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EIGENSOLUTION OF SPECIAL COMPOUND MATRICES AND APPLICATIONS

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ABSTRACT

In this paper, previously developed matrices with canonical forms are generalized. These matrices arise in problems of structural mechanics such as stability and dynamics of structures.

Application of these matrices to graphs leads to efficient methods for the the eigensolution of Laplacian matrices. The use of the present method is especially benificial for large-scale models with repeated submodels.

Keywords: compound matrices; eigenvalues; graphs; repeated subgraphs; tensor product

1. INTRODUCTION

Many engineering problems require the calculation of the eigenvalues and eigenvectors of matrices. As an example, the eigenvalues correspond to natural frequencies in vibration of systems and buckling loads in the stability analysis of structures [1-4]. Eigenvalues and eigenvectors of matrices associated with adjacency and Laplacian of graphs form the basis for algebraic graph theory [5-7]. These eigensolutions have found many applications in sparse matrix technology, and are particularly employed in ordering and partitioning of graphs, and decomposition of large-scale finite element meshes for parallel computing [8-10]. General methods are available in literature for such calculations [11-14]. However, for matrices with special structures, it is beneficial to make use of their extra properties.

Methods were developed for factorization of symmetric graphs into smaller subgraphs using decompostion and healings, Refs. [15-18].

In this paper, some matrices having canonical forms are generalized. These matrices arise in many problems of structural mechanics such as stability analysis and dynamics of structures.

Application of these matrices to graphs leads to efficient methods for the the eigensolution of Laplacian matrices. The use of the present method is especially benificial for large-scale models with repeated subgraphs.

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2. A COMPOUND MATRIX OF SPECIAL PATTERN

Consider a compound matrix **M** containing submatrices **A** and **B**. Let **A** and **B** be m×m submatrices. The matrix **M** is an n×n rectangular block matrix having **A** on its diagonal entries and **B** with different multipliers in all the entries of **M** as:

$$[\mathbf{M}]=\begin{bmatrix} \mathbf{A}+q_{11}\mathbf{B} & q_{12}\mathbf{B} & \cdots & q_{1n}\mathbf{B} \\ q_{21}\mathbf{B} & \mathbf{A}+q_{22}\mathbf{B} & \cdots & q_{2n}\mathbf{B} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1}\mathbf{B} & q_{n2}\mathbf{B} & \cdots & \mathbf{A}+q_{nn}\mathbf{B} \end{bmatrix}_{(n\times m).(n\times m)} \quad (1)$$

where q_{ij} (i,j=1,2,..n) can be real or complex numbers.

The coefficients of the submatrices **B** form an n×n matrix as:

$$[\mathbf{Q}]=\begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}_{n\times n} \quad (2)$$

It can be proved that submatrices $\mathbf{s}_i$ can be formed as

$$\mathbf{s}_i = \mathbf{A}+\lambda_i\mathbf{B}, \quad (3)$$

such that $\lambda_i (i=1,2,...,n)$ are the eigenvalue of the matrix **Q**, and:

$$\{\mathrm{eig}[\mathbf{M}]\} = \bigcup_{i=1}^{n} \{\mathrm{eig}[\mathbf{s}_i]\} \quad (4)$$

$\mathbf{s}_i$ s are called *condensed submatrices* of **M**, and $\lambda_i s\,(i=1,2,...,n)$ are the *decomposition coefficients*.

Proof. Using the notation from tensor product of matrices, **M** can be written as:

$$\mathbf{M}_{m\times n} = \mathbf{I}_n \otimes \mathbf{A}_m + \mathbf{Q}_n \otimes \mathbf{B}_m. \quad (5)$$

Here $\mathbf{I}_n$ is an n×n identity matrix. $\mathbf{A}_m$ and $\mathbf{B}_m$ are m×m submatrices and $\mathbf{Q}_n$ contains the coefficients (multipliers) of $\mathbf{B}_m$ in **M**.

Now consider the eigenvalues of $\mathbf{A}_m$ as $\mu_i (i=1,2,...,n)$. Then the first term in the right-hand side of Eq. (5) is eigen-equivalent to:

$$\begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \cdots & \\ & & & \mu_n \end{bmatrix}, \tag{6}$$

Two matrices are called *eigen-equivalent* if there is a similarity transformation between these matrices (see Appendix for definition).

The second term is eigen-equivalent to

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_n \end{bmatrix} \otimes \mathbf{B}_m, \tag{7}$$

where $\lambda_i (i = 1,2,...,n)$ are the eigenvalues of **Q**.

Now **M** is a block diagonal matrix where in each diagonal entry, $\lambda_i \times \mathbf{B}_m$ is summed with $\mathbf{A}_m$. The eigenvalues of **M** are formed as the eigenvalues of its diagonal blocks matrices $\bigcup_{i=1}^{n} \text{eig}(\mathbf{A}_m + \lambda_i \mathbf{B}_m)$.

Now an important question arises which asks when $\text{eig}(\mathbf{A}_m + \lambda_i \mathbf{B}_m)$ can be expressed in terms of the eigenvalues of $\mathbf{A}_m$ and $\mathbf{B}_m$. In a special case it can be observed that if $\mathbf{A}_m$ and $\mathbf{B}_m$ can be expressd as a linear combination of **I** and **Q**, then $\text{eig}(\mathbf{A}_m + \lambda_i \mathbf{B}_m) = \text{eig}(\mathbf{A}_m) + \lambda_i \text{eig}(\mathbf{B}_m)$. If $\mathbf{A}_m$ and $\mathbf{B}_m$ are considered as the genertors of two graph product, then such a condition is always fulfilled for the adjacency matrices, in Cartesian, strong Cartesian abd direct products. This combination also holds for Cartesian product when Laplacian matrices are considered. However, for strong Cartesian and direct products, members should be added to the free edges of the graph product to provide this property. Thus the eigenvalues can be expressed in terms of the eigenvalues of their generators, Ref. [19].

Special Case: Consider **M** as a 2×2 block matrix,

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B} & \mathbf{A} \end{array}\right], \tag{8}$$

which corresponds to the coefficient matrix:

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{9}$$

Eigenvalue of **Q** are found as

$$|\mathbf{Q}| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow (-\lambda)^2 - 1 = 0, \tag{10}$$

Leading to $\lambda_1 = 1$ and $\lambda_2 = -1$. Therefore

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{A} + \lambda_1 \mathbf{B} = \mathbf{A} + \mathbf{B}, \\ \mathbf{s}_2 &= \mathbf{A} + \lambda_2 \mathbf{B} = \mathbf{A} - \mathbf{B}. \end{aligned} \tag{11}$$

These are the same relationships developed in Refs [15-16] as Form II symmetry.

Example: Consider the following 4×4 block matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}-\mathbf{B} & \mathbf{0} & \mathbf{B} \\ \mathbf{B} & \mathbf{0} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{B} & \mathbf{0} & \mathbf{A}-3\mathbf{B} \end{bmatrix}, \tag{12}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}, \text{ and } \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{13}$$

For this compound matrix, the coefficient matrix **Q** is formed as:

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 \end{bmatrix}, \tag{14}$$

leading to:

$$\{\text{eig}[\mathbf{Q}]\} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{0, 1.1284, -0.7984, -3.3301\}. \tag{15}$$

Therefore

$$\begin{array}{ll} \mathbf{s}_1 = \mathbf{A} & \text{eig}[\mathbf{s}_1] = \{-1.8730, 5.8730\} \\ \mathbf{s}_2 = \mathbf{A} + 1.1284\mathbf{B} & \text{eig}[\mathbf{s}_2] = \{10.0405, 1.8584\} \\ \mathbf{s}_3 = \mathbf{A} - 0.7984\mathbf{B} \quad \Rightarrow & \text{eig}[\mathbf{s}_3] = \{-4.9455, 3.3569\} \\ \mathbf{s}_4 = \mathbf{A} - 3.3301\mathbf{B} & \text{eig}[\mathbf{s}_4] = \{-16.2340, -3.0765\} \end{array} \tag{16}$$

and

$$\begin{aligned} \text{eig}[\mathbf{M}] &= \bigcup_{i=1}^{4} \text{eig}[\mathbf{s}_i] \\ &= \{-1.8730, 5.8730, 10.0405, 1.8584, -4.9455, 3.3569, -16.2340, -3.0765\}. \end{aligned} \tag{17}$$

3. APPLICATIONS IN GRAPH THEORY

Consider the Laplacian matrix **L** of a graph (see Appendix for definitions) composed of submatrices **s** and **h**, such that the submatrix **s** together with fraction of **h** are on the diagonal entrie and multiples of **h** in the other entries, as follows:

$$[\mathbf{L}] = \begin{bmatrix} \mathbf{s} + q_{11}\mathbf{h} & q_{12}\mathbf{h} & \cdots & q_{1n}\mathbf{h} \\ q_{21}\mathbf{h} & \mathbf{s} + q_{22}\mathbf{h} & \cdots & q_{2n}\mathbf{h} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1}\mathbf{h} & q_{n2}\mathbf{h} & \cdots & \mathbf{s} + q_{nn}\mathbf{h} \end{bmatrix}, \tag{18}$$

where q_{ij} (i,j=1,2,..n) can be real numbers.

The coefficients of the submatrices **h** form an n×n matrix as:

$$[\mathbf{Q}] = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}. \tag{19}$$

If the eigenvalues of **Q** are shown by $\lambda_i (i = 1, 2, ..., n)$, then the condensed cores of the graph G can be found as follows:

$$\mathbf{s}_i = \mathbf{A} + \lambda_i \mathbf{B}, \tag{20}$$

and

$$\{\text{eig}[\mathbf{L}]\} = \bigcup_{i=1}^{n} \{\text{eig}[\mathbf{s}_i]\} \tag{21}$$

The subgraphs corresponding to $\mathbf{s}_i$ (i=1,2,…,n) are called the *condensed cores* of G. Each such a core contains part of the eigenvalues of the Laplacian matrix of the graph G.

One of the application of latter formula is the calculation of the spectra of those graphs which are formed from repeated subgraphs. Another application corresponds to the decompositon of unsymmetric graphs in special cases.

4. APPLICATION TO FORM IV SYMMETRY

Form IV symmetry is develop and discussed in Ref. [20]. In this section the application of the theorem proved in this paper is discussed.

Consider the following 6m×6m matrix in a tri-diagonal block form:

$$[\mathbf{L}] = \begin{bmatrix} \mathbf{s}-\mathbf{h} & -\mathbf{h} & & & & \\ -\mathbf{h} & \mathbf{s} & -\mathbf{h} & & & \\ & -\mathbf{h} & \mathbf{s} & -\mathbf{h} & & \\ & & -\mathbf{h} & \mathbf{s} & -\mathbf{h} & \\ & & & -\mathbf{h} & \mathbf{s} & -\mathbf{h} \\ & & & & -\mathbf{h} & \mathbf{s}-\mathbf{h} \end{bmatrix}_{6m\times 6m} \tag{22}$$

The coefficient matrix $\mathbf{Q}$ is as follows:

$$[\mathbf{Q}] = \begin{bmatrix} -1 & -1 & & & & \\ -1 & 0 & -1 & & & \\ & -1 & 0 & -1 & & \\ & & -1 & 0 & -1 & \\ & & & -1 & 0 & -1 \\ & & & & -1 & -1 \end{bmatrix}, \tag{23}$$

and

$$\text{eig}[\mathbf{Q}] = \{-\sqrt{3}, -2, -1, 0, 1, \sqrt{3}\}\,. \tag{24}$$

Therefore the condensed cores of the graph are as:

$$\begin{aligned}
\mathbf{s}_1 &= \mathbf{s} - \sqrt{3}\mathbf{h} \\
\mathbf{s}_2 &= \mathbf{s} + \sqrt{3}\mathbf{h} \\
\mathbf{s}_3 &= \mathbf{s} - \mathbf{h} \\
\mathbf{s}_4 &= \mathbf{s} + \mathbf{h} \\
\mathbf{s}_5 &= \mathbf{s} - 2\mathbf{h} \\
\mathbf{s}_6 &= \mathbf{s}
\end{aligned} \tag{25}$$

5. LAPLACIAN OF A GRAPH COMPOSED OF REPEATED SUBGRAPHS

Consider a graph G as shown in Figure 1. This graph consists of three subgraphs g_1, g_2 and g_3 connected by link members. The Laplacian of G can be expressed as:

$$[\mathbf{L}(G)] = \begin{bmatrix} \mathbf{s}(g_1) & \mathbf{h}(g_1,g_2) & \mathbf{h}(g_1,g_3) \\ \mathbf{h}(g_2,g_1) & \mathbf{s}(g_2) & \mathbf{h}(g_2,g_3) \\ \mathbf{h}(g_3,g_1) & \mathbf{h}(g_3,g_2) & \mathbf{s}(g_3) \end{bmatrix}, \tag{26}$$

where $\mathbf{s}(g_i)$ is the Laplacian matrix of the subgraph g_i and $\mathbf{h}(g_i,g_j)$ expresses the manner in which two subgraphs g_i and g_j are connected to each other. In general $\mathbf{h}(g_i,g_j) = \mathbf{h}^t(g_j,g_i)$ and in special case such as symmetry, $\mathbf{h}(g_i,g_j) = \mathbf{h}(g_j,g_i)$.

The link matrix $\mathbf{h}(g_i,g_j)$ is the same as the adjacency matrix of the two subgraphs g_i and g_j with reverse sign. The column-indices of the matrix correspond to nodal labels of g_i and the row-indices are those of g_j.

Example: Consider the graph G composed of three subgraphs g_1, g_2 and g_3 connected by link members. The subgraphs are shown in bold lines. The submatrices of $\mathbf{L}(G)$ are as follows:

$$[\mathbf{s}(g_1)] = \begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \end{matrix}, \quad [\mathbf{s}(g_2)] = \begin{matrix} & \begin{matrix} C & D & E & F \end{matrix} \\ \begin{matrix} C \\ D \\ E \\ F \end{matrix} & \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{matrix}, \quad [\mathbf{s}(g_3)] = \begin{matrix} & \begin{matrix} H & J & I \end{matrix} \\ \begin{matrix} H \\ J \\ I \end{matrix} & \begin{bmatrix} 4 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{matrix} \tag{27}$$

Link members are as follows:

$$[\mathbf{h}(g_1,g_2)]=\begin{array}{c} \\ A \\ B \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C & D & E & F \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \end{array},\ [\mathbf{h}(g_2,g_1)]=\begin{array}{c} \\ C \\ D \\ E \\ F \end{array}\!\!\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \tag{28a}$$

$$[\mathbf{h}(g_3,g_2)]=\begin{array}{c} \\ H \\ J \\ I \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},\ [\mathbf{h}(g_2,g_3)]=\begin{array}{c} \\ C \\ D \\ E \\ F \end{array}\!\!\begin{array}{c} \begin{array}{ccc} H & J & I \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{array}, \tag{28b}$$

$$[\mathbf{h}(g_3,g_1)]=\begin{array}{c} \\ H \\ J \\ I \end{array}\!\!\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \text{ and } [\mathbf{h}(g_1,g_3)]=\begin{array}{c} \\ A \\ B \end{array}\!\!\begin{array}{c} \begin{array}{ccc} H & J & I \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{array}. \tag{28c}$$

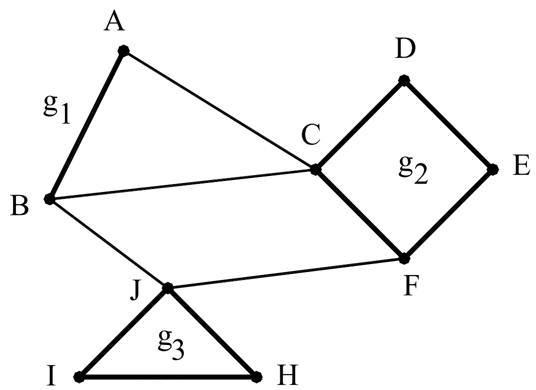

Figure 1. A graph composed of three subgraphs and link members

Now **L**(G) is formed as:

$$[\mathbf{L}(G)] = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ J \\ H \\ I \end{array} \begin{array}{c} \begin{array}{ccccccccc} A & B & C & D & E & F & J & H & I \end{array} \\ \left[\begin{array}{cc|cccc|ccc} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline -1 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{array}\right] \end{array}. \tag{29}$$

Special Case 1: For some graphs with repeated subgraphs $\mathbf{h}(g_i,g_j) = \mathbf{h}(g_j,g_i) = \mathbf{h}$, then the Laplacian has the pattern as given in Eq. (22).

As an example, consider a graph G comprising of a repeated subgraph g_i connected to the nodes of a path graph. The Laplacian of the entire graph will have the following pattern:

$$[\mathbf{L}(G)] = \begin{bmatrix} \mathbf{s}-\mathbf{h} & -\mathbf{h} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{h} & \mathbf{s} & -\mathbf{h} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{h} & \mathbf{s} & \dots & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{s} & -\mathbf{h} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{h} & \mathbf{s} & -\mathbf{h} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & -\mathbf{h} & \mathbf{s}-\mathbf{h} \end{bmatrix}, \tag{30}$$

where $\mathbf{s} = \mathbf{s}(g_i)$ is the Laplacian matrix of a repeated subgraph of G, and $\mathbf{h}$ is the connection matrix. Forming $\mathbf{Q}$ and calculating its eigenvalues $\lambda_i (i = 1,2,...,n)$, the condensed matrices of $\mathbf{L}$(G) can be found, and the corresponding eigenvalues can easily be calculated using Eq. (20). For the graph shown in Figure 2, $\mathbf{s}$ and $\mathbf{h}$ are as:

$$\mathbf{s} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{h} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

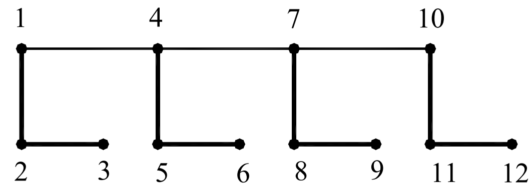

Figure 2. A graph with linear repeated subgraphs

The Laplacian of G has the following pattern:

$$\mathbf{L}(\mathrm{G}) = \begin{bmatrix} \mathbf{s-h} & \mathbf{-h} & \mathbf{0} & \mathbf{0} \\ \mathbf{-h} & \mathbf{s} & \mathbf{-h} & \mathbf{0} \\ \mathbf{0} & \mathbf{-h} & \mathbf{s} & \mathbf{-h} \\ \mathbf{0} & \mathbf{0} & \mathbf{-h} & \mathbf{s-h} \end{bmatrix}_{12\times 12}. \quad (32)$$

The matrix **Q** is formed as

$$\mathbf{L}(\mathrm{G}) = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad (33)$$

leading to

$$\mathrm{eig}[\mathbf{Q}] = \{0, 1.4142, -1.4142, -2\}. \quad (34)$$

Therefore the condensed cores of the graph are as,

$$\begin{array}{ll} \mathbf{s}_1 = \mathbf{s} & \mathrm{eig}[\mathbf{s}_1] = \{3.7321, 2, 0.2679\} \\ \mathbf{s}_2 = \mathbf{s} + 1.4142\mathbf{h} & \mathrm{eig}[\mathbf{s}_2] = \{4.8072, 2.2979, 0.3091\} \\ \mathbf{s}_3 = \mathbf{s} - 1.4142\mathbf{h} & \mathrm{eig}[\mathbf{s}_3] = \{1.3217, 0.1420, 3.1221\} \\ \mathbf{s}_4 = \mathbf{s} - 2\mathbf{h} & \mathrm{eig}[\mathbf{s}_4] = \{0, 1, 3\} \end{array} \quad (35)$$

leading to:

$$\{\mathrm{eig}[\mathbf{L}(\mathrm{G})]\} = \bigcup_{i=1}^{n} \{\mathrm{eig}[\mathbf{s}_i]\} \quad (36)$$

$$= \{3.7321, 2, 0.2679, 4.8072, 2.2979, 0.3091\}, 1.3217, 0.1420, 3.1221, 0, 1, 3\}.$$

Special Case 2: Consider a graph G comprising of a repeated subgraph g_i connected to the nodes of a cycle graph, Figure 3. The Laplacian of the entire graph will have the following pattern:

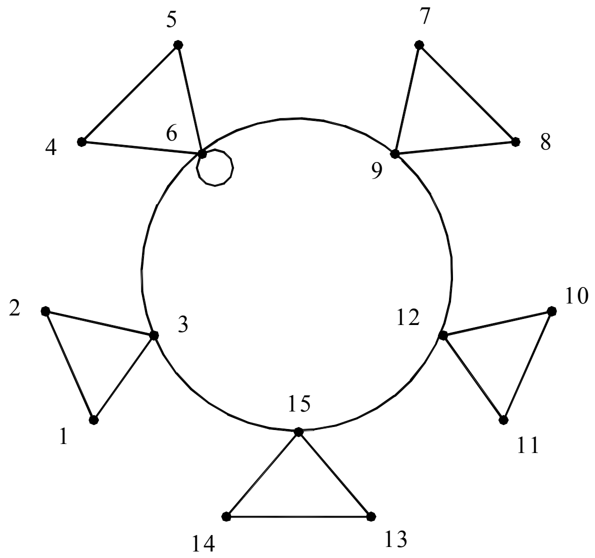

Figure 3. A graph with circular repeated subgraphs

$$\mathbf{L}(G)=\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 6 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 4
\end{array}\right] \quad (37)$$

The structure of the Laplacian matrix, suggests that the matrix of the repeated core and the link matrix are as follows:

$$\mathbf{s}=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 4\end{bmatrix}_{3\times 3}\qquad \mathbf{h}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}_{3\times 3} \tag{38}$$

The Laplacian matrix has the following pattern:

$$\mathbf{L}(\mathrm{G})=\begin{bmatrix}\mathbf{s} & \mathbf{h} & \mathbf{0} & \mathbf{0} & \mathbf{h}\\ \mathbf{h} & \mathbf{s}-2\mathbf{h} & \mathbf{h} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{h} & \mathbf{s} & \mathbf{h} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{h} & \mathbf{s} & \mathbf{h}\\ \mathbf{h} & \mathbf{0} & \mathbf{0} & \mathbf{h} & \mathbf{s}\end{bmatrix}_{15\times 15} \tag{39}$$

The matrix **Q** is formed:

$$\mathbf{Q}=\begin{bmatrix}0 & 1 & 0 & 0 & 1\\ 1 & -2 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1\\ 1 & 0 & 0 & 1 & 0\end{bmatrix}_{5\times 5} \tag{40}$$

and its eigenvalues are calculated as

$$\mathrm{eig}[\mathbf{Q}]=\{-2.7013,-1.6180,0,0.6180,1.7913\}\,. \tag{41}$$

Therefore, the condensed cores of the graph are as

$$\mathbf{s}_i=\mathbf{s}+\mathrm{eig}[\mathrm{Q}_i]\mathbf{h}. \tag{42}$$

Substitution of these eigenvalues in Eq. (42) leads to:

$$\begin{aligned}
&\mathbf{s}_1=\mathbf{s}-2.7913\mathbf{h} && \mathrm{eig}[\mathbf{s}_1]=\{3,0.6731,7.1182\}\\
&\mathbf{s}_2=\mathbf{s}-1.6180\mathbf{h} && \mathrm{eig}[\mathbf{s}_2]=\{3,0.6013,6.0167\}\\
&\mathbf{s}_3=\mathbf{s} && \mathrm{eig}[\mathbf{s}_3]=\{3,0.4384,4.5616\}\\
&\mathbf{s}_4=\mathbf{s}+0.6180\mathbf{h} && \mathrm{eig}[\mathbf{s}_4]=\{3,4.0399,0.3421\}\\
&\mathrm{s}_5=\mathbf{s}+1.7913\mathbf{h} && \mathrm{eig}[\mathbf{s}_5]=\{3,3.1423,0.0664\}
\end{aligned} \tag{43}$$

Thus the eigenvalues of **L**(G) are as follows:

$$\{\mathrm{eig}[\mathbf{L}(G)]\} = \bigcup_{i=1}^{n} \{\mathrm{eig}[\mathbf{s}_i]\} \tag{43}$$

={3,0.6731,7.1182,3,0.6013,6.0167,3,0.4384,4.5616,3,4.0399,0.3421,3,3.14230.0664}.

In fact the cores are triangles with different nodal weights.

6. CONCLUDING REMARKS

The main aim of this paper is to generalize the previously developed canonical forms and to provide a powerful means for decomposing symmetric graphs with repeated subgraphs.

Civil engineering structures contain a large number of members and nodes with different symmetries. The present method simplifies the numerical operations required for calculating the eigenvalues and eigenvectors of the corresponding matrices.

Applications of the present methods can be extended to different civil engineering problems, such as calculating the natural frequencies of vibrating systems and the buckling load of structures. The method can also be employed in other fields of engineering where eigenvalues and eigenvectors of matrices are involved.

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APPENDIX: DEFINITIONS FROM ALGEBRAIC GRAPH THEORY

A *graph* G (N,E) consists of a set of elements, N, called *nodes* and a set of elements, E, called *edges (members)*, together with a relation of incidence which associates two distinct nodes with each edge, known as its *ends*. Two nodes of a graph are called *adjacent* if these nodes are the end nodes of an edge. An edge is called *incident* with a node if it is an end node of the edge. The *degree* of a node is the number of edges incident with the node. A *subgraph* S_i of a graph S is a graph for which $N(S_i) \subseteq N(S)$ and $E(S_i) \subseteq E(S)$, and each member of S_i has the same ends as in S.

The adjacency matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ of a labelled graph G, containing n nodes, is defined as

$$a_{ij} = \begin{cases} 1 & \text{if node } n_i \text{ is adjacent to } n_j \\ 0 & \text{otherwise} \end{cases} \tag{A1}$$

The degree matrix $\mathbf{D} = [d_{ij}]_{n\times n}$ is a diagonal matrix of node degrees. d_{ij} is equal to the degree of the ith node.

The Laplacian matrix $\mathbf{L} = [l_{ij}]_{n \times n}$ is defined as

$$\mathbf{L} = \mathbf{D} - \mathbf{A} \tag{A2}$$

Therefore, the entries of L are as

$$l_{ij} = \begin{cases} -1 & \text{if node } n_i \text{ is adjacent to } n_j \\ \deg(n_i) & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{A3}$$

Further definitions and concepts from theory of graphs and applications in structural mechanics may be found in Ref. [10].

Two matrices **P** and **R** are said to be *similar* if there exists a nonsingular **S** such that

$$\mathbf{R} = \mathbf{S}^{-1}\mathbf{P}\mathbf{S}. \tag{A4}$$

This equation is a *similarity transformation* and **S** is the *transforming matrix*. The above equation is equivalent to **SR** = **PS**.