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A GENERAL METHOD FOR THE GEOMETRICALLY NONLINEAR ANALYSIS OF STRUCTURES

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ABSTRACT

A method is presented which permits the geometrically nonlinear analysis of structures undergoing arbitrarily large displacements, rotations and strains. This method is based on the general nonlinear theory of elasticity and can be applied consistently to different types of structural elements such as beams and plates. The method has so far been implemented for two-dimensional trusses and frames. For several examples of structures with very large displacements and rotations as well as snap-through, the results which are obtained with the method are compared to exact solutions. The convergence of the method to the exact results is demonstrated.

Keywords: geometric nonlinearity, structural analysis, finite element method

1. INTRODUCTION

The increasing size of the world population and the necessity to raise the standard of living in disadvantaged areas place new demands on the civil engineering profession. Since the available amount of material, energy and labour is limited, their use must be optimized. One of the strategies which may contribute to this goal is a reduction in the amount of material that is contained in the structural systems. This strategy reduces the self-weight of the structures (which is a very significant part of the total load), thus further reducing the amount of material that is required to support the structure. In addition, the behaviour of the structure during earthquakes can be improved by the reduced mass, and the amount of energy required for the production of building materials and for the construction of buildings can be reduced.

The strategy of reducing the amount of material for a structure with given shape and function can require knowledge and skills which are not treated in depth in traditional civil engineering education and practise. As the amount of material in a structure is reduced, the stiffness of the structure is reduced and therefore its deformation under load increases. It is

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no longer sufficient to compute the forces in and the displacements of the structural members with traditional linear theories of structural analysis, which are emphasized in the traditional professional education of civil engineers. Instead, the structural behaviour must be predicted with nonlinear theories which take into account the influence of the displacements on the forces in the structure and on its stability. These theories are considerably more complex than the corresponding theories for geometrically linear behaviour. For example, the principle of superposition for loading conditions, which is extensively applied in linear analysis, no longer holds in nonlinear analysis. While reliable computer-based methods of linear structural analysis are available, the methods for non-linear analysis have not yet been fully generalized and are not yet supported by suitable software. Practising engineers have limited intuitive insight into the nonlinear behaviour of their structures.

This paper reports a line of research which is intended to contribute to a unification of the nonlinear analysis of different types of structures. All types of structural components are to be treated in a similar fashion. The limitations on the range of validity of the theories will be reduced. Different component types are to be integrated into a single application package for nonlinear structural analysis. At the same time, it is hoped that the education of civil engineers in nonlinear analysis will become easier and more efficient, thus laying a sound foundation of the application of nonlinear methods of structural analysis in civil engineering practise. The concept is being pursued by the authors in a cluster of research projects in close cooperation with Professor P. Dunaiski and Dr. G. van Rooyen of Stellenbosch University in South Africa.

The nonlinear analysis of structural systems differs in three main aspects from the linear analysis of these structures:

- The elastic strains (deformations) of the structure are non-linear functions of the derivatives of their displacements.
- The equilibrium equations must be formulated in the displaced geometric state of the structure, which is unknown at the start of the analysis and remains to be determined.
- There can be plastic strains which depend on the load history. The stresses (forces) in the structure are no longer linear functions of the elastic strains in the structure.

The physical nonlinearity described in the last point will be treated indirectly in this paper by considering tension-only bars in trusses. The emphasis of the paper is placed on the geometric nonlinearity of structural behaviour.

2. STATE OF THE ART

The future significance of nonlinear structural analysis was clearly recognized in the 1970's, as is shown by a large number of ground-breaking publications such as the papers by Wood and Zienkiewicz [1] on the geometrically nonlinear finite element analysis of beams, frames, arches and axisymmetric shells; Wunderlich and Beverungen [2] on a geometrically nonlinear theory for plane curved rods; Argyris, Dunne and Scharpf [3] on large displacement – small strain analysis of structures; Bathe and Polourch [4] on large displacement analysis of three-dimensional beam structures. The Europe-U.S. Workshop [8] at Bochum in 1980 was devoted to "Nonlinear Finite Element Analysis in Structural Mechanics" and treated topics such as the general formulations, geometrically nonlinear rods and shells, physically nonlinear structures, nonlinear dynamics, methods of solution and computational algorithms. A total of 63 authors presented 38 papers.

In 1981, the German Research Association installed a Schwerpunktprogramm (focal program) "Nonlinear Computations in Structural Engineering" with Professor E. Stein as coordinator. Most of the German technical universities participated with research projects. Experiences were exchanged at five colloquia. At the colloquium which concluded the Focal Program in 1989, a total of 72 authors presented 30 papers [9] which summarize substantial research efforts extending over many years. The main areas of research presented were the geometri-cally and physically nonlinear behaviour of frames and of thin-walled surface structures as well as the physically nonlinear behaviour of concrete structures.

Since that time, a steady sequence of papers and books has been published, dealing with geometrical and physical nonlinearities in the analysis of many types of structural components and presenting a wealth of different approaches to the problem [10] to [21]. Software products such as [22] to [25] offer nonlinear analysis capabilities.

In view of the substantial volume and high quality of the research which was performed in the past, the question arises whether additional research in the coming years is likely to achieve significant progress in nonlinear structural analysis. In our opinion, this progress can be expected, since research in the past was not able to resolve two issues which stand in the way of a broad and beneficial application of nonlinear analysis in everyday engineering by normally trained and experienced professionals: the limitations on the range of validity of the nonlinear theories and the lack of a unified treatment of all types of structural components.

The limitations on the range of validity of the nonlinear theories stem primarily from approximations in the strain-displacement relations, particularly the replacement of trigonometric functions of angles of rotation by series approximations and the neglect of displacement derivatives which are small under specific circumstances. It is difficult for the practising engineer to judge reliably whether the behaviour of his specific structure satisfies the limitations of a specific theory. Another limitation is the inability of many algorithms to follow the displacement - load curve through points of singularity of the system stiffness matrix. This precludes the reliable identification of snap-through effects.

The Focal Progrmam [9] demonstrated the difficulties encountered in a unified treatment of nonlinear structural analysis. It contained a substantial effort to create DFGBIB, a Fortran-based Software-Library for nonlinear structural analysis. DFGBIB consists of a basic library and a general library. The basic library contains programs for finite element topology and geometry, matrix algebra and parsing of alphanumeric input. It is intended to be used by all components of the general library. The general library contains many different types of finite elements, subroutines for different types of material behaviour as well as alphanumeric pre- and postprocessors for finite element models. It was decided, against advice to the contrary, not to use a unified data base and an integrated interactive graphic user interface. A set of benchmarks was created for testing of the finite elements.

In spite of the high level research effort that created DFGBIB, the library did not have the desired effect in engineering practise. From the perspective of today, it is evident that this can be attributed to the problems described above. The theoretical approach of the different contributors to DFGBIB was not unified. In addition, coherence and generalisation in the data structure, the algorithms and the user interfaces are prerequisites for the success of any computer-assisted engineering application. The DFGBIB did not meet this requirement.

It may be argued that the software with nonlinear capabilities for structural analysis, which is available on the market, provides the desired capabilities. Yet this commercial software has black-box structure, so that it is very difficult for an engineer who carries the responsibility for a design to form an independent opinion of the suitability of the provided algorithms for his specific task. Since the reliability of the results frequently depends on the tuning of the algorithm with suitable step sizes and error bounds, the degree of allowable automation of the computations is considerably less than for linear analyses, and insight is essential.

3. A NEW CONCEPT FOR NONLINEAR STRUCTURAL ANALYSIS

The aim of the research reported in his paper is to reduce the complexity and to increase the reliability of geometrically nonlinear structural analysis, and thus to improve the attractiveness of the nonlinear approach for practising engineers. This effort will benefit from several advances that have been achieved in the past decades:

- A thorough mathematical formulation of the mechanics of the nonlinear behaviour of structures can be found in the cited literature.
- Advances and experiences in Bauinformatik provide a platform for the design of coherent and adaptable application software for nonlinear structural analysis.
- The advances in computer technology provide significantly extended data storage capacities both at runtime and in the files, as well as processing speeds that exceed those of 1989 by the order of 1000.
- The advances in software technology provide programming platforms such as Java which are object-oriented and contain many functions for data bases and for user interfaces that had to be programmed with great effort in earlier years.

In order to achieve the desired reduction in complexity and increase in reliability, a new generalized concept for nonlinear structural analysis must be invented which is readily understandable, computationally efficient and computer-oriented. The volume of the theory that must be presented in engineering education for nonlinear behaviour should be reduced by treating all types of structural components in a common manner, which is implemented in a common software whose theoretical foundations the users understand.

The authors believe that they are following a concept which has the desired characteristics. In this concept, the theory for each type of structural component is derived as a special case of the general nonlinear theory of elasticity by introducing additional equations, which reflect the special aspects of the behaviour of that type of component. These additional equations make the problem over-determined: the number of equations exceeds the number of unknowns, which leads to contradictions. This situation is well-known from linear analysis, for instance in the case of plate theory, but the consequences may be different for linear and nonlinear behaviour. The deformation method is used for all element types. As in the usual approach, the incremental solution algorithm is based on the finite element method and the arc length control method. The concept can be applied to every type of structural component for which the following conditions can be satisfied:

- If approximate values for the displacements and reactions of the structure are known for given values of the prescribed displacements and loads on the structure, the error vector in the governing equations can be determined by an exact theory which does not contain any approximation as to the magnitude of the displacements, the rotations and the strains in the structure.
- If the displacements, the reactions and the error vector are known for a state of the structure, the displacements and reactions in the next state of the structure can be predicted with approximate governing equations. The approximate theory and the displacement trial functions must be such that rigid body motions of a finite element do not cause forces which act on the element.

It will be shown in this paper that the new concept can be applied to two-dimensional trusses and frames and leads to accurate results even if the displacements and rotations are very large. This suggests that there is sufficient probability that the new line of research can lead to sufficiently simple and reliable analyses of structures consisting of different types of compo-nents for arbitrarily large strains, displacements and rotations.

4. EXAMPLES OF NONLINEAR STRUCTURAL BEHAVIOUR

As an introduction to the nonlinear theory for trusses and frames, two examples of nonlinear behaviour are presented for which exact solutions are known. The 2-bar-truss illustrates snap-through behaviour, the cantilever beam is characterized by a rotation through an angle of 2π at its tip. Thus both examples show great differences between the linear and the nonlinear behaviour of the structure. These examples will be used in later sections to test the accuracy of the results obtained with finite element analyses following the new concept. They will also be used to study the sensitivity of the computational algorithms to the values of their control parameters, such as step size in the displacement-load diagram and error limits for the termination of iterations.

4.1 Snap-Through of a 2-Bar-Truss

Figure 1 shows a symmetric 2-bar-truss CAD which carries a vertical load P at its apex A. The dimensions of the truss are shown in the figure. The supports C and D are fixed. Let the area of the cross-section of each bar be A and its modulus of elasticity E. If the coordinates of the displacement of the apex A to point B under load P are specified in global space x, the displacement is denoted by **u**. If the coordinates of the displacement are specified in the local space y, the vector is denoted by **v.** The value of the load P is to be determined as a function of the displacement **u**.

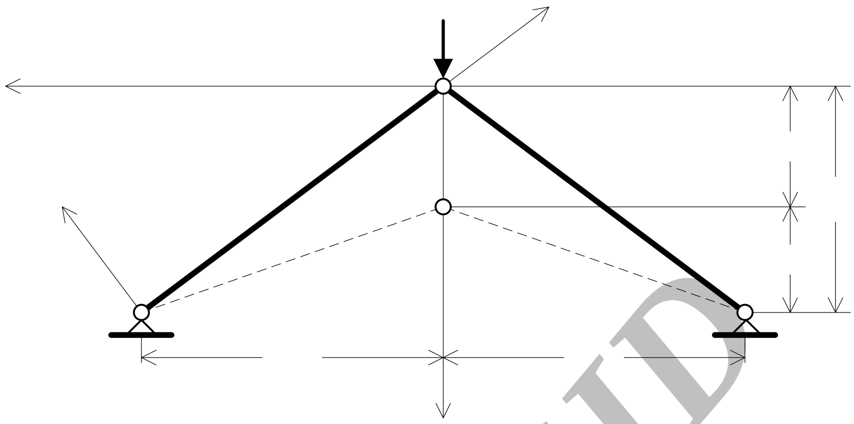

Figure 1. Nonlinear analysis of a 2-bar-truss

Consider the base vectors in the direction of the axis of a bar in the reference and in the instant configuration under load P. Then the axial strain ε in the bar is given by the strain tensor of Green:

$$\mathbf{e_1} = \frac{d\mathbf{y}}{dy_1}$$

$$\mathbf{g_1} = \frac{d(\mathbf{y}+\mathbf{v})}{dy_1} = \mathbf{e_1} + v_{1,1}\,\mathbf{e_1} + v_{2,1}\,\mathbf{e_2}$$

$$\varepsilon = \frac{1}{2}(\mathbf{g}_1^T\mathbf{g}_1 - \mathbf{e}_1^T\mathbf{e}_1) = \frac{1}{2}((1+v_{1,1})^2 + (v_{2,1})^2 - 1) = \frac{1}{2}((\frac{L}{L_0})^2 - 1)$$

The equilibrium at the apex A in the instant configuration yields the physical stress $\hat{\sigma}$ and the corresponding 2.Piola-Kirchhoff stress σ in the bar:

x 1

$$\sigma = \frac{L_0}{L}\hat{\sigma} = -\frac{P}{2A}\frac{L}{h}\frac{L_0}{L} = -\frac{PL_0}{2Ah}$$

The stress-strain relationship σ = E ε and the geometrical relationship h = $h_o - u_2$ yield:

y 2

$$P = AE(\frac{h_0}{L_0})^3(\frac{u_2}{h_0})(1-\frac{u_2}{h_0})(2-\frac{u_2}{h_0})$$

Equilibrium at node C leads to the horizontal component R_1 of the reaction at C:

$$R_1 = \frac{AE}{2}(\frac{h_0}{L_0})^2(\frac{a}{L_0})(\frac{u_2}{h_0})(2-\frac{u_2}{h_0})$$

Define the normalized displacement s, load p and reaction r as follows:

$$s := \frac{u_2}{h_0} \qquad p := \frac{P}{AE}(\frac{L_0}{h_0})^3 \qquad r := \frac{R_1}{AE}(\frac{L_0}{h_0})^2(\frac{L_0}{a})$$

The function p(s) is shown in Figure 2. The load p reaches local extrema of p=±0.384900 for the displacements s = 0.422650 and s = 1.577350. The truss has three equilibrium configurations for load p = 0, corresponding to displacements s = 0.00, s = 1.00 and s = 2.00. In the range between the local extrema, the load decreases with increasing displacement. The transition at constant load from a displacement state a in this range to a displacement state b, where the load increases with increasing displacement, is called snap-through.

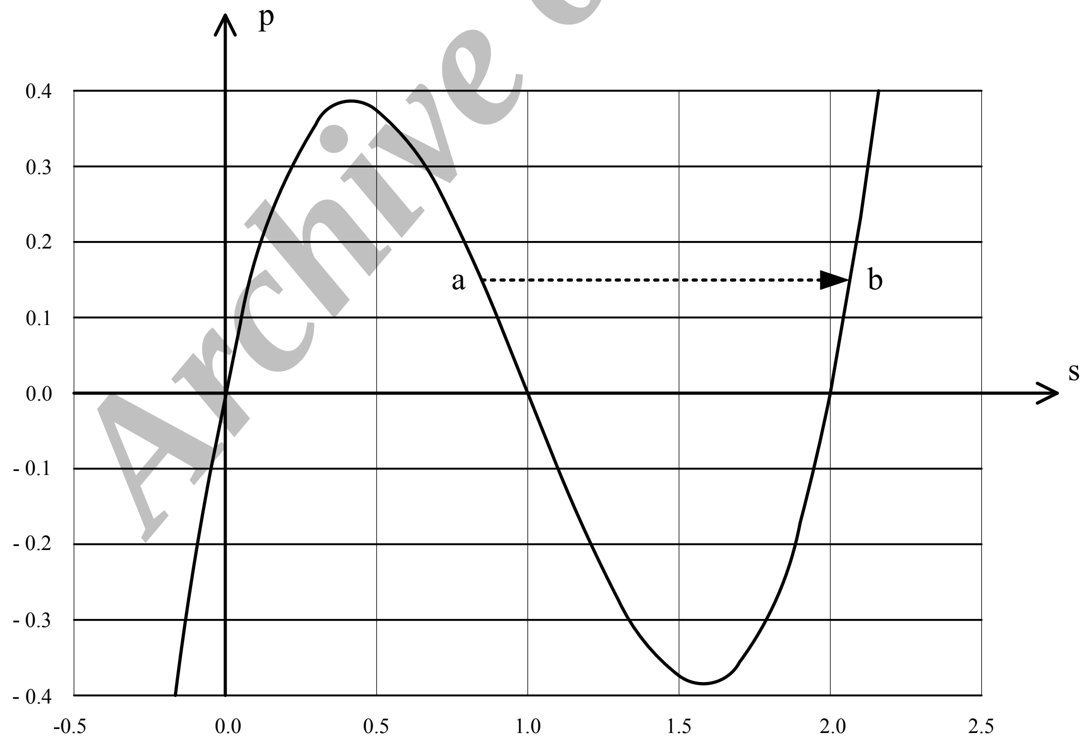

Figure 2. Displacement - load diagram of a 2-bar-truss with snap-through a-b s: normalised displacement p: normalised load

The function r(s) is shown in Figure 3. The value of the reaction for the displacements under the extreme loads p = ±0.384900 is r = 0.333333. The maximum value of the reaction is r = 0.50. It is reached for s = 1.00 and p = 0.00.

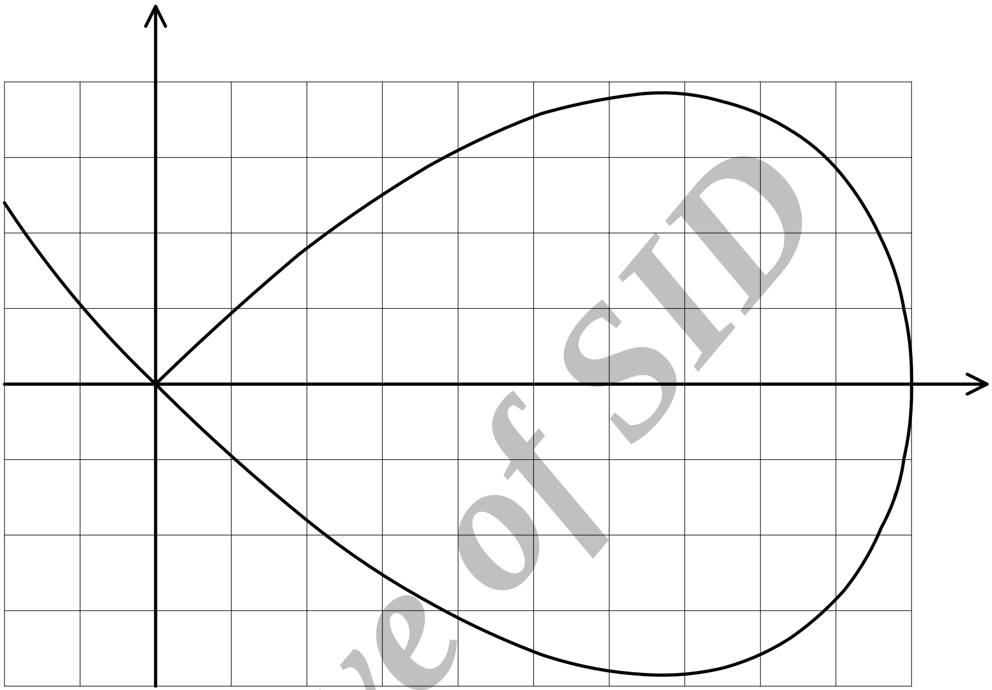

Figure 3. Horizontal reaction – load diagram of a 2-bar-truss r: normalised reaction
p: normalised load

4.2 Large Rotations of a Cantilever Beam with End Moment

Figure 4 shows a cantilever beam with span L and modulus of elasticity E, whose cross-section has area A and moment of inertia I. The displacements due to an end moment M acting at the tip of the cantilever are to be determined as a function of the end moment.

Consider a point P with coordinate s on the axis of the reference configuration of the beam. Let the displacement vector of point P for an instant configuration of the beam be denoted by **u**(s) if its coordinates are referred to global space. Let the rotation of the axis of the beam at point P in the instant configuration be ω. The change in the angle ω with the distance s on the axis of the beam is determined with the Kirchhoff hypothesis. Integration with respect to s and the boundary condition ω = 0 at s = 0 lead to:

p

0.4

0.3

0.2

$$\frac{d\omega}{ds} = -\frac{M}{EI}$$
$$\omega = -\frac{M}{EI}s$$

The differential equations for the displacement coordinates follow from the triangle at $\hat{P}$. Define the normalised moment α and the normalised displacement **v**:

$$1-\frac{du_1}{ds} = \cos\omega \quad \rightarrow \quad \frac{du_1}{ds} = 1-\cos\frac{\alpha s}{L}$$
$$\frac{du_2}{ds} = \sin\omega \quad \rightarrow \quad \frac{du_2}{ds} = \sin\frac{\alpha s}{L}$$
$$\alpha = \frac{ML}{EI} \qquad v_i = \frac{u_i}{L}$$

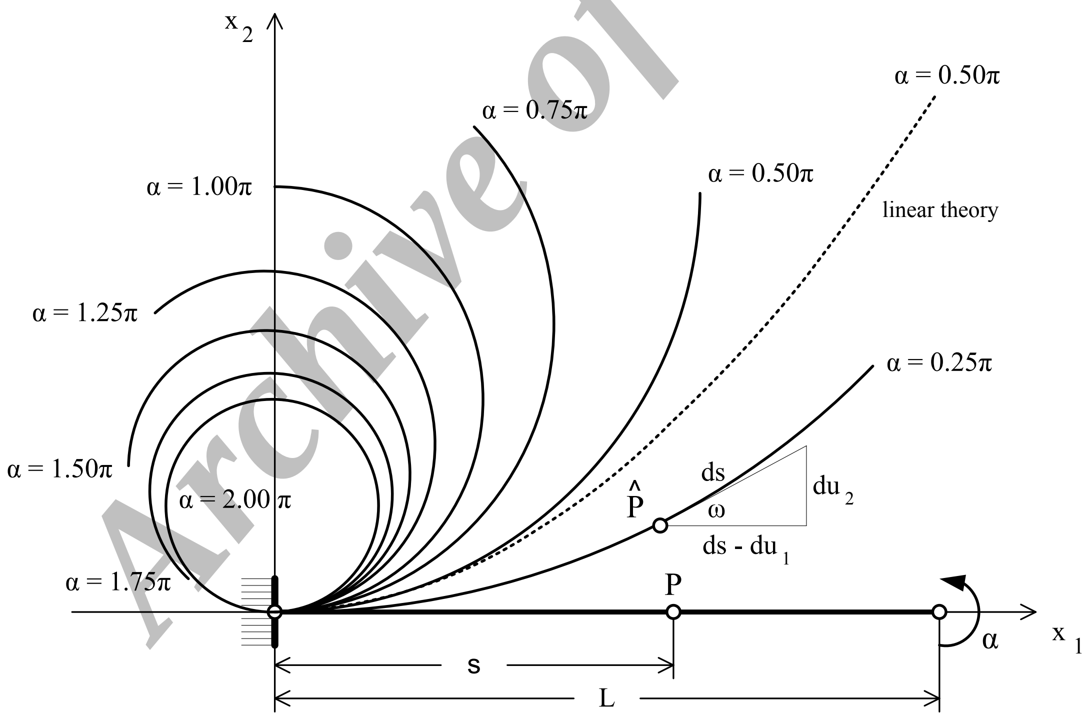

Figure 4. Deformation of a cantilever beam subjected to end moment α

Integration with respect to s and the boundary condition **v**=**0** at s=0 yield the displacement of the axis of the bar:

$$v_1 = \frac{1}{\alpha}\sin\frac{\alpha s}{L} - \frac{s}{L}$$
$$v_2 = \frac{1}{\alpha}(1-\cos\frac{\alpha s}{L})$$

The instant configurations of the bar for $\alpha = 0.25\pi, 0.50\,\pi,\ldots,2.00\,\pi$ are shown in Figure 4. The rotation at the tip of the cantilever is 2π. The displacement computed with linear theory for the end moment $\alpha = 0.50\pi$ is shown as a dotted line

5. THEORY OF NONLINEAR STRUCTURAL BEHAVIOUR

5.1 Nonlinear Theory of Elasticity

The following relations of the nonlinear theory of elasticity are used in the nonlinear theory for trusses and frames [26-27]. Material points are identified by their location in the reference configuration. The coordinates of the strain tensor **E** of Green are computed as functions of the derivatives of the displacement coordinates with respect to the coordinates:

$$e_{km} = 0.5(u_{k,m} + u_{m,k} + \sum_i u_{i,k}u_{i,m})$$

$u_{i,k}$ derivative of u_i with respect to x_k

The Cauchy stress tensor **T** describes the state of stress in the instant configuration. Its coor-dinates are referred to reference space. The coordinates of the 2. Piola-Kirchhoff stress tensor **S** are related to the Cauchy stress tensor through the material deformation gradient **F**:

$$\mathbf{S} = (\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-1}$$

Every instant configuration of the body must satisfy the following integral form:

$$\int_C \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}\,dv = \int_C \delta\mathbf{u}^T\mathbf{p}_V\,dv + \int_{\partial C_u}\delta\mathbf{u}^T\mathbf{p}\,da + \int_{\partial C_p}\delta\mathbf{u}^T\mathbf{p}_0\,da$$
$$\mathbf{x}\in C_u : \mathbf{u} = \mathbf{u}_0$$

$\boldsymbol{\varepsilon}$ engineering strain coordinates
$\boldsymbol{\sigma}$ coordinates of 2.Piola-Kirchhoff stress tensor
C reference configuration of the body
∂C_u surface on which displacement coordinates are prescribed
∂C_p surface on which stress coordinates are prescribed

5.2 Nonlinear Theory of Trusses

Consider a plane truss which satisfies the following hypotheses:

- The axis of the bar in the instant configuration is a straight line.

- Every plane cross-section normal to the axis of the bar in the reference configuration remains plane and normal to the axis in the instant configuration.
- The shape of the cross-section does not change.

Figure 5 shows the reference location and the instant location of a bar AB of the truss, as well as the local coordinate system y of the bar.

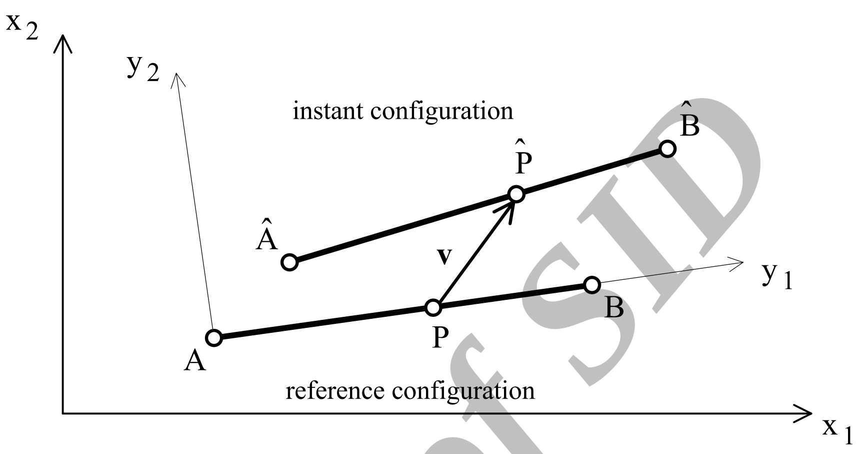

Figure 5. Configurations of a bar AB of a truss

The displacement **v** depends only on the axial coordinate. All components of strain except the axial strain are null. The axial strain ε at a point P of the bar follows from the general theory:

$$\varepsilon = v_{1,1} + 0.5(v_{1,1}^2 + v_{2,1}^2)$$

It is assumed that the truss is loaded at its nodes. The integral form of the governing equation for a state s is derived from the general theory by integration over the cross-section of the bar:

$$\sum_e \int_{L_e} A_e \delta\varepsilon\, s\, dy_1 = \sum_{i \in C_q} \delta u_i\, q_i + \sum_{i \in C_f} \delta f_i\, r_i$$

C_q set of prescribed node load coordinates (global space)

C_f set of prescribed node displacement coordinates (global space)

The changes Δv and Δr in the displacements and reactions of the truss from state s to state s+1 are predicted with corresponding incremental governing equations:

$$\sum_{e}\int_{L_e} EA_e\big((1+\nu_{1,1})\delta(\Delta\nu_{1,1})+\nu_{2,1}\delta(\Delta\nu_{2,1})\big)\big((1+\nu_{1,1})\Delta\nu_{1,1}+\nu_{2,1}\Delta\nu_{2.1}\big)dy_1 +$$
$$\sum_{e}\int_{L_e} sA_e\big(\Delta\nu_{1,1}\delta(\Delta\nu_{1,1})+\Delta\nu_{2,1}\delta(\Delta\nu_{2,1})\big)dy_1 = \delta e + \sum_{i\in C_q}\delta(\Delta u_i)\Delta q_i + \sum_{i\in C_f}\delta(\Delta f_i)\Delta r_i$$

The term δe accounts for the error in the governing equations for state s due to errors in the displacements and reactions of state s. The concept requires that the error term must be computed without approximations in the strain-displacement relations.

$$\delta e \;=\; \varepsilon EA\,\delta\mathbf{v}_e^{\,T}\begin{bmatrix} -(1+v_{1,1}) \\ -v_{2,1} \\ 1+v_{1,1} \\ v_{2,1} \end{bmatrix}$$

The displacements on the axis of the bar are interpolated linearly between the nodal values. The left hand side of the incremental governing equations yields the incremental matrix **K** for the bar, which is used in the solution algorithm as described in section 6.

$$\mathbf{K} \;=\; \frac{EA}{L}\begin{bmatrix} 1+v_{1,1}(2+v_{1,1})+\varepsilon & v_{2,1}(1+v_{1,1}) & -(1+v_{1,1}(2+v_{1,1})+\varepsilon) & -v_{2,1}(1+v_{1,1}) \\ v_{2,1}(1+v_{1,1}) & v_{2,1}^2+\varepsilon & -v_{2,1}(1+v_{1,1}) & -v_{2,1}^2-\varepsilon \\ -(1+v_{1,1}(2+v_{1,1})+\varepsilon) & -v_{2,1}(1+v_{1,1}) & 1+v_{1,1}(2+v_{1,1})+\varepsilon & v_{2,1}(1+v_{1,1}) \\ -v_{2,1}(1+v_{1,1}) & -v_{2,1}^2-\varepsilon & v_{2,1}(1+v_{1,1}) & v_{2,1}^2+\varepsilon \end{bmatrix}$$

5.3 Nonlinear Theory of Frames

Consider a plane frame which satisfies the following hypotheses:

- The displacement of all points on a plane cross-section of a member is the vector sum of the displacement of the point P on the axis and the displacement due to rigid body rotation of the cross-section about an axis through point P normal to the plane.
- The angle of rotation of the cross-section is such that a fibre of the beam which is normal to the cross-section in the reference configuration remains normal to the rotated cross-section in the instant configuration.

Figure 6 shows the reference and the instant locations of a member of the frame with length 2a, as well as its reference coordinate system y and instant coordinate system z. The angle of rotation β of the axis at a node is decomposed into a chord rotation θ of the member and a rotation ψ of the tangent to the axis relative to the chord. The displacement of a point P on the axis is the sum of a bar displacement **v** of the chord of the member, referred to reference space y, and a beam displacement **w** relative to the chord, referred to instant space z. These lead to the following derivatives with respect to the distance y on the axis, where $-1 \le z \le 1$ is the normalised axial coordinate:

$$v_{i,1} = \frac{1}{2a}(v_{iB} - v_{iA}) \qquad i \in \{1,2\}$$

$$w_{2,11} = \frac{1}{2a}(3z(\psi_A + \psi_B) - (\psi_A - \psi_B))$$

The displacement of a point Q on the normal through P is the vector sum of the translation of point P and a rotation of point Q about point P through an angle β:

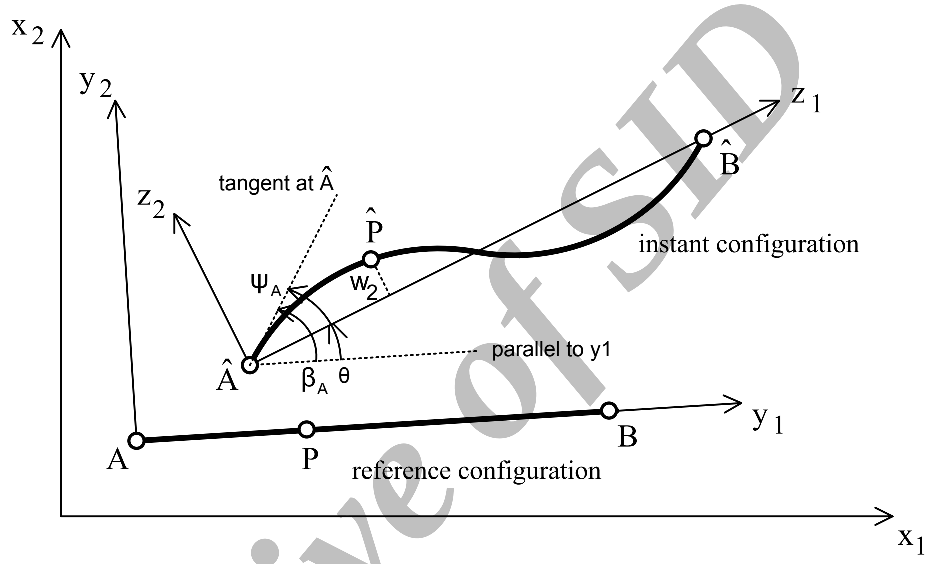

Figure 6. Configurations and node rotations of a member of a frame

$$\mathbf{v}_Q = \mathbf{v}_P - y_2 \dot{\mathbf{v}}_P \qquad \mathbf{w}_Q = \mathbf{w}_P - y_2 \dot{\mathbf{w}}_P$$

$$\begin{bmatrix} v_{1Q} \\ v_{2Q} \end{bmatrix} = \begin{bmatrix} v_{1P} \\ v_{2P} \end{bmatrix} - y_2 \begin{bmatrix} \sin\beta \\ 1-\cos\beta \end{bmatrix} \qquad \begin{bmatrix} w_{1Q} \\ w_{2Q} \end{bmatrix} = \begin{bmatrix} w_{1P} \\ w_{2P} \end{bmatrix} - y_2 \begin{bmatrix} \sin\psi + \sin\theta \\ -\cos\psi + \cos\theta \end{bmatrix}$$

The angle of rotation β is chosen so that the normal of the instant cross-section is tangent to the instant axis at point P:

$$\tan\beta = \frac{v_{2,1}}{1 + v_{1,1}}$$

Due to the frame hypotheses, only the axial strain ε differs from null:

$$\varepsilon = v_{1,1} + 0.5(v_{1,1}^2 + v_{2,1}^2) - \cos\psi \; w_{2,11}\, y_2$$

The governing equations for a state s of the frame and the error vector are derived in analogy to the corresponding expressions for a truss. For the incremental algebraic equations, it is assumed that the chord rotation does not change during a step. The bar displacement is interpolated linearly in reference space, the beam displacement with Hermite polynomials in instant space. The element size should be chosen so that the relative rotation ψ in the instant state does not exceed 15 degrees. The incremental strain is then given by:

$$\Delta\varepsilon = (1+\nu_{1,1})\Delta\nu_{1,1} + \nu_{2,1}\Delta\nu_{2,1} + 0.5(\Delta\nu_{1,1}^2 + \Delta\nu_{2,1}^2) - \Delta w_{2,11} y_2$$

The strain increment Δε is used to derive a bar stiffness matrix in reference space and a beam stiffness matrix in instant space. Both are transformed to global space and then added. This leads to the following algebraic equation for the incremental step from state s to state s+1:

$$K_s \Delta u_s = e_s + \Delta q_s + \Delta r_s$$

$$\begin{bmatrix} K_{11S} & K_{12S} \\ K_{21s} & K_{22s} \end{bmatrix} * \begin{bmatrix} \Delta u_{1s} \\ \Delta u_{2s} \end{bmatrix} = \begin{bmatrix} e_{1s} \\ e_{2s} \end{bmatrix} + \begin{bmatrix} \Delta q_{1s} \\ \Delta q_{2s} \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta r_{2s} \end{bmatrix}$$

Δu_{1s} increment of the unkonwn displacements
Δu_{2s} increment of the prescribed displacements
Δr_{2s} increment of the unknown reactions

6. ALGORITHM FOR THE NONLINEAR ANALYSIS OF STRUCTURES

The method of solution is a modification of the arc length method which accounts for a nearly singular stiffness matrix of the structure. Consider a displacement-load diagram with measure a (containing a scale factor μ) for displacement and measure p for load:

$$a = \mu\sqrt{\mathbf{u}_s^T \mathbf{u}_s} \qquad p = \sqrt{(\mathbf{q}_s + \mathbf{r}_s)^T (\mathbf{q}_s + \mathbf{r}_s)}$$

The increments of the loads and the prescribed displacements are assumed to be fractions Δλ of the total prescribed loads and displacements:

$$\Delta q_{1s} = \Delta\lambda_s q_{1t} \qquad \Delta u_{2s} = \Delta\lambda_s u_{2t}$$

The solution of the incremental governing equations for these external influences yields the unknown displacements and reactions as functions of Δλ:

$$\Delta u_{1s} = \Delta\lambda_s a_{1s} + b_{1s} \qquad a_{1s} = K_{11s}^{-1}(q_{1t} - K_{12s} u_{2t}) \qquad b_{1s} = K_{11s}^{-1} e_{1s}$$

$$\Delta r_{2s} = \Delta\lambda_{2s} + b_{2s} \qquad a_{2s} = K_{12s} a_{1s} + K_{12s} u_{2t} - q_t \qquad b_{2s} = K_{12s} b_{1s} - e_{2s}$$

The condition that the arc length in the a-p diagram for step s equals that for step 0 leads to a quadratic equation for $\Delta\lambda$. The root which maximizes the distance of the new state from the last state is chosen. This approach is not suitable if one of the following conditions is violated:

- The norm of the error vector **e** is small compared to the norm of the load and reaction vector increment $\Delta\mathbf{q} + \Delta\mathbf{r}$ in the first step.
- The norm of the displacement correction **b** due to **e** is small compared to the norm of the displacement increment $\Delta\mathbf{u}$ in the first step.

Near singular states of the system stiffness matrix, these conditions are not satisfied. The arc length method is therefore modified as shown the Figure 7:

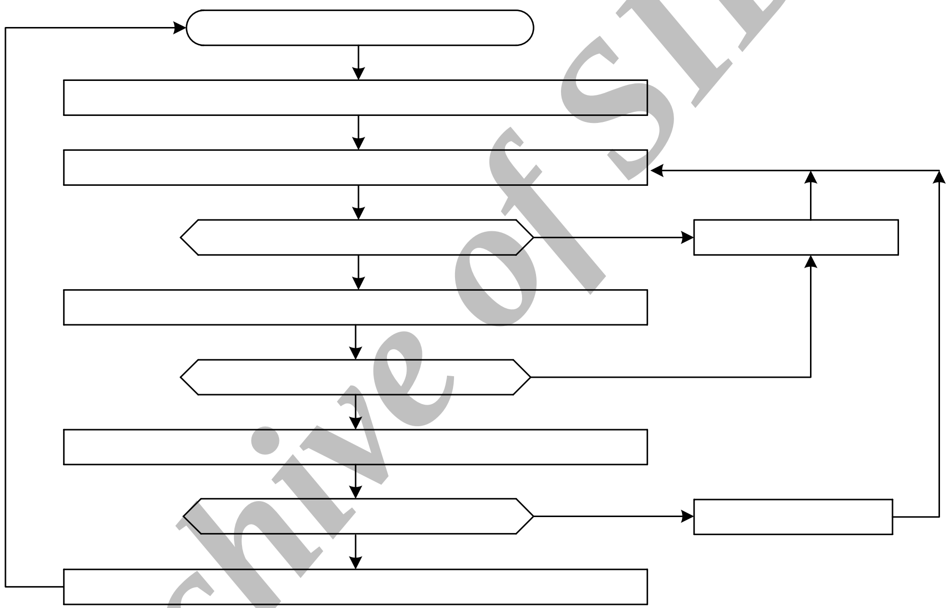

Figure 7. Modification of the arc length method for nonlinear structural analysis

7. ACCURACY OF THE METHOD

The method which is described in the preceding sections has been implemented in the test bed shown in Figure 7. The results of the analysis of the 2-bar-truss in section 4.1 with 180 load steps up to a maximum normalised displacement of s = 2.2183 are shown Table 1. All quantities are normalised, as defined in section 4.1. The errors in the computed displacements and reactions are very small and nearly independent of the magnitude of the displacements.

Table 1. Comparison of computed and exact displacements and reactions of a 2-bar-truss

State	Displacement	Load computed	Load exact	Reaction computed	Reaction exact
1	0.015685	0.030635	0.030635	0.015562	0.015562
10	0.172206	0.260553	0.260554	0.157378	0.157378
20	0.443521	0.385849	0.384155	0.345522	0.345165
30	0.739480	0.242838	0.242839	0.466064	0.466065
40	1.111042	-0.109673	-0.109673	0.493834	0.493835
50	1.531531	-0.381004	-0.381360	0.358673	0.358737
60	1.754113	-0.325259	-0.325260	0.215657	0.215657
70	1.931760	-0.122828	-0.122828	0.065912	0.065912
80	2.081590	0.183694	0.183694	-0.084918	-0.084918
90	2.211980	0.568293	0.56893	-0.234448	-0.234448

The results of an analysis of the cantilever beam in section 4.2 for end moments up to $\alpha = 2\pi$ are shown in Table 2. The structural model consists of 20 members and is loaded in 200 steps. The moments and displacements are normalised, as defined in section 4.2. The errors in the computed horizontal and vertical displacements of the tip of the cantilever are of the order of 0.1 percent and nearly independent of the magnitude of the displacements.

Table 2. Comparison of computed and exact tip displacements for a cantilever beam with end moment

State	Displacement	Load computed	Load exact	Reaction computed	Reaction exact
1	0.031180	-0.000162	-0.000162	0.015586	0.015589
20	0.623600	-0.063529	-0.063564	0.301844	0.301826
40	1.247479	-0.239858	-0.239918	0.547078	0.546932
60	1.871916	-0.489916	-0.489825	0.692997	0.692654
80	2.497150	-0.759918	-0.759425	0.720989	0.720595
100	3.123329	-0.995139	-0.994152	0.640326	0.640290
120	3.751022	-1.153789	-1.152598	0.484368	0.485194
140	4.380044	-1.216459	-1.215815	0.300922	0.302796
160	5.010254	-1.190025	-1.190802	0.138535	0.141015
180	5.641497	-1.103434	-1.106098	0.033308	0.035259
200	6.241945	-1.002556	-1.006605	0.000022	0.000136

8. CONCLUSIONS

The investigation of the nonlinear behaviour of plane trusses and frames has shown that the concept for the analysis of the nonlinear behaviour of structures which the authors have adopted can be applied successfully. A truss and a frame, for which exact analyses are available, have been studied in a Java-based test bed. The results show in both cases that the magnitude of the displacements and rotations does not influence the accuracy of the results of the numerical analysis substantially.

The results for the truss show that the displacements in a structure with snap-through can be calculated to an accuracy of 6 figures, except near points of singularity of the stiffness, where the accuracy can drop to 4 figures. The number of load steps of the example can be reduced significantly if the usual engineering accuracy of about 3 percent error is acceptable.

The results for the frame show that bending action is analysed correctly even if the rotations are of the order of 360 degrees. In spite of the very large rotations, the error in the tip displacement due to a given end moment is of the order of 0.1 percent of the maximum displacement. For rotations below 180 degrees, the error is substantially less. It can be reduced further by increasing the number of elements in the cantilever.

On the basis of these results, it has been decided to proceed with the line of investigation and to extend the application of the concept to other types of structural components, particularly three-dimension trusses and frames as well as membranes and plates. The implementation of the two-dimension trusses and frames has shown that the concept is very suited for systematic treatment of data base, algorithms and user interface in a software package.

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