ASIAN JOURNAL OF CIVIL ENGINEERING (BUILDING AND HOUSING) VOL. 8, NO. 1 (2007)
PAGES 1-12

EXPERIMENTAL AND NUMERICAL FORM-FINDING OF MEMBRANE STRUCTURES

T.S. Loong[a], H. Abdul Razak[*a], Z. Ismail[a] and C.K. Keong[b]
[a]Department of Civil Engineering, University of Malaya, Malaysia
[b]Department of Civil Engineering, University of Science, Malaysia

ABSTRACT

This paper describes a study conducted to determine the surface shape of membrane structures with fixed edges. Results obtained experimentally by taking coordinate measurements in three dimensions from scale down physical models are compared with computed values based on the formulation of the force density method for determining the surface shape. A program in C++ is developed in order to assist in the computation. It is observed that the results under minimal surface conditions are in close agreement and the formulation can thus be applied practically to predict the surface shape of membrane structures.

1. INTRODUCTION

Membrane structures refer to those structures of which the main structural components are thin sheets of membrane made of resin coated fabrics. Design of membrane structures typically involves three stages i.e. form finding, cutting pattern determination and structural analysis. Form finding is a process to predict the membrane shape or the behavior of the membrane shape when subjected to a given boundary or loading condition.

The variety of possible shapes applicable to the design of large span membrane structures demand new ideas in structural analysis. Generally, forces and deformations are evaluated based on a known shape or form. However, in membrane structures the shape or form is an unknown entity prior to computation except in a few simple cases. Thus, the search for a suitable shape plays an important role since it influences to a great extent the forces and deformations in the unloaded and loaded condition.

There are quite a number of experimental techniques used in form finding involving direct physical measurements of the surface on scale models. Materials such as tulle, spandex, thin rubber sheet, cloth and even soap film have been used to construct models. Models using soap film will form the so called minimal or equivalently equally stressed surface. Such surfaces are frequently used in the design of membrane structures. However, the drawback of soap film models is that the surfaces formed are extremely if not impossible

* Email-address of the corresponding author: hashim@um.edu.my

to measure using conventional techniques. Photographs are normally taken in order to reconstruct the surface for further numerical work. On the other hand, models built using materials such as tulle or thin rubber sheet can be measured more easily. In general, by using these materials the models have to be constructed by combining several pieces of flattened sheets and subsequently stretching the combined pieces used. The outcome of the surface formed is therefore very much dependant on the cutting pattern of the flattened pieces of material used. The experimental method of determining surfaces helps in visualizing the problem but suffers a severe disadvantage when compared to the numerical method. The number of variants in a physical model is very limited and measurement data has to be processed to digital format prior to further computational work and for display using an interactive system.

Numerical methods are highly suitable for form finding of membrane structures since they allow an unlimited number of shape variants, boundary and loading conditions. The formulation presented herein was developed by Schek [1] and is based on the concept of minimal surface and does not take into account any external loading. This situation is applicable to suspension type structures whereby the stresses induced in the membrane are equal in all directions and only one minimal surface is generated in the state of equilibrium. Structures whose membrane are under unequal stresses are problematic. They tend to from folds and are strained more at the points, which are subjected to high loads until they tear or adopt forms that deviate strongly from the original cutting pattern. The shape without minimal surface will deform unduly under external loading and can be subjected to severe flapping. Thus, correctly stressed structures exhibit almost equal stresses in all directions and therefore forming one minimal surface. Minimal surface structures are the least problematic with respect to cutting patterns and also the most resistant to deformation.

2. FORCE DENSITY METHOD

This method is based on the notion that in the state of equilibrium, one system of linear equations can be obtained to represent the shape for equilibrium. This system is constructed using the force-length ratios (f/l) or namely, force-densities in the branches of a membrane structure's network. Single quantity of force-densities is prescribed for each branch, and a unique result for the appropriate state of equilibrium can be obtained by solving this system of linear equations.

Schek [1] suggested that several different shapes can be computed and displayed in a short time, by varying the prescribed value of force-densities in each branches. Thus the opinion that the computation of general shape of membrane structures is only possible by trial and error methods can no longer be accepted.

3. FORMULATION

A vector, $\boldsymbol{a}$ can be represented by a one-column-matrix and it correlates with a diagonal matrix $\boldsymbol{A}$ in the following way, e.g.

$$a = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{1}$$

then

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \tag{2}$$

A network is constructed using some branches and nodes, as shown in Figure 1.

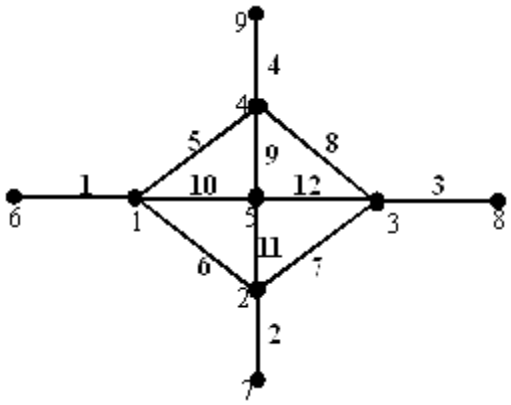

Figure 1. Branch-node network

Nodes can be divided into two categories, namely “free” and “fixed”, Schek [1]. In branch-node numbering, it is advisable to number the free points followed by the fixed points.

If,

$$\boldsymbol{n_s} = \text{total number of points}$$

then,

$$\boldsymbol{n_s} = \boldsymbol{n} + \boldsymbol{n_f}$$

where,

$$\boldsymbol{n} = \text{total number of free points}$$
$$\boldsymbol{n_f} = \text{total number of fixed points}$$

Each branch in the branch-node network has two matched node number, $\boldsymbol{i}(\boldsymbol{j})$ and $\boldsymbol{k}(\boldsymbol{j})$. Therefore, a usual branch-node matrix, $\boldsymbol{C_s}$ is defined by

$$C_s(j,i) = \begin{cases} +1 \text{ for } i(j) = 1 \\ -1 \text{ for } k(j) = 1 \\ 0 \text{ in the other cases.} \end{cases} \quad (3)$$

where,

i, k = node number,
j = branch number.

Applying the above to the network in Figure 1, the branch-node matrix can be written in the following form,

	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	–1	0	0	0
2	0	1	0	0	0	0	–1	0	0
3	0	0	1	0	0	0	0	–1	0
4	0	0	0	1	0	0	0	0	–1
5	1	0	0	–1	0	0	0	0	0
6	1	–1	0	0	0	0	0	0	0
7	0	1	–1	0	0	0	0	0	0
8	0	0	1	–1	0	0	0	0	0
9	0	0	0	1	–1	0	0	0	0
10	1	0	0	0	–1	0	0	0	0
11	0	1	0	0	–1	0	0	0	0
12	0	0	1	0	–1	0	0	0	0
			C					***C*** $_f$	

Figure 2. Typical branch-node matrix

Hence,

$$\boldsymbol{C}_s = \boldsymbol{C} + \boldsymbol{C}_f$$

where,

C = Branch-node matrix for free points,
C $_f$ = Branch-node matrix for fixed points.

q is the force-density vector for the branch node network, and ***Q*** is the diagonal force-density matrix in equation (1) and (2).

$$\boldsymbol{q} = \boldsymbol{L}^{-1}\boldsymbol{s} \quad (4)$$

where,

$\boldsymbol{L}$ = branch length matrix,
$\boldsymbol{s}$ = branch internal-force vector.

The solution for the branch-node network is given below,

$$\begin{aligned} \boldsymbol{C}^T\boldsymbol{QCx} + \boldsymbol{C}^T\boldsymbol{QC}_f\boldsymbol{x}_f &= \boldsymbol{p}_x \\ \boldsymbol{C}^T\boldsymbol{QCy} + \boldsymbol{C}^T\boldsymbol{QC}_f\boldsymbol{y}_f &= \boldsymbol{p}_y \\ \boldsymbol{C}^T\boldsymbol{QCz} + \boldsymbol{C}^T\boldsymbol{QC}_f\boldsymbol{z}_f &= \boldsymbol{p}_z \end{aligned} \tag{5}$$

For simplification,

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{C}^T\boldsymbol{QC} \\ \boldsymbol{D}_f &= \boldsymbol{C}^T\boldsymbol{QC}_f \end{aligned}$$

Therefore, for equilibrium state,

$$\begin{aligned} \boldsymbol{Dx} &= \boldsymbol{p}_x - \boldsymbol{D}_f\boldsymbol{x}_f \\ \boldsymbol{Dy} &= \boldsymbol{p}_y - \boldsymbol{D}_f\boldsymbol{y}_f \\ \boldsymbol{Dz} &= \boldsymbol{p}_z - \boldsymbol{D}_f\boldsymbol{z}_f \end{aligned} \tag{6}$$

where,

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ = free nodes coordinate vector,
$\boldsymbol{x}_f, \boldsymbol{y}_f, \boldsymbol{z}_f$ = fixed nodes coordinate vector,
$\boldsymbol{p}_x, \boldsymbol{p}_y, \boldsymbol{p}_z$ = external force vector on each free nodes respectively.

From equation (6), given the positions of the fixed points, external loads and a set of prescribed force densities, the shape generated under equilibrium is given by the free point coordinates,

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{D}^{-1}(\boldsymbol{p}_x - \boldsymbol{D}_f\boldsymbol{x}_f) \\ \boldsymbol{y} &= \boldsymbol{D}^{-1}(\boldsymbol{p}_y - \boldsymbol{D}_f\boldsymbol{y}_f) \\ \boldsymbol{z} &= \boldsymbol{D}^{-1}(\boldsymbol{p}_z - \boldsymbol{D}_f\boldsymbol{z}_f) \end{aligned} \tag{7}$$

The solution for equation (7) is unique for the prescribed force-densities. Variation in vector q will result in a variation in the state of equilibrium. Thus the force densities are suitable for network description parameters. This set of equation will be utilized to predict the coordinates of the membrane surface.

4. PROGRAMMING STRUCTURE

The choice of programming language is governed by the built-in features or tools provided, so as to perform the required computational work efficiently. Additionally, the language chosen should also be able to support future modification easily which are necessary when other parameters are to be considered. These parameters may well include different boundary conditions, high points and supports. The utilities such as matrix solver and Window's graphic user interface (GUI) which are incorporated into C++ [3], makes it the logical choice for programming the formulation mentioned earlier.

Relevant information must be provided in order to start the form-finding process. The information required includes interconnection condition, force densities of the branches, the external force acting onto the structure and fixed point or fixed boundary, as illustrated in Figure 3.

Several computer programs were written and computer aided analysis tools were used to complete the form-finding process [2]. These includes,

4.1 Formation of Branch-node and Force Density Matrix

The program written will form the branch-node and force density matrixes. A sub-routine is written by using the result from equation (3). If the connection between the branch and node is provided, the sub-routine will then check the condition of the connection provided and form the branch-node matrix accordingly.

A sub-routine is also written to generate the force density matrix. Generally, a targeted force density in all branches for the structure has to be selected for the analysis.

4.2 Formation of Fixed Point Coordinate Matrix

The program written will help the user in determining three extra fixed points on each fixed boundary. A sub-routine is written to calculate the respective fixed point. The method used here is direct interpolation. In such case, the coordinate difference between two given fixed support is divided evenly into four parts, which will generate the coordinates of three intermediate fixed points.

4.3 Free Point Determination

The mathematical software MATLAB, is used to determine nine free points in the next step. Equation (7) is used and three matrixes (***x***, ***y*** and ***z*** respectively), which provide the information of the free point coordinates are generated.

4.4 Display of Equilibrium Shape

The display of the equilibrium shape is made possible using the plotting software SURFER. MATLAB generates the free point and fixed point coordinate matrices and then store them into the SURFER directory. Using the appropriate menu in SURFER, the membrane surface can be visualized either as a three-dimensional or a contour plot.

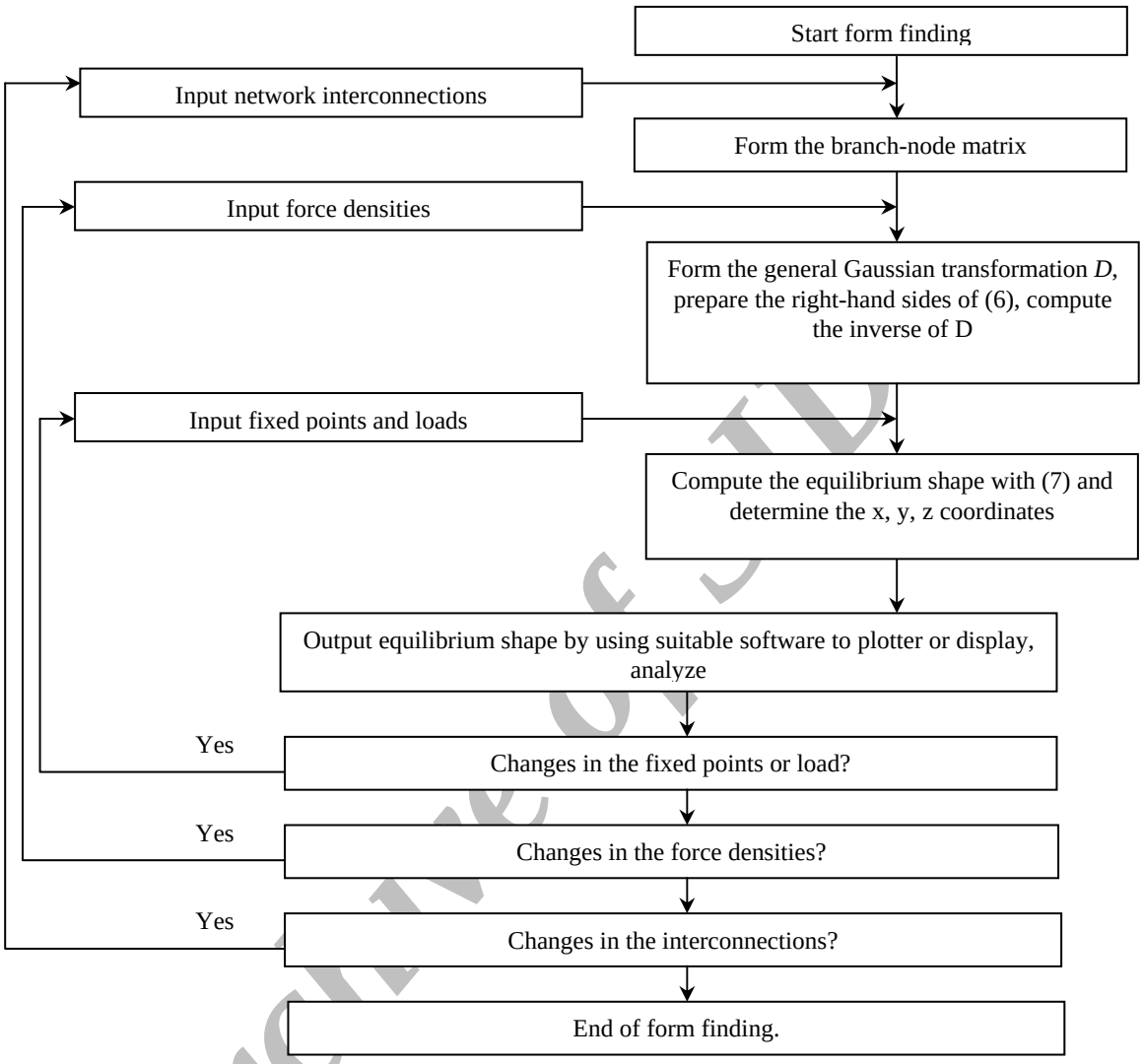

Figure 3. Form finding process [1]

5. MODEL CONSTRUCTION

The models used in this investigation were constructed using thin rubber sheets and restrained all along the four edges by plywood forming a box-like shape. Each model consists of two high points and two low points forming a shape which was frequently adopted for membrane structures when they were firstly introduced.

During the process of model construction, extreme care was exercised so as to avoid having a model without minimal surface. In order to avoid this, all stresses in the stretched rubber sheet must be equal in all directions. This was achieved by drawing some circles on the sheet before stressing as shown in Figure 4. Subsequently, the shape of the circles was

checked after the stressing process and it was assumed that the stresses were equal in all directions if the shape of the circle remained perfectly round as shown in Figure 5. Under such a situation, the minimal surface was formed.

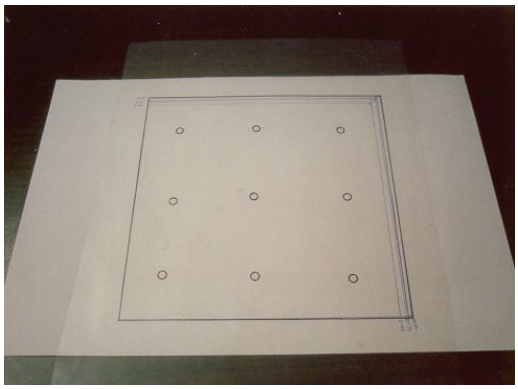

Figure 4. Before stressing

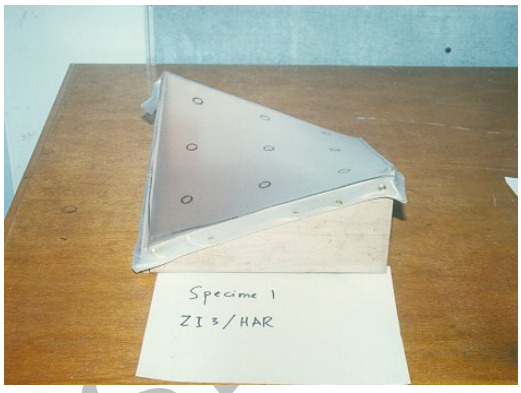

Figure 5. After stressing

6. CASE STUDY [2]

Results from the computer program written based on the force density formulation are compared with measured values obtained from scale models. The basic model with a base dimension of 300mm by 300mm, comprised of two high points and two low points with fixed edges. In order to obtain variation in the shape of the membrane surface, differences in height at the high and low points were introduced. For the purpose of this case study, results from three constructed models with different profiles obtained by varying the height of the low and high points are presented.

6.1 Model 1

For Model 1, the high and low points were maintained at the same height but a height difference of 100mm was introduced between them. Upon examining the contour plots as shown in Figure 6, the theoretical and experimental results matched very well especially at the four corners, i.e. the area marked as A, B, C and D. In particular, the region around the low points at levels below 190 mm and the high points at levels above 225 mm, the contours are indistinguishable. However, the right center region marked as E at levels around 210.0mm, the deviation between the results becomes significant. This may be due to the errors occurring while measurements were taken and factors such as incorrect penetration of the measuring head onto the membrane at point E, inaccuracy of the measuring machine and human factor such as eyesight. Since this model was the first to be built, the lack of care and skill exercised while measuring proved to be vital for purpose of accuracy. The contours at the center for Model 1 reflect a different pattern for the theoretical and measured plots. This is due to the entire area being very close to the contra-flexure point, which is highly sensitive to the measuring technique.

6.2 Model 2

For Model 2, a height difference of 50mm between the two low points was introduced while maintaining the high points at the same height. The height between the high points and the lowest point was 100mm. Upon examining the contour plots as shown in Figure 7, the theoretical and experimental results also matched very well. The contours are almost indistinguishable at the four corners, i.e. the area marked as A, B, C, D and the center region, marked as E. Overall, the contours obtained from this model gave the best results in terms of the proximity of the theoretical and experimental values. This is due to the well built model whereby the membrane was stressed evenly.

6.3 Model 3

For Model 3, a height difference of 25 mm was introduced between the two low points and two high points. Consequently a height difference of 75mm was obtained between the lowest and highest point of the model, as shown in Figure 8. The general trend in which the two contour plots were almost indistinguishable especially at the four fixed point regions marked as A, B, C and D were observed. A slight deviation in the experimental and predicted contour plots occurred at the center region. This is due to the fact that this region is located near the contra-flexure point and errors during measuring contributed to the divergence.

Furthermore, the three dimensional side views of both the theoretical and experimental results for all the models considered were very similar in profile. Typically this is shown in Figure 9 for the results obtained from Model 1. Therefore, it can be concluded that the results are in close agreement with each other.

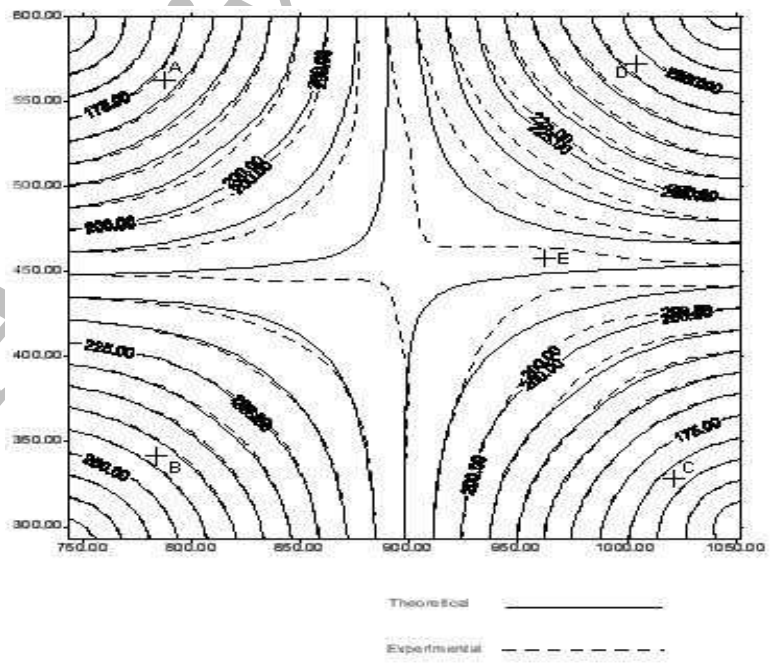

Figure 6. Contour plots for model 1

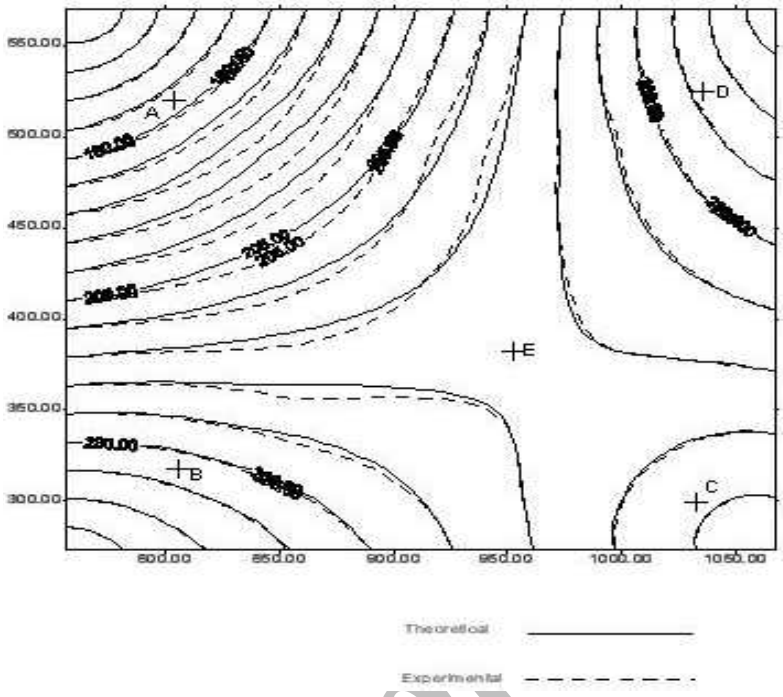

Figure 7. Contour plots for model 2

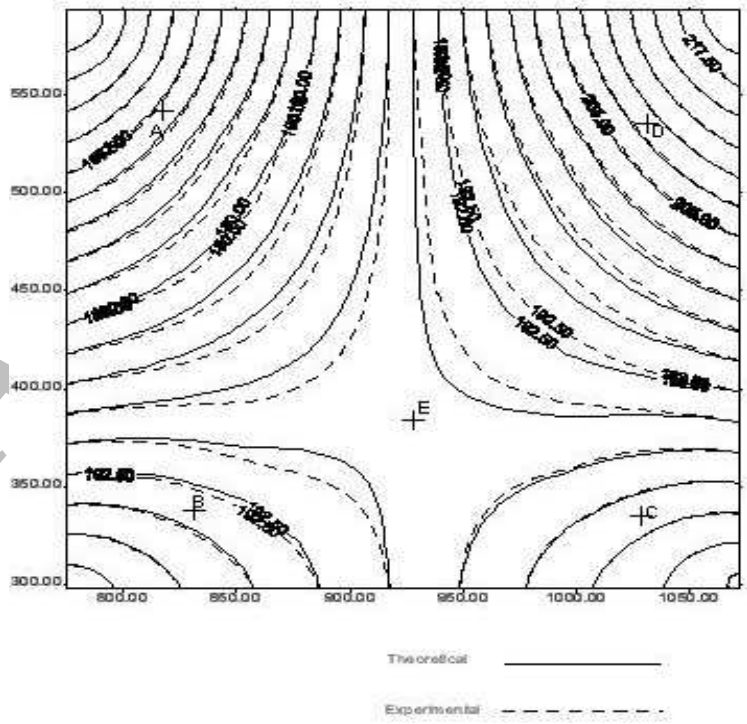

Figure 8. Contour plots for model 3

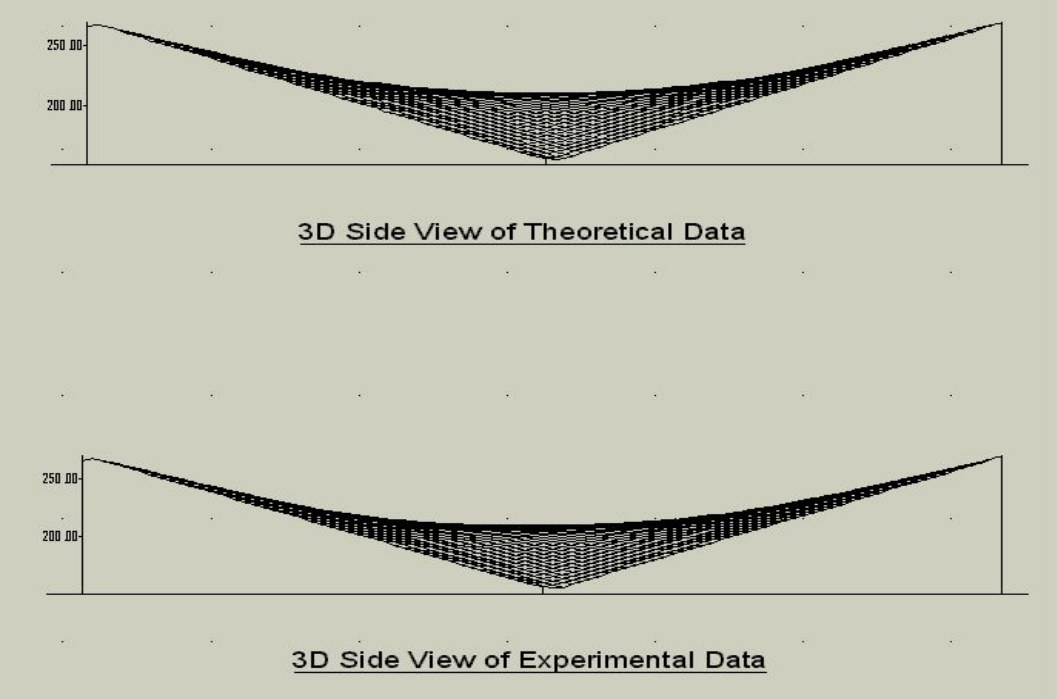

Figure 9. 3D side view for model 1

7. CONCLUSIONS

In this study, it can be concluded that the numerical form finding technique using force density method with network analogy can be used to determine the surface shape of the structure with certain constraints.

The constraints are: -

1. Applicable to structure with fixed boundaries and uniform force density.
2. No consideration of external force.
3. Limited to four-point structure only.

Several conclusions can be derived from the observations and results from this investigation.

1. The experimental contours are almost indistinguishable with the theoretical contours. All the contours formed by the experimental results almost coincided with the contours of the numerical results. The numerical results give the predicted surface shape of the membrane structure.
2. The measured values obtained experimentally are in random order and can be situated at any location in the structure. Thus, no comparison can be made from

three coordinates measured at a particular point on the model. Therefore, the comparison between theoretical and experimental results can only be based on the contour plots. Under such circumstances, the results obtained were satisfactory with respect to the quality and accuracy of the measuring technique.

3. The experimental results have shown an almost identical profile in three-dimensional view when compared to the numerical results. A very smooth curvature is formed in all the constructed models.
4. The good experimental and theoretical agreement obtained in this study showed that the prediction of the surface shape based on minimal surface can be applied to a membrane structure. A minimal surface can be formed only if the structure is in the state of equilibrium and this is reflected well in the surface shape of the models constructed. Therefore, the shape of any membrane structure within the constraints of the assumptions made in the formulation can be predicted using the computer program developed herein.

REFERENCES

1. Schek, H.J., The force density method for form finding and computation of general network, *Computer Methods in Applied Mechanics and Engineering*, **3**(1974) 115-134.
2. Loong, T.S., Experimental and Numerical Form-finding of Membrane Structure, Faculty of Engineering, University of Malaya, 1998.
3. Hyman, M. and Arncon, B., Borland C++ 5 for Dummies, 2nd edition, IDG Book Worldwide Inc., 1996.