB) for short-term loading gives a parabola and a straight line up to ultimate strain ε_u as

$$\frac{\sigma_c}{\sigma_u} = \frac{k\eta - \eta^2}{1 + (k-2)\eta} \tag{4}$$

where σ_c is the cylindrical compressive strength of concrete.

σ_u = Ultimate stress

$$\eta = \frac{\varepsilon_u}{0.002},\ k = \frac{[0.0022(1.1E)]}{\sigma_u}$$

The value of ε_u is given between 0.003 and 0.0035

A monotonically increasing uniaxial stress and axial strain equation proposed by Sargin and modified by Attard and Setunge [8]

$$Y = \frac{AX + BX^2}{1 + CX + DX^2} \tag{5}$$

X, Y refers to stress and strain non-dimensional zed with respect to the corresponding values at peak stress. Where A, B, C and D are material constants [173, 197, 154].

Richard and Abbott [165] proposed a three parameter stress-strain relation

$$\sigma = \frac{E_1 \varepsilon}{\left(1 + \left(\frac{E_1 \varepsilon}{\sigma_o}\right)^n\right)^{\frac{1}{n}}} + E_p \varepsilon \tag{6}$$

where E_p is plastic modulus, σ_o is a reference plastic stress, $E_1 = E - E_p$ and n is a shape parameter of stress-strain curve.

Carreira and Chu [44] proposed a stress-strain relation for reinforced concrete in tension

$$\frac{\sigma_t}{\sigma_t^{'}} = \frac{\beta\left(\frac{\varepsilon}{\varepsilon^{'}}\right)}{\beta - 1 + \left(\frac{\varepsilon}{\varepsilon_t^{'}}\right)^{\beta}} \tag{7}$$

where stress corresponding to the strain ε, $\sigma_t^{'}$ point of maximum stress, $\varepsilon_t^{'}$ straincorresponding to maximum stress $\sigma_t^{'}$, β is a parameter depends on the shape of the stress-strain diagram.

Mander et al. [132]

$$\sigma = \frac{\sigma_{pc} \frac{\varepsilon}{\varepsilon_{pc}} r}{r - 1 + \left(\frac{\varepsilon}{\varepsilon_{pc}}\right)^{r}} \tag{8}$$

where σ_{pc} and ε_{pc} are peak stress and strain of confined concrete.

$$r = \frac{E_c}{E_c - E_s}$$

$$E_c = 5000\sqrt{\sigma}\ , \quad E_s = \frac{\sigma_{pc}}{\varepsilon_{pc}}$$

Gerstle [78] proposed a biaxial stress-strain relation by conducting biaxial compression tests

$$\tau_{oct} = \tau_p (1 - e^{\left(\frac{-2G_o}{\tau_{ou}}\gamma_{oct}\right)}) \tag{9}$$

G_o = Initial shear modulus.

τ_{oct} = Octahedral shear stress.

γ_{oct} = Octahedral shear strain.

τ_p = Peak octahedral shear stress obtained from the failure envelope.

Equivalent uniaxial stress-strain relations Chen [52] are also available for biaxial and triaxial stress conditions of concrete. For biaxial compression

$$\sigma = \frac{E_o \varepsilon_{iu}}{1 + [\frac{E_o}{E_s} - 2]\frac{\varepsilon_{iu}}{\varepsilon_{ic}} + [\frac{\varepsilon_{iu}}{\varepsilon_{ic}}]^2} \tag{10}$$

$E_o =$ Initial tangent modulus of elasticity.

$E_s = \frac{\sigma_{ic}}{\varepsilon_{ic}} =$ Secant modulus at the maximum (peak) compressive stress.

$\varepsilon_{ic} =$ Equivalent uniaxial strain corresponding to peak compressive principal stress.

$\varepsilon_{iu} =$ Equivalent uniaxial strain.

For triaxial tension and compression

$$\sigma = \frac{E_o \varepsilon_{iu}}{1 + [R + \frac{E_o}{E_s} - 2]\frac{\varepsilon_{iu}}{\varepsilon_{ic}} - (2R - 1)[\frac{\varepsilon_{iu}}{\varepsilon_{ic}}] + R(\frac{\varepsilon_{iu}}{\varepsilon_{ic}})^3} \tag{11}$$

where

$$R = \frac{E_o(\frac{\sigma_{ic}}{\sigma_{if}} - 1)}{E_s(\frac{\varepsilon_{ic}}{\varepsilon_{if}} - 1)^2} - \frac{\varepsilon_{ic}}{\varepsilon_{if}} \tag{12}$$

σ_{if} , ε_{if} Coordinates of some point on the descending branch of the stress-equivalent strain curve.

Apart from the above many stress-strain relations specific for ascending branch and for different kind of loading are available in the literature (Popovics [159] and Chen [52]).

3. LINEAR ELASTIC MODELS

Linear elastic models are the simplest constitutive models available in the literature Chen [52], Bangash [10]. In linear elastic models concrete is treated as linear elastic until it reaches ultimate strength and subsequently it fails in brittle manner. For concrete under tension, since the failure strength is small, linear elastic model is quite accurate and sufficient to predict the behaviour of concrete till failure. Linear elastic stress-strain relation using index notation can be written as (Ahmad and Shah [3])

$$\sigma_{ij} = F_{ij}(\varepsilon_{kl}) \tag{13}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{14}$$

where F_{ij} is a function and C_{ijkl} represents material stiffness.

But this simple linear elastic constitutive law is often inappropriate as concrete falls under pressure sensitive group of materials whose general response under imposed load is highly nonlinear and inelastic. Also, in case of reversal of loading, these models fail to predict the concrete behaviour.

4. NONLINEAR ELASTIC MODELS

Concrete under multiaxial compressive stress states exhibit significant nonlinearity and linear elastic models fail in these situations. Significant improvements can be made in this situation using nonlinear constitutive models. There are two basic approaches followed for nonlinear modelling namely secant formulation (Total stress-strain) and tangential stress-strain (Incremental) formulation. Incremental stress-strain relation using index notation can be written in the following form [78, 79].

$$d\sigma_{ij} = C_{ijkl}{}^{t} d\varepsilon_{kl} \tag{15}$$

Here $C_{ijkl}{}^{t}$ is the tangent material stiffness.

Secant formulations are reversible and path independent and are applicable primarily to monotonic or proportional loading situations. These models are simple extensions of linear elastic models and formulated by assuming functional relations for secant bulk modulus, secant shear modulus [52] and assuming stresses and strains are derived as gradients of stress and strain potentials [139]. Especially the incremental or hypoelastic models using variable tangent moduli for describing the material stiffness can handle inelastic deformations and cyclic loading.

In the elasticity based models, a suitable failure criterion is incorporated for a complete description of the ultimate strength surface. Defining failure itself is a difficult task. Criteria such as yielding, load carrying capacity and initiation of cracking have been used to define failure. Failure can be defined as the ultimate load carrying capacity of concrete and represents the boundary of the work-hardening region. Many failure criterion are available in the literature for normal, high strength, light weight and steel fibre concrete. The most commonly used failure criteria are defined in stress space by a number of constants varying from one to five independent control parameters. Various criteria are available for concrete [86, 33, 34, 27, 57, 137, 122, 30, 143, 205] and more familiar criterion like Mohr-Coulomb criteria, Drucker-Prager, Chen and Chen [55], Ottosen [151], Hsieh-Ting-Chen [89], Willam and Warnke [198], Menetrey and Willam [136], Sankarasubrsmanian and Rajasekaran [172], Fan and Wang [71], etc. Out of the available failure models Ottosen [151] four parameter and Willam and Warnke [198] five parameter models are very popular in the literature (see Figure 4). A more sophisticated criterion was developed by Menetrey and Willam [136] by modifying the well-known Hoek and Brown criterion for rock masses. This criterion predicts the behaviour of concrete in a better manner and is expressed by the following expression.

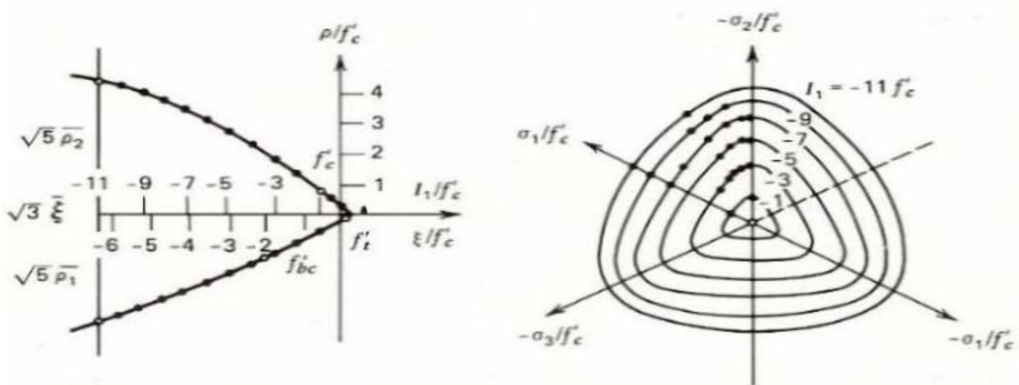

Figure 4. Willam and Warkne five-parameter model [198]

$$f(\xi,\rho,\theta) = [\sqrt{1.5}\,\frac{\rho}{f_c^{'}}] + m[\frac{\rho}{\sqrt{6}f_c^{'}}\,r(\theta,e) + \frac{\xi}{\sqrt{3}f_c^{'}}] - c = 0 \qquad (16)$$

where ξ = Hydrostatic stress invariant, ρ = Deviatoric stress invariant and θ = Deviatoric polar angle and $r(\theta,e)$ is an elliptic function.

$$\xi = \frac{I_1}{\sqrt{3}},\ I_1 = \sigma_{ii}$$

$$\rho = \sqrt{2J_2}\ ,\ J_2 = \frac{1}{2} S_{ij} S_{ji}$$

$$\cos 3\theta = \frac{3\sqrt{3}J_3}{2J_2^{\frac{3}{2}}}\ ,\ J_3 = \frac{1}{3} S_{ij} S_{jk} S_{ki}$$

5. PLASTICITY BASED MODELS

Classical plasticity based models form a big group in literature in the recent past. The mechanism of material non-linearity in concrete consists of both plastic slip and micro cracking. The large variety of models which are available to characterize the stress-strain and failure behaviour of material under multidimensional stress states (Domingo et al. [64], Chuan-Zhi et al. [56], Tsai [187], Richard et al. [165]) have certain advantages and disadvantages, which depend, to a large extent on their particular application. Yield criteria, flow rule and hardening rule are the three corner stones of any plasticity model.

In plasticity theory the total strain increment tensor is assumed to be the sum of the elastic and plastic strain increment tensors

$$d\sigma_{ij} = d\sigma_{ij}^{\ e} + d\sigma_{ij}^{\ p} \tag{17}$$

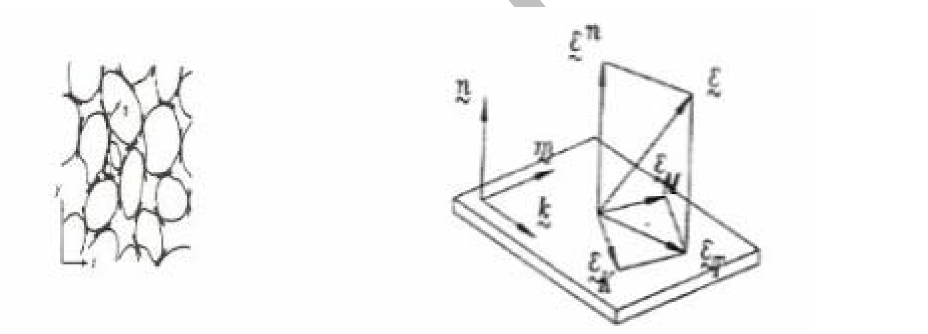

Figure 5. Microplane and stress-strain components on a microplane [24]

Hooke's law provides the necessary relationship between incremental stress and elastic strain. The plastic part of the strain increment tensor needs a flow rule to define the direction of plastic flow as explained bellow.

5.1 Yield criteria

Yield criteria of material should be known from experiments. Bridgman in his experiments pressure showed that hydrostatic pressure has negligible effect on the yield point but this is not the case with all the materials. Concrete is one such material whose behaviour is influenced by the effect of hydrostatic pressure. Yield criterion, which are hydrostatic pressure dependent and hydrostatic pressure independent, are available in the literature (see Table 2).

Table 2. Yield criterion

Yield criterion	References
Pressure independent	Tresca yield criterion or Maximum shear stress
	Von-Mises criterion or Maximum distortion energy criterion.
Pressure dependent	Rankine or Maximum tensile stress criterion
	Mohr-Coulomb criterion
	Drucker-Prager criterion
	Mises-Schleicher criterion.

Some failure models, developed specifically for concrete (Ottosen [151], Willam and Warnke [198], Menetrey and Willam [136]) are also used as yield function by applying some corrections and integrated into the theory of plasticity to compute strains and stresses in the yielded materials [82, 140]. Apart from the hydrostatic pressure, the directional dependence of material is also considered while formulating yield criteria. For isotropic materials orientation of the principal stresses is immaterial while for an anoisotropic material the material properties are highly direction dependent. Hill [88] proposed a yield criteria for anisotropic materials.

Any yield surface needs to satisfy certain physical requirements which ensure uniqueness of solution for the boundary value problem, such as condition of irreversibility of plastic deformation and the work which is expended on plastic deformation in a cycle is positive [130]. These requirements impose some restrictions on the shape of the yield surface such as smoothness, convexity and non-circular deviatoric section. But non-smooth yield surfaces are often included in the constitutive description of a material for the mere fact that an appropriate, smooth yield function is simply not available (Bigoni et al. [31], Jiang et. al [98]). But these non-smooth yield surfaces (Tresca or Mohr-Coulomb)cause an indeterminate situation while determining the direction of the plastic strain increment. de Borst[60] described an algorithm to handle the integration of stress-strain laws with singular yield point.

Out of the above criteria, Menetrey and Willam [136] three-parameter model predicts concrete behaviour better than any other with less number of parameters and includes the effect of intermediate principal stress.

5.2 Flow rules

A stress increment $d\sigma$ to the current state of stress σ results in elastic as well as plastic strain, if the stress state falls outside the elastic region. To describe the stress-strain relationship for an elastic-plastic deformation, we must define flow rule which define the direction of the plastic strain increment without any information regarding magnitude. Flow rule may or may not be associated with the yield criteria.

$$d\varepsilon_{ij}{}^{p} = d\lambda \frac{\partial Q}{\partial \sigma_{ij}} \tag{18}$$

where $d\lambda$ Is a non-negative scalar; Q is plastic potential function.

The above equation is similar to fluid flow equation, so this is called flow rule. When $Q=f$ (f is yield function), a special form of the above equation will be obtained and this is called an associated flow rule, meaning that the form is associated with the yield surface. Experimental data, however, indicate that associated flow may not be the most appropriate assumption for characterizing the response of concrete. Researchers like Smith et al. [181], Grassel [82] and Vermeer et al. [192], Frantziskonis et al. [75] have noted that concrete displays shear dilatancy characterized by volume change associated with shear distortion of the material. For typical yield functions, this characteristic is contrary to the assumption of associated flow. Additionally, data show that concrete subjected to compressive loading exhibits non-linear volume change, displaying contraction at low load levels and dilation at higher load levels. These characteristics of concrete response may be difficult than the assumption of associated flow. In order to improve modelling of concrete material response, non-associated flow models, in which the yield and plastic potential functions are not identical,in the form of equation 18, were used. Various forms of the plastic potentials were tried in literature with a general form as [53].

$$Q = Q(\sigma_{ij}, \alpha_1, \alpha_2, ..., \alpha_N) \tag{19}$$

$\alpha_1, \alpha_2, ..., \alpha_N$ are functions of hardening parameters.

Han and Chen [84], Dvorkin et al. [67, 68] used a plastic potential in the form of Drucker-Prager type as

$$Q(\sigma_{ij}, \alpha) = \alpha I_1 + \sqrt{J_2} + \text{Constant} \tag{20}$$

α can be obtained from an uniaxial compression test as

$$\alpha = \frac{1}{\sqrt{3[1 - \frac{\varepsilon^p}{\varepsilon_v{}^p}]}}$$

$\varepsilon_v{}^p$ is the volumetric part of the plastic strain and is the second invariant of stress tensor.

Onate et al. [149], Vermeer and de Borst [192] used a Mohr-Coulomb type plastic potential with angel of dilatancy Ψ instead of the internal friction angle ϕ.

$$Q(\sigma,\psi)=\frac{I_1}{3}\sin\psi+\sqrt{J_2}[\cos\theta-\frac{sin\theta\sin\psi}{\sqrt{3}}] \quad (21)$$

where Ψ is the angle of dilatancy obtained by experiments. Grassl et al. [82] expressed a plastic potential in the Haigh-Westergaard space as:

$$Q=-A(\frac{\rho}{\sqrt{q(k)}})^2-B\frac{\rho}{\sqrt{q(k)}}+\frac{\xi}{\sqrt{q(k)}}=0 \quad (22)$$

A and B are parameters determined from the axial strain state in uniaxial and triaxial compression. $\sqrt{q(k)}$ is the hardening/softening law.

Materilas exhibitig nonassociated flow voilates the Drucker's stability postulates. Lade et al. [120, 121] studied the material stability during nonassociated flow and the possible consequences of nonassociate flow.

5.3 Hardening rules

The law, which governs the phenomenon of configuration change in yield surface which occurs during loading process, is hardening rule. One of the major problems of work/strain hardening plasticity is finding the evolution of the yield surface (Ohtami and Chen [147]). Several hardening rules have been proposed in the literature. Depending on the hardening rule used, the material response after initial yielding differs considerably. The hardening rules available in the literature are isotropic hardening, kinematic hardening, independent hardening and mixed hardening. In isotropic hardening, the basic assumption is uniform expansion of the yield surface. Yield surface do not under go any distortion or translation. The concrete behaviour under monotonic loading has been modelled by many Imran et al. [91], Smith et al. [181] using isotropic hardening.

Prager [160] proposed a model in connection with his kinematic model to predict the translation of the yield surface. Kinematic model assumes that, during plastic loading, the yield surface translates as a rigid body in stress space without any expansion.

Suppose the initial yield surface is described by

$$f(\sigma)\text{-}k=0 \quad (23)$$

Due to kinematic hardening the subsequent yield surface takes the form as

$$f(\sigma\text{-}\alpha)\text{-}k=0 \quad (24)$$

where α= Back stress, that represents the centre of the yield surface.

K = Material constant representing the size of the yield surface.

Prager proposed a linear constitutive equation for the back stress as

$$d\alpha=cd\varepsilon^p \quad (25)$$

where c = Material constant.

The main deficiency of this model is its inconsistency when applied to subspaces. Ziegler [208] rectified this inconsistency by modifying

$$d\alpha=(\sigma-\alpha)d\mu \tag{26}$$

where $d\mu$=Proportionality scalar constant determined by the yield criterion. Later several models were developed to predict the yield center movement,on the basis of thermodynamic principles and to further simplify kinematic hardening rule (Voyiadjis and Rashid [193], Wang [196], Jiang [95, 96, 97], Lade and Kim [119], Phillips et al. [158]).

Mixed hardening rule, which is a combination of isotropic hardening and kinematic hardening. In mixed hardening, the increment of plastic strain can be split in to the following two components

$$d\varepsilon^{p} = d\varepsilon^{pi} + d\varepsilon^{pk} = Md\varepsilon^{p} + (1-M)d\varepsilon^{p} \tag{27}$$

where $d\varepsilon^{pi}$ and $d\varepsilon^{pk}$ are isotropic and kinematic strains respectively.

M is a mixed hardening parameter and varies between $0 \le M \le 1$

M = 0, Kinematic hardening and M = 1, Isotropic hardening.

Mixed hardening can be used to simulate the Baushinger effect. Bathe [13] derived Prager's mixed hardening parameter while dealing with the computational plasticity.

In addition to the above mentioned hardening rules, several models such as Single hardening model (Lade et al. [119]), Multiple hardening model first proposed by Murray et al. [140] further developed by Ohtami and Chen [147] and Novel hardening model by Grassl et al. [82], etc. have been proposed in the literature by different researchers and used with partial sucesses in different loading situations. Different number and type of hardening parameters has been used for modelling concrete. In multiparameter hardening parameter model each hardening parametere characterizes a loading surface starting from intial yield surface to the failure surface. By introducing a shape factor k the initial yield and subsequent loading surfaces can be written as [85].

$$f = \rho - k\rho_f = 0 \tag{28}$$

where $\rho = \sqrt{2J_2}$, $k(k_o, \sigma_m)$ is a shape factor.

k_o is hardening parameter, ρ_f defines the failure envelope.

In the above equation, shape factor depends on the hardening parameter and the hardening parameter ranging between intial yield surface and the failure surface. Typically hardenig parameters commonly used in practice are effective plastic strain or plastic work defined as follows:

Effective plastic strain

$$\varepsilon_p = \int \sqrt{d\varepsilon^p d\varepsilon^p} \tag{29}$$

Plastic work

$$W_p = \int \sigma d\varepsilon^p \tag{30}$$

Apart from the plastic work/effective plastic strain, many other hardening parameters have been used to model concrete.

Han and chen [85] used , $k_o, (k_o = f(W^p))$, as hardening parameter in their nonuniform hardening plasticity model to model inelastic behaviour of concrete including brittle failure in tension, ductile behaviour in compression and volumetric dilation under compressive loading. In this model the range of the hardening parameter is taken as $k_y \le k_o \le 1$. When $k_o = k_y$, the loading surface corresponds to initial yield surface and when the yield surface reaches the ultimate/failure surface the hardening parameter becomes $k_o = 1$. When $k_o = 1$ the loading surface must match with the failure surface and the intersection point of the loading surface with hydrostatic axis can be written as

$$\overline{\rho} = \frac{A}{1 - k_o} \tag{31}$$

where A is a constant.

Han and Chen have given the importance of hardening parameter in defining the loading surface. And also proposed a relation between hardening parameter and base plastic modulus. Base plastic modulus $H_b{}^p$ is obtained from uniaxial compressive test and related to the plastic modulus as

$$H^p = M(\sigma_m, \theta) H_b{}^p \tag{32}$$

where $M(\sigma_m, \theta)$ is a modification factor.

This model is flexible and can fit wide range of experimental data. The parameters such as shape factor, plastic modulus, modification factor, etc. can be adjusted and calibrated against additional experimental data.

Grassl [82] used volumetric part of the plastic strain as hardening parameter in his hardening law to model the influence of multiaxial stress states on the deformation capacity of concrete.

$$k(d\varepsilon_v{}^p) = d\varepsilon_v{}^p = d\lambda \delta_{ij} \frac{\partial g}{\partial \sigma_{ij}} \tag{33}$$

where δ_{ij} is the kronecker delta.

In this model he used only volumetric part of the plastic strain instead of plastic strain itself because plastic strain as the hardening parameter cannot describe the increase of plastic deformation in multiaxial compression stress states. In this study the representation of the behaviour of concrete in uniaxial, biaxial and triaxial compression with single calibration is achieved.

Murry et al. [140] used multiple hardening parameters such as current values of uniaxial compression, equal biaxial compression and uniaxial tension in his model to predict the behaviour of prestressed concrete tension structures. In this study the hardening rule proposed in the form of

$$\sigma_c = \sigma_c{}^o + g(\lambda) \tag{34}$$

$$\sigma_{t1} = \sigma_{t1}{}^o + h(\mu_1) \tag{35}$$

$$\sigma_{t2} = \sigma_{t2}{}^o + h(\mu_2) \tag{36}$$

where g and h are hardening functions of the equivalent plastic strain parameters, λ is compressive plastic strain parameter, μ_1 and μ_2 are tensile plastic strain parameters $\sigma_c, \sigma_c{}^o$ Current and initial compressive yield stresses, current and initial tensile yield stresses in directions 1 and 2.

Ohtani and Chen [147] in their multiple hardening parameter model proposed a concept of *N* hardening parameters with each hardening mode associates with corresponding damage parameter as

$$f(\sigma_{ij}, \mu_1, \mu_2, ..., \mu_N) = 0 \tag{37}$$

$$\mu_M = \mu_M(\xi_M) \tag{38}$$

where ξ_M is damage parameter related to plastic strain tensor and no way related to the damage parameter used in continuum damage models described in the later sections of this article.

Lin et al. [129] proposed a two stage hardening rule based on work hardening hypothesis. In the first stage the current yield stress in the uniaxial compression increases from its initial value $\sigma_{c0} = 0.6 f_c$ to the value corresponding to the peak of the uniaxial compressive curve $\sigma_c = f_c$. In this model the initial hardening is assumed to be deviatoric and characterized in terms of effective strain $\bar{\varepsilon}^p$ and by work equality the rate form of the effective plastic strain defined as

$$d\bar{\varepsilon}^p = \frac{\sigma_{ij} d\varepsilon^p}{\sigma_c} \tag{39}$$

This model satisfies Drucker's postulate and the equation (39) is always positive. Once the value of σ_c reaches the peak of the uniaxial compressive strength the material follow either hardening or softening depending on the sign of the volumetric plastic strain rate $d\varepsilon_v{}^p = \frac{d\varepsilon^p}{3}$

The hardening-softening law valid for the second stage is given by
if $d\varepsilon_v{}^p \leq 0$ (hardening)

$$d\sigma_c = -\lambda_1 d\sigma_c d\varepsilon_v{}^p \tag{40}$$

if $d\varepsilon_v{}^p \succ 0$ (softening)

$$d\sigma_c = -2\lambda d\sigma_c \varepsilon_v{}^p d\varepsilon_v{}^p \tag{41}$$

where λ and λ_1 are empirical parameters.

The advantage of this model is that the material parameters can be identified in a sequential manner from a set of well defined characteristic states rather than optimizing the data fits. Kang and William [104] proposed a concrete model based on an intermediate loading surface of the form

$$F(\xi,\rho,\theta)_{fail} = F(\xi,\rho,\theta)_{fail} + F(\xi,\rho,k(q_h))_{hardg} + F(\xi,\rho,c(q_s))_{softg} = 0 \tag{42}$$

where ξ, ρ, θ are Haigh-Westergaard coordinates.

k, c are variables which parameterize the loading surface.

The hardening is incorporated through the function

$$F(\xi,\rho,k(q_h))_{hardg} = \frac{-\rho_1}{f_c^{'}}[(\frac{\xi-\xi_1}{\xi_o-\xi_1})-1] \tag{43}$$

where β = $0.25[\frac{1-k^2}{1-k_o{}^2}]$ β is a function of the hardening parameter *k*. In this model when $k = k_o$ and *c*=1, f_s vanishes and the initial yield surface is described by $f_{fail} + f_{hardh} = 0$. This model capture the main deformation characteristics of concrete such as pressure sensitivity, nonlinear behaviour, deviatoric evolution, strain softening, etc.

Table 3 shows some of the plasticity based models for concrete along with the hardening parameter used.

Table 3. Plasticity models for concrete

Constitutive model	Remarks
Grassl (2002)	Menetrey and Willam (1995) yield surface Non-associative flow rule Proposed a novel hardening law is used as hardening parameter
Imran et al.(2001)	Four-parameter Hsieh-Ting-Chen Criterion Non-associative flow rule Isotropic hardening plastic strain is used as hardening parameter
Kang and Willam (1999)	Derived from the proposed intermediate loading surface Non-associated plastic flow rule Proposed a hardening function $0 \leq k \leq 1$ is used as hardening parameter
Feenstra and de Borst (1996)	Drucker-Prager yield surface Non-Associative flow rule are hardening parameters (defined in [72]).
Onate et al. 1988	Modified version of Mohr-Coulomb yield surface Non-associative flow rule changes in intergranular cohesion as hardening parameter
Ohtani and Chen (1988)	Chen and Chen yield surface (1975) Associative flow rule N number of hardening parameters hardening parameters:
Han and Chen (1987)	Yield surface: Willam-Warnke five-parameter, Hsieh-Ting-Chen four parameter Non-Associative flow rule and Non uniform hardening rule effective plastic strain as hardening parameter
Han and Chen (1985)	Chen and Chen yield surface (1975) Associative flow rule Multiple hardening parameters effective plastic strain as hardening parameter
Vermeer and de Borst (1984)	Mohr-Coulomb yield surface Non-Associative flow rule cohesion and internal friction as hardening paremeters.
Murry (1979)	Arrived from a biaxial failure surface Associative flow rule Multiple hardening parameters are hardening parameters
Chen and Chen(1975)	Initial discontinuous surface, loading surface and failure surfaces proposed

In summary, the classical elasto-plasticity theory of concrete employs the chosen yield criterion, flow and hardening rules alongwith the observed material behaviour under uniaxial compressive stress. The resulting incremental stress-strain relations are then used to obtain the material response to any chosen stress or strain history.

6. STRAIN SOFTENING AND STRAIN SPACE PLASTICITY

For pressure dependent materials like concrete, the slope of the stress-strain curve decreases steadily and monotonically with the load and eventually becomes negative (Frantziskonis et al. [73, 74, 148]). The behaviour after the peak, where a further deformation requires a decrease in load is called the strain softening. Capturing the stress-strain response after peak (strain softening) depend on many factors like test equipment, test procedure, sample dimensions and stiffness of the machine, etc. (Lubliner [130], Shah [173], Chen [53], Read and Hegemier [162]).

Classical plasticity theories are developed in stress space where stress and its increments are treated as independent variables. Eventhough stress space formulation is commonly accepted in engineering practice this approach has some inherent disadvantages: (a) For strain softening materials, there is no clarity in defining the criteria of loading-unloading. (b) For many structural materials, the slope of the uniaxial stress-strain curve becomes zero at the ultimate strength point (peak) where the stress space formulation may not offer reliable results.

These disadvantages of stress space formulation can be eliminated with the help of strain space formulation. Drucker's stability postulate which is the basis for the stress based formulation is invalid in strain softening portion where a weaker stability criterion proposed by Il'yushin [90], provides necessary basis for the strain space formulation.

Il'yushin's postulate states that the work done by the external forces in a closed cycle of deformation of an elastic-plastic material is nonnegative.

$$dW = \oint d\sigma_{ij}{}^{p} d\varepsilon_{ij} \geq 0 \tag{44}$$

However, Il'yushin did not formulate a complete plasticity theory in strain space, which was proposed later by Naghdi and coworkers. The basic formulation of strain space formulation as well as equivalance or otherwise with the stress-space formulation have been discussed in the literature (Naghdi and Trapp [142], Casey and Naghdi [45], Pekau et al. [157], Kiousis [107], Mizono and Hatanaka [138], and Barbagelata [11]). The application of the strain space formulation for strain softening materials for different situations are also discussed in the literature (Stevens [185], Iwan and Chelvakumar [94], Han and Chen [84, 83], Dafalias [58]). Computational algorithms based on the strain space formulation was examined by some researchers like Iwan and Yoder [94], Runesson et al. [169] and Lee [127], etc.

7. ENDOCHRONIC THEORY OF INELASTICITY

In the classical plasticity-based models, finding the yield surface pose many problems and an attempt was made to develop a continuous model for inelastic behaviour which did not require the existence of the yield condition. This model is based on the concept of intrinsic (or endochronic) time, defined in terms of strain or stress and used to measure the degree of damage occurred to the internal structure of the material. This model was primarily developed for metals by Valanis [188, 189, 190, 191]. Sandler [170] studied its stability and uniqueness and Rivlin [167] critically evaluated the theory. It has been extended to concrete by Bazant et al. [27, 28, 29], to fibre reinforced concrete by Reddy and Gopal [163]. Endochronic model can describe inelastic volume dilatancy, unloading, strain softening, hydrostatic pressure sensitivity and pinching of hysteresis loops under cyclic loading. Eventhough this model gives superior results, its popularity is restricted by its complexity. The numerous numerical coefficients required for the development of a constitutive law are estimated by curve fitting of available experimental data. The main obstacle in the development and application of this method is the large number of parameters required. As a result, this model has not undergone further development in the last 15-20 years.

The intrinsic time ξ (on pseudo-time scale) introduced by endochronic theory is

$$\xi = \int_0^{\zeta} \frac{d\zeta}{f(\zeta)} \tag{45}$$

where $f(\zeta)>0$ and $d\zeta>0$.

The value of $f(\zeta)$ is a history-dependent material function. A typical constitutive equation for linear endochronic theory with pseudo-time measure ξ is as follows (which is similar to a linear viscoelastic model)

$$\sigma_{ij} = \int_0^{\xi} E_{ijkl}(\xi - \xi^{'}) \frac{\partial \varepsilon_{kl}}{\partial \varepsilon^{'}} \tag{46}$$

8. FRACTURING AND CONTINUUM DAMAGE MODELS

These models are based on the concept of propagation and coaleesence of microcracks, which are present in the concrete even before the application of the load. Damage based models are often used to describe the mechanical behaviour of concrete in tension. In the earlier class of models (Dougill [65, 66]), plastic deformation is defined by usual flow theory of plasticity and the stiffness degradation is modelled by fracturing theory. The second class of models is based on the use of a set of state variables quantifying the internal damage resulting from a certain loading history. The fundamental assumption in these models is that the local damage in the material can be averaged and represented in the form of damage variables, which are related to the tangential stiffness tensor of the material. The models of this category can describe progressive damage of concrete occurring at the microscopic level, through variables defined at the level of the macroscopic stress-strain relationship Krajcinovic and Fonseka [112]. Continuum damage mechanics was introduced by Kachanov in 1958 for creep related problems and has been applied to the progressive failure of materials. In 1980s, it was established that damage mechanics could model accurately the strain-softening response of concrete (Krajcinovic [113, 114], Lemaitre [123, 124], Chaboche [49, 51]). Considering the material as a system described by a set of variables and a thermodynamic potential, constitutive law is derived which has to obey the kinematics of damage. Various models of gradually increasing complexity with choice of potential and damage parameter (Scalar, Tensor, etc.) are proposed (Mazars and Cabot [135], Kratzig and Polling [115]) and implemented for concrete (see Table 4). Various damage models such as elastic damage, plastic damage (Ju [102], Lee et al. [126]), damage model using bounding surface concept (Voyiadjis [195, 194]),Wu and Komarakulnanakorn [200] presented an endochronic theory of continuum damage mechanics, models for cyclic loading, etc. [183, 1, 126] are available in the literature. Continuum damage mechanics based material models in the literature basically followed two approaches one inspired by plasticity and the other followed the thermodynamic fundamentals and energy balance.In the first approach, similar to plasticity, assumes a damage surface, damage loading function and a consistency condition [195, 203] where as in the second approach [101, 100, 102, 176, 177] assumes a free energy potential in the form of Helmholtz or Gibbs subjected to the satisfaction of Clausius-Duhem inequality.

Table 4. Representation of damage (Singh [178])

Damage variable as	**References**
Scalar	Kachnov [103], Rabotov [161], Simo and Ju [176, 177], Ju [101], Lemaitre [123, 124, 125], Chaboche [47, 48] Mazars [135], Krajcinovic
Vector	Kachnov, Hayhurst and Storakers, Davison and Stevens, Krajcinovic and Foneska [112], Krajcinovic [111]
Second rank tensor	Rabotov, Murakami and Ohno, Vakulenko and Kachanov, Dragon and Mroz, Cordebois and Sidoroff, Betten.
Fourth order tensor	Chaboche [50], Sidroff, Chow and Wang, Chow and Wei, Ortiz [150]
Eight order tensor	Chaboche.
Strain tensor	Rudnicki and Rice [168], Singh and Digby [179], Bazant and Kim [30], Nicholson [145]

Kratzig (1998) derived a strain based damage theory by assuming a Helmholtz free energy expression of the form

$$\psi(\varepsilon_{ij}, C_{ijkl}, p) = \frac{1}{2\rho} \quad (47)$$

where ρ= Material density.

C_{ijkl} Current stiffness tensor, ε_{ij} = Strain tensor.

P= an internal variable describing the radius of the limit state surface.

The assumed free energy potential should satisfy the Clausius-Duhem inequality and assumed a damage evolution law of the form

$$-\rho\dot{\psi} + \sigma : \dot{\varepsilon} \geq 0 \quad (48)$$

$$\dot{p} = \dot{\lambda}\frac{1}{H} \quad (49)$$

where λ= Consistency parameter.

H= Hardening/softening modulus.

Thus, the following incremental stress-strain law is obtained

$$\dot{\sigma}_{ij} = [C_{ijkl} + \frac{4H}{\partial\Gamma/\partial\rho}\frac{\partial\Gamma}{\partial(\varepsilon_{ij}\varepsilon_{mn})}\varepsilon_{mn}\varepsilon_{pq}\frac{\partial\Gamma}{\partial(\varepsilon_{pq}\varepsilon_{kl})}] \quad (50)$$

where $\dot{\Gamma}(\varepsilon_{ij}\varepsilon_{kl}, p) = 0$ is a consistency condition.

Among the variety of theories that describe the behaviour of concrete, CDM has the advantage to be founded on a rational frame work of the material theory, therefore having a sound physical background (Suanno and Ramm [184]). The CDM alone is not able to reproduce all facets of the behaviour of quasibrittle materials. It works rather as the missing link between the theories like plasticity or elasticity. Therefore a fully coupled model is more able to describe the realistic material behaviour. The CDM formulations also are finite element oriented.

9. MICROPLANE MODELS

Micromechanical models attempt to develop the macroscopic stress-strain relationship from the mechanics of the microstructure. The only popular model in this category, which reached up to implementation stage, is the microplane model proposed by Bazant and his associates (Bazant [24]). The microplane model, first proposed by Budianski for metals in the name of slip theory of plasticity and later extended to concrete and other geomaterials like rock (Bazant et al. [14], Gambarova and Floris [76], Caner et al. [38], Carol et al. [40], Pande and

Sharma [155]). Unlike the other constitutive models, which characterize the material behaviour in terms of second order tensors, the microplane model characterize in terms of stress and strain vectors. The macroscopic strain and stress tensors are determined as a summation of all these vectors on planes of various orientations (Microplanes) under the assumption of static or kinematic constraint. The static constraint (the stress vector acting on a given plane is the projection of the macroscopic stress tensor) used in the earlier models, act as an obstruction for the generalization of the microplane model for post peak strain softening quasi-brittle materials (Bazant [23], Bazant and Oh [25, 22]). In the later models this shortcoming was rectified by using kinematic constraint (the strain vector on any inclined plane is the projection of the macroscopic strain tensor). Later the microplane formulation was generalized for nonlinear triaxial behaviour of concrete by Bazant and Prat [20, 21], implemented into nonlocal finite element code by Bazant and Ozbolt [18], Carol and Prat [40] and successfully used in the analysis of compression failure by Bazant and Ozbolt [19]. This model was further modified by introducing stress-strain boundaries by Bazant et al. [15, 16, 17], introduced damage by Carol et al. [41] and plasticity concepts in to the microplane model by Carol and Bazant [43] and the numerical algorithm was developed by Caner and Bazant [38]. The latest effective formulation of microplane for concrete is named model M4 by Bazant et al. [14]. These microplane formulations were thermodynamically inconsistent in some loading situations and this was rectified by Carol et al. [42, 117]. Further research in microplane theory is still an active area and some recent studies such as vertex effect at rotating principle axes by Caner et al. [37], the application of microplane plane model to model triaxial compression for low confinement Ghazi et al.[81], cyclic triaxial behaviour Ozbolt and Bazant [153], for reinforced planar members by Park and Kim [156], development of elastic and elasto-plastic micropolar microplane models by Etse et al. [69],and application to large strain problems Carol et al. [39] were reported in literature.

The basic relations of the microplae model are briefly explained below The normal strain on the microplane is

$$\varepsilon_N = N_{ij}\varepsilon_{ij} \tag{51}$$

where $N_{ij} = n_i n_j$.

The shear strains on each microplane are characterized by their components in the chosen directions M and L

$$\varepsilon_M = M_{ij}\varepsilon_{ij}; \;\; \varepsilon_L = L_{ij}\varepsilon_{ij} \tag{52}$$

Static equivalence of stresses between the macro and micro levels can be enforced by principle of virtual work written for the surface Ω of a unit hemisphere.

$$\frac{2\pi}{3}\sigma_{ij}\delta\varepsilon_{ij} = \int(\sigma_N\delta\varepsilon_N + \sigma_L\delta\varepsilon_L + \sigma_M\delta\varepsilon_M)d\Omega \tag{53}$$

This equation means that the virtual work of macro-stresses within a unit sphere must be equal to the work of micro-stresses regarded as the tractions on the surface of the sphere. The integral physically represents a homogenization of different contributions coming from planes of various orientations within the material. The kinematic constraint links the microplane strains to the macroscopic strain tensor, and the same constraint hold for the virtual strains. The volumetric and deviatoric components are

$$\delta\varepsilon_v = \frac{\delta\varepsilon_{kk}}{3} = \left(\frac{\delta_{ij}}{3}\right)\delta\varepsilon_{ij} \tag{54}$$

$$\delta\varepsilon_D = \delta\varepsilon_N - \delta\varepsilon_v = (N_{ij} - \frac{\delta_{ij}}{3})\delta\varepsilon_{ij} \tag{55}$$

$$\delta\varepsilon_L = L_{ij}\delta\varepsilon_{ij}\,;\ \delta\varepsilon_M = M_{ij}\delta\varepsilon_{ij} \tag{56}$$

The following incremental constitutive equations for the microplane are suggested by Bazant et al. [14, 99]:

$$\dot{\sigma}_v = E_v\dot{\varepsilon}_v;\dot{\sigma}_D = E_D\dot{\varepsilon}_D;\dot{\sigma}_M = E_T\dot{\varepsilon}_M;\dot{\sigma}_L = E_T\dot{\varepsilon}_L \tag{57}$$

where E_v, E_D and E_T are microplane elastic moduli defined from the macroscopic material behaviour.

Substitution of equations (54), (55) and (56) in to equation (53) yields a modified integral formula for the macroscopic stress (Carol et al. [42])

$$\begin{aligned}\sigma_{ij} &= \frac{3}{2\pi}\int_\Omega [\sigma_v\frac{\delta_{ij}}{3} + \sigma_D(N_{ij} - \frac{\delta_{ij}}{3}) + \sigma_M M_{ij} + \sigma_L L_{ij}]d\Omega \\ &= (\sigma_v - \bar{\sigma}_D)\delta_{ij} + \frac{3}{2\pi}\int_\Omega (\sigma_D N_{ij} + \sigma_M M_{ij} + \sigma_L L_{ij})d\Omega\end{aligned} \tag{58}$$

where

$$\bar{\sigma}_D = \frac{1}{2\pi}\int_\Omega \sigma_D d\Omega$$

In conclusion the three major steps of microplane model are projecting macro stress/stain tensor to microplane using static or kinematic constraint, defining a constitutive law at microplane level and getting the constitutive law at macro level by summing up all the stress/strain vectors on microplane.

The main advantage of microplane models is its conceptual clarity as the model is formulated in terms of vectors and the inherent nature of satisfying tensorial invariance requirements. Microplane model treats apparent corners (Vertex) which appear in the conventional yield surface based material models. The disadvantage in the microplane model is the huge computational work and storage requirements (4 to 10 times more than the conventional models).

10. COMPUTATIONAL IMPLEMENTATION

Formulation and solution of inelastic constitutive equations has been an active area of investigation for so many years. Integration of constitutive equation, solution of nonlinear equation and derivation of the tangent operator are the main concerns among researchers. There exists a whole field spread across many disciplines with roots in numerical analysis, finite element method (FEM) and boundary element method (BEM) etc. Extensive studies have been reported in the literature (Matzenmiller and Taylor [134], Malavar et al. [131], Ristinmaa and Tryding [166], Hartmann and Haupt [87], Owen and Hinton [152], Dodds [62], Simo and Taylor [175], Wissmann and Hauck [199], Nyssen [146], Yoder and Whirly [204], de Borst [60], Al-Rasby [6]) related to the development of accurate, sTable, consistent and convergent algorithms, implementation of various yield criterion and failure criterion into various computational codes such as linear and non-linear finite element method (LFEM, NLFEM), Boundary element method (BEM), etc. These developments leads to various analytical tools in the form of commercial finite element codes such as ANSYS, ABAQUS, ADINA, ASKA, DYPLAS(Dynamic Plasticity),FLAC-3D, LS-DYNA, NISA, etc. These software packages often implement many of the available material models in the literature as these packages are not material specific. In the case of FLAC-3D (Fast Lagrangian Analysis of Continua in 3-Dimensions), only one failure criterion-Drucker-Prager criterion was implemented.

The fast development of digital computers has resulted in a tendency toward more complicated models with number of material parameters. But Krieg and Krieg [116] note that even the simplest traditional models with von Mises yield criterion is implemented with considerable error in structural analysis.

The user has to take enough care while selecting the particular material model because these softwares often use familiar von Mises or Drucker-Prager models as default material models. These models do not represent the properties of the failure surface of concrete. In the most of the cases the failure surface obtained by these FE codes gives straight meridians and ignores the effect of intermediate principal stress in contrast to the original concrete behaviour with curved meridians and influence of intermediate principal stress component.

Artificial neural networks(ANNs) has been used as an alternatives for characterizing the behaviour of concrete (Basneer [12], Shin and Pande [174], Ghaboussi et al. [77], Zhao et al. [205, 206, 207], Ren and Zhao [164] Sankarasubrsmanian and Rajasekaran [172]). An adequately trained and validated ANN can represent general rules governing the material behaviour and can predict the constitutive behaviour of material. Theory of fuzzy sets have also been used for material modelling by Klisinski [108].

11. CONCLUSIONS

In this article concrete constitutive modelling based on various approaches, their implementation and the aspects related to strain space formulation are discussed.

Elasticity based models are simple and material is modelled up to peak. Many attempts for proposing a suitable failure criterion for concrete can be found in literature. These efforts resulted in a realistic failure model such as Willam and Warnke five parameter and subsequently a three parameter model of Menetrey and Willam. These models represent concrete behaviour in a realistic manner.

One advantage of theory of plasticity is the simple and direct calibration of the stress state. The yield surface corresponds to a certain stage of hardening to the strength envelop of concrete, and thus has a strong physical meaning. The theory of plasticity has a very long tradition and hence implementation of the formulation is efficient and thermodynamic validity is assured. One of the disadvantages is the indirect calibration of the deformation behaviour in the form of plastic potential.

Plasticity theory heavily depends on the assumption of existence of a yield surface. This assumption poses a problem while applying plasticity theory to concrete, where a well defined yield surface and experimental data related to yield surface are insufficient. This difficulty gives rise to new theories such as endochronic theory, microplane theory, etc.

Concrete structures subjected to complex stress states exist widely. Modern analytical tools like finite element method demands a realistic constitutive model. This need has given researchers a chance to explore various approaches such as endochronic theory, continuum damage mechanics, micromechanics, etc. Each of these models has their own strengths and weaknesses as discussed in the above sections.

It is very important to choose a reasonable constitutive model in research and design as it affects the design accuracy to a great extent. More experimental results in complex stress states and more realistic material models are demanded for research and engineering application in the future.

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