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FORCE METHOD FOR FINITE ELEMENT MODELS WITH INDETERMINATE SUPPORT CONDITIONS

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ABSTRACT

Graph theoretical force methods are highly efficient for the generation of sparse and banded null bases and flexibility matrices, however, these methods require special considerations when the support conditions are indeterminate. These considerations with special methods are presented in this paper which lead to efficient utilization of graph theoretical force method for indeterminate support conditions with no substantial decrease in sparsity.

Keywords: Finite elements, indeterminate support condition, sparse, flexibility matrix, null basis, graph theory

1. INTRODUCTION

The force method of structural analysis, in which the redundant forces are used as unknowns, is appealing to engineers, since the properties of members of a structure most often depend on the member forces rather than joint displacements. This method was used extensively until 1960. After this, the advent of the digital computer and the amenability of the displacement method for computation attracted most researchers. As a result, the force method and some of the advantages it offers in non-linear analysis and optimization has been neglected.

Four different approaches are adopted for the force method of structural analysis, which are classified as:

1. Topological force methods,
2. Algebraic force methods,
3. Mixed algebraic-combinatorial force methods,
4. Integrated force method.

Topological methods have been developed by Henderson [1], Maunder [2] and Henderson and Maunder [3] for rigid-jointed skeletal structures using manual selection of

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the cycle bases of their graph models. Methods suitable for computer programming are due to Kaveh [4-6]. These topological methods are generalized to cover different types of skeletal structures, such as rigid-jointed frames, pin-jointed planar trusses and ball-jointed space trusses [7,8].

Algebraic methods have been developed by Denke [9], Robinson [10], Topçu [11], Kaneko et al. [12], Soyer and Topçu [13] and mixed algebraic-topological methods have been used by Gilbert et al. [14], Coleman and Pothen [15-16], and Pothen [17]. The integrated force method has been developed by Patnaik [18-19], in which member forces are used as variables, the equilibrium equations and the compatibility conditions are satisfied simultaneously in terms of these variables.

Graph theoretical methods for the formation of sparse and banded null basis for the model consisting of triangular plane stress and strain elements and rectangular plate bending and tetrahedron elements are developed by Kaveh et al. [20-22]. Graph theory is also used for a flexibility analysis of thin plated structures, Maunder [23].

In this paper, two methods are presented for the inclusion of the support conditions in such manner that the sparsity of the flexibility matrices is not substantially altered.

2. ALGEBRAIC FORCE METHODS

Consider a discrete or discretized structure S, which is assumed to be statically indeterminate. Let **r** denote the m-dimensional vector of generalized independent element (member) forces, and **p** the n-vector of nodal loads. The equilibrium conditions of the structure can then be expressed as,

$$\mathbf{H}\mathbf{r} = \mathbf{p}, \tag{1}$$

where **H** is an n×m *equilibrium matrix*. The structure is assumed to be geometrically stable (rigid), and therefore **H** has a full rank, i.e. t = m – n > 0 and rank **H** = n.

The member forces can be written as

$$\mathbf{r} = \mathbf{B}_0\mathbf{p} + \mathbf{B}_1\mathbf{q}, \tag{2}$$

where $\mathbf{B}_0$ is an m×n matrix such that $\mathbf{HB}_0$ is an n×n identity matrix, and $\mathbf{B}_1$ is an m×t matrix such that $\mathbf{HB}_1$ is an n×t zero matrix. $\mathbf{B}_0$ and $\mathbf{B}_1$ always exist for a structure, and in fact many of them can be found for a structure. $\mathbf{B}_1$ is called a *self-stress matrix* as well as *null basis matrix*. Each column of $\mathbf{B}_1$ is known as a *null vector*. Notice that the null space, null basis and null vectors correspond to complementary solution space, statical basis and self stress systems, respectively, when *S* is taken as a general structure.

Minimizing the complementary potential energy requires that **r** minimize the quadratic form,

$$\frac{1}{2}\mathbf{r}^t\mathbf{F}_m\mathbf{r}, \tag{3}$$

subject to the constraint as in Eq. (1). $\mathbf{F}_m$ is an m×m block diagonal element flexibility matrix. Using Eq. (2), it can be seen that **q** must satisfy the following equation,

$$(\mathbf{B}_1^t \mathbf{F}_m \mathbf{B}_1)\mathbf{q} = -\mathbf{B}_1^t \mathbf{F}_m \mathbf{B}_0 \mathbf{p}, \tag{4}$$

where $\mathbf{B}_1^t \mathbf{F}_m \mathbf{B}_1 = \mathbf{G}$ is the *overall flexibility matrix* of the structure. Computing the redundant forces **q** from Eq. (4), **r** can be found using Eq. (2). The structure of **G**, is again important, and its sparsity, bandwidth and conditioning govern the efficiency of the force method. For the sparsity of **G** one can search for a sparse $\mathbf{B}_1$ matrix, which is often referred to as the *sparse null basis* problem.

Many algorithms exist for computing a null basis $\mathbf{B}_1$ of a matrix **H**. For the moment, let **H** be partitioned so that,

$$\mathbf{HP} = [\mathbf{H}_1, \mathbf{H}_2], \tag{5}$$

where $\mathbf{H}_1$ is n×n and non-singular, and **P** is a column permutation matrix that may be required in order to ensure that $\mathbf{H}_1$ is non-singular. One can write:

$$\mathbf{B}_1 = \mathbf{P}\begin{bmatrix} -\mathbf{H}_1^{-1}\mathbf{H}_2 \\ \mathbf{I} \end{bmatrix}. \tag{6}$$

$$\mathbf{B}_0 = \mathbf{P}\begin{bmatrix} \mathbf{H}_1^{-1} \\ \mathbf{0} \end{bmatrix} \tag{7}$$

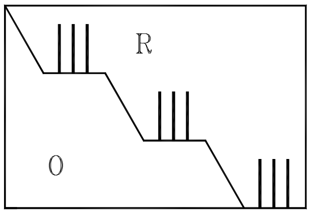
Obviously, a permutation **P** that yields a non-singular matrix $\mathbf{H}_1$, can be chosen purely symbolically, but this says nothing about the possible numerical conditioning of $\mathbf{H}_1$ and the resulting $\mathbf{B}_1$.

In order to control the numerical conditioning, pivoting must be employed. There are many methods based on various matrix factorizations, including the Gauss-Jordan elimination, **QR**, **LU**, **LQ** and Turn-back method. The latter method is briefly described in the following:

Turn-Back LU Decomposition Method: Topçu developed a method, the so-called Turn-back **LU** procedure, which is based on **LU** factorization and often results in highly sparse and banded $\mathbf{B}_1$ matrices. Heath et al. [24] adopted this method for use with **QR** factorization. Due to the efficiency of this method, a brief description of their approach will be presented in the following.

Write the matrix $\mathbf{H} = (h_1, h_2, ..., h_n)$ by columns. A *start column* is a column such that the ranks of $(h_1, h_2, ..., h_{s-1})$ and $(h_1, h_2, ..., h_s)$ are equal. Equivalently, h_s is a start column if it is linearly dependent on lower-numbered columns. The coefficients of this linear dependency give a null vector whose highest numbered non-zero is in position s. It is easy to see that the number of start columns is m – n = t, the dimension of the null space of **H**.

The start column can be found by performing a **QR** factorization of **H**, using orthogonal transformations to annihilate the sub-diagonal non-zeros. Suppose that in carrying out the **QR** factorization we do not perform column interchanges but simply skip over any columns that are already zero on and below the diagonal. The result will then be a factorization of the form



(8)

The start columns are those columns where the upper triangular structure jogs to the right; that is, h_s is a start column if the highest non-zero position in column s of **R** is no larger than the highest non-zero position in earlier columns of **R**.

The Turn-back method finds one null vector for each start column h_s by "turning back" from column s to find the smallest k for which columns $h_s, h_{s-1}, ..., h_{s-k}$ are linearly dependent. The null vector has a non-zero only in position s–k through s. Thus, if k is small for most of the start columns, then the null basis will have a small profile. Notice that the turn-back operates on **H**, and not on **R**. The initial **QR** factorization of **H** is used only to determine the start columns, and then discarded.

The null vector that Turn-back finds from start column a_s may not be non-zero in position s. Therefore Turn-back needs to have some way to guarantee that its null vectors are linearly independent. This can be accomplished by forbidding the left-most column of the dependency for each null vector from participating in any later dependencies. Thus, if the null vector for start column a_s has its first non-zero in position s–k, every null vector for a start column to the right of a_s will be zero in position s–k.

3. INDETERMINATE SUPPORT CONDITIONS IN THE FORCE METHOD

The effect of support conditions can generally be included in two different manners in finite element analysis by the force method. These methods are described in this section.

3.1 First method

In this approach, the support reactions are not selected as redundant forces and thus the supports are present in the primary determinate structure. Using this process, all the redundant forces are selected from structural elements. From mathematical point of view for such a modeling, the rows corresponding to fixed degree of freedom of supports should be removed from the rectangular equilibrium matrix **H**. If the problem is solved using this process, then there will be no one to one mapping between the interface graph and the associate graph and the degree of statical indeterminacy of structure, leading to inefficient usage of these transformations. In fact, such a modeling for applying support conditions in the force method is useful for algebraic methods and should not be utilized in graph theoretical approaches.

3.2. Second method

In this method all the support reactions except the forces which are required for a determinate supports, are selected as redundant forces and thus will not be present in the primary determinate structure. It should be noted that the number of such supports depends on the type of problem. In this case, the degree of statical indeterminacy is divided to internal DSI and external (support) DSI. Now all the topological transformations and their relations to the DSI which have been introduced previously in references [20-22] will be valid. This will lead to an efficient usage of graph theoretical force methods. However, for efficient utilization of this method, some considerations should be taken into account for the formation of the rectangular equilibrium **H** and the flexibility matrices **G** , which can also be stated in two different manners.

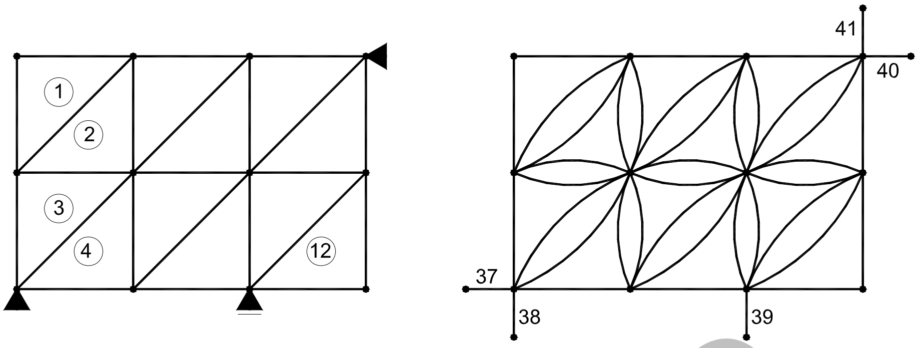
In the first case, in the equilibrium matrix the columns related to the support conditions are located between columns which are related to the elements. As an example, if such a style is employed, then the element forces of a triangular finite element with element number i , will not be $\{F_{3i-2},...,F_{3i}\}$ anymore. Then the effect of this variation in all part of analysis including, making the equilibrium, null space, and flexibility matrices as well as the report of element forces, should be considered.

In the second case, these columns are positioned at the end of columns which are related to the elements.

The first case is complicated and does not lead to localized self stress systems. Though the second case does not lead to localized self stress systems, however, it is better than the former one, because it is simpler and does not change the order of elements, and also it results in flexibility matrices which have more suitable pattern. The latter case is comprehensively studied in the following.

4. A COMBINED ALGEBRAIC AND GRAPH THEORETICAL METHOD

A triangular finite element model with indeterminate support conditions is shown in Figure 1(a), and the corresponding interface graph is illustrated in Figure 1(b). Each support reaction force is modeled as a simple graph member. Thus a simple support and a roller support are modeled using two and one simple graph member, respectively. These members should be numbered and used in the formation of the equilibrium matrix. It is obvious that, for each of these members, a column is added to the equilibrium matrix from right-hand side. In fact, these columns are vectors with all of their entries being zero except the entries in the rows corresponding to the freedom restraints, which have unit values in their entries. In this case, all the nodes of structure are considered as free nodes and there is no need for removal of rows.



(a) (b)

Figure 1. (a) A finite element mesh with indeterminate support condition; (b) Interface graph of the model

It is important to note that, in the present approach the end nodes of added members, which are artificially connected to the ground, should not be considered as structural nodes and should ultimately be excluded from nodal numbering and ordering. As an example, the equilibrium matrix for the structure shown in Figure 1(a), has 19 rows and 36 columns, having DSI=17, while this matrix for Figure 1(b) has 24 rows and 41 columns having DSI=17.

Using such a modeling approach, the DSI of structure shown in Figure 1(b), is internally 15 and externally 2. If an algebraic procedure is used for calculating the null basis, columns 37 to 39 will be selected as independent columns and columns 40 and 41 will be taken as dependent ones. Despite of this, the selection of a statically determinate support condition for any type of structures is very simple and needs no additional procedures. This simple example shows that the selection of dependent and independent columns from equilibrium matrix is straightforward. Using the above procedure, generally, dependent columns are made from all internal dependent columns (these columns are corresponding to the generators of self stress systems) and all the added columns, except the columns related to a statical determinate support condition. However it seems that, the definition of special graphs with respect to the type of problems and supports conditions for the calculation of sparse null vectors (related to supports) is possible but not simple. This means that, an algebraic procedure for finding this type of null vectors can be more suitable.

The equilibrium matrix $\mathbf{H}$ is schematically shown in Figure 2, in its general form, in which $\mathbf{h}_1$ is the related submatrix of internally determinate structure, $\mathbf{h}_{1e}$ corresponds to the submatrix of determinate support conditions, $\mathbf{h}_2$ is related submatrix of internal redundant forces and $\mathbf{h}_{2e}$ is the submatrix corresponding to the external redundant forces. Then

$$\mathbf{H} = [\mathbf{H}_1 \quad \mathbf{H}_2] \tag{9}$$

In which

$$\mathbf{H}_1 = [\mathbf{h}_1 \quad \mathbf{h}_{1e}] \quad , \quad \mathbf{H}_2 = [\mathbf{h}_2 \quad \mathbf{h}_{2e}] \tag{10}$$

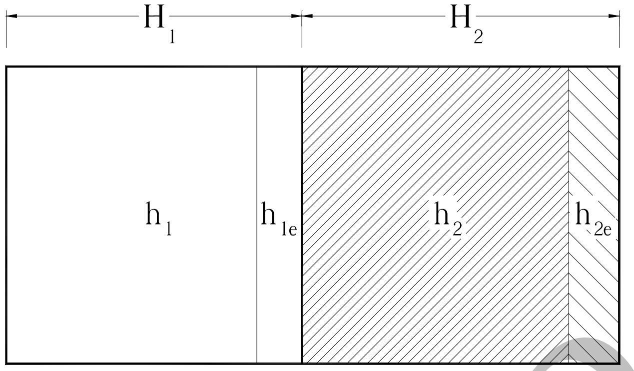


Figure 2. Schematic view of a partitioned rectangular equilibrium matrix

It should be noted that, though identifying the determinate support condition is simple and can separately be performed, however, if the matrix $[\mathbf{h}_1 \quad \mathbf{h}_e]$ (submatrix $\mathbf{h}_e = [\mathbf{h}_{1e} \quad \mathbf{h}_{2e}]$ includes all columns which are related to support conditions) enters in an LU-decomposition, all independent columns can automatically be selected. It is $\mathbf{h}_1$ plus α columns (α being 3 for planar and 6 for three dimensional cases) which will be identified out of the columns of $\mathbf{h}_e$. By this decomposition performed on $\mathbf{H}$, the corresponding null space can be generated using the following procedure:

Substituting $\mathbf{H}_2$ from Eq. (10) into Eq. (6) leads to:

$$\mathbf{B}_1 = \mathbf{P}\begin{bmatrix} -\mathbf{H}_1^{-1}\mathbf{H}_2 \\ \mathbf{I} \end{bmatrix} = [\mathbf{B}_{1i} \quad \mathbf{B}_{1e}] \tag{11}$$

In which

$$\mathbf{B}_{1i} = \mathbf{P}\begin{bmatrix} -\mathbf{H}_1^{-1}\mathbf{h}_2 \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \text{ and } \mathbf{B}_{1e} = \mathbf{P}\begin{bmatrix} -\mathbf{H}_1^{-1}\mathbf{h}_{2e} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \tag{12}$$

In Refs. [20-22], efficient graph theoretical methods are presented for the calculation of highly sparse and banded $\mathbf{B}_{1i}$ matrix for three different types of elements. Then in Eq. (11) this part can be substituted using the null vectors which are calculated using these methods. Thus

$$\mathbf{B}_{1i} = \begin{bmatrix} \mathbf{b}_{1g} \\ \mathbf{0} \end{bmatrix} \tag{13}$$

where $\mathbf{b}_{1g}$ is the computed null space by the graph theoretical methods. It is obvious that for matching the size of matrices, some rows with zero entries should be added to $\mathbf{b}_{1g}$ matrix. The number of these rows is equal to the number of added members for modeling of the support conditions. Such zero rows and columns should be added to unassembled flexibility matrix because fixed supports have infinite stiffness and zero flexibility. Now the second part of null space matrix, $\mathbf{B}_{1e}$, which is related to the external redundant forces is studied. This part can be calculated using two different approaches as follows:

1. The formation of $\mathbf{B}_0$ matrix is necessary for the finite element analysis using the force method. This matrix can be calculated using Eq. (7), thus the inverse of $\mathbf{H}_1$ is required for finding $\mathbf{B}_0$. On the other hand $\mathbf{B}_{1e}$ can be calculated employing Eq. (12), however, in practice there is no need for the calculation of $-\mathbf{H}_1^{-1}\mathbf{h}_{2e}$, since the special characteristic of $\mathbf{h}_{2e}$, it is sufficient to select columns related to constrained degrees of freedom and add zero and unit matrices of appropriate sizes, and exchanging the rows. Thus the null vectors corresponding to redundant support forces which are not necessarily sparse can simply be calculated.
2. The submatrix $\mathbf{B}_{1e}$ can directly be calculated without using the inverse of $\mathbf{H}_1$. This can be done using Turn-Back method. Since the submatrix $\mathbf{h}_{2e}$ is determined, thus from each of its columns and to the left, **LU** decomposition can be performed. Using this process for each column, a minimum dependency set is found and the submatrix $\mathbf{B}_{1e}$ is directly obtained. It should be noted that, the numbering of elements which are related to support conditions is performed at the end of the numbering process of finite elements. Therefore, using the Turn-Back method will not usually leads to sparse null vectors and it is possible to require more computational time than first process.

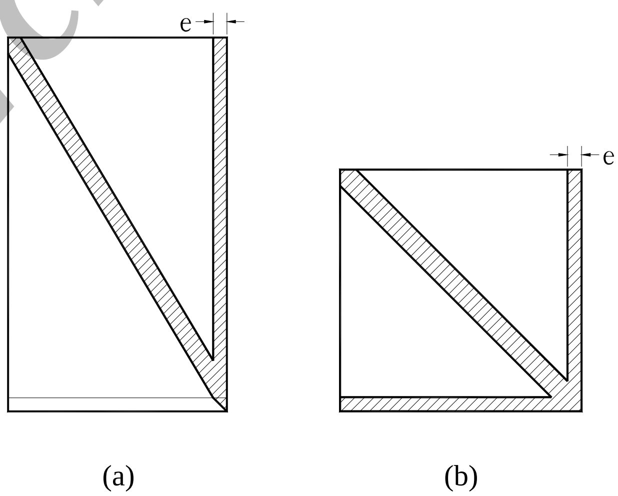


Figure 3. Matrix patterns (a) Null basis $\mathbf{B}_1$ (b) Flexibility matrix $\mathbf{G}$

However each of above processes for the calculation of $\mathbf{B}_{1e}$, leads to general null basis $\mathbf{B}_1$ and flexibility matrix **G**, where their patterns are depicted in Figure 3.

The pattern of the flexibility matrix which is shown in Figure 3(b) is a well-known *Doubly Bordered Banded Form* but in the real models the degree of external indeterminacy is considerably less than the degree of internal indeterminacy. Then the width of non-sparse part of the null basis and the flexibility matrix (e) will be insignificant. This pattern can effectively be stored using *Skyline* data structure. Also another effective approach is the usage of partitioned form of the flexibility matrix as shown in the following:

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_{11} & \mathbf{g}_{12} \\ \mathbf{g}_{12}^{t} & \mathbf{g}_{22} \end{bmatrix} \tag{14}$$

In this form the submatrix $\mathbf{g}_{11}$ is highly sparse and banded and effective solutions can be performed on it.

The combination of these two approaches will lead to an efficient usage of graph theoretical force methods for the generation of sparse and banded null basis and flexibility matrices. Three examples of triangular plane stress, rectangular plate bending and tetrahedron finite elements with indeterminate support conditions are presented in the following section.

5. NUMERICAL RESULTS

In this section three examples are studied having different types of elements and indeterminate support conditions.

Example 1: Half of a continuous beam (because of symmetry) which is modeled using triangular plane stress finite elements is shown in Figure 4. The specifications of the model are as follows:

Number of triangular elements = 200, Thickness = 0.05m, Elastic module = E = 2e+8kN/m^2, Poisson's ratio= v = 0.3, Number of nodes = 126, DSI=357 (Internal DSI=351, External DSI=6)

As shown in Figure 4, support conditions are indeterminate and the degree of external indeterminacy is 6, which is approximately 1.7% of the total DSI. Using the procedure presented with the combination of graph theoretical force method, the null basis and the flexibility matrices are calculated that their patterns are depicted in Figure 5. Her nz shows the number of non-zero entries of the matrices.

As it is shown in Figure 5, the width of non-sparse part is insignificant compared to total width, and both matrices have suitable patterns. Since each node has only two degrees of freedom (displacement in x and y direction), in rectangular finite element models with indeterminate support conditions, the process of applying support conditions is identical to the triangular finite elements models. On the other hand, in the interface graph, a simple support and roller support will be modeled using two members and one member, respectively.

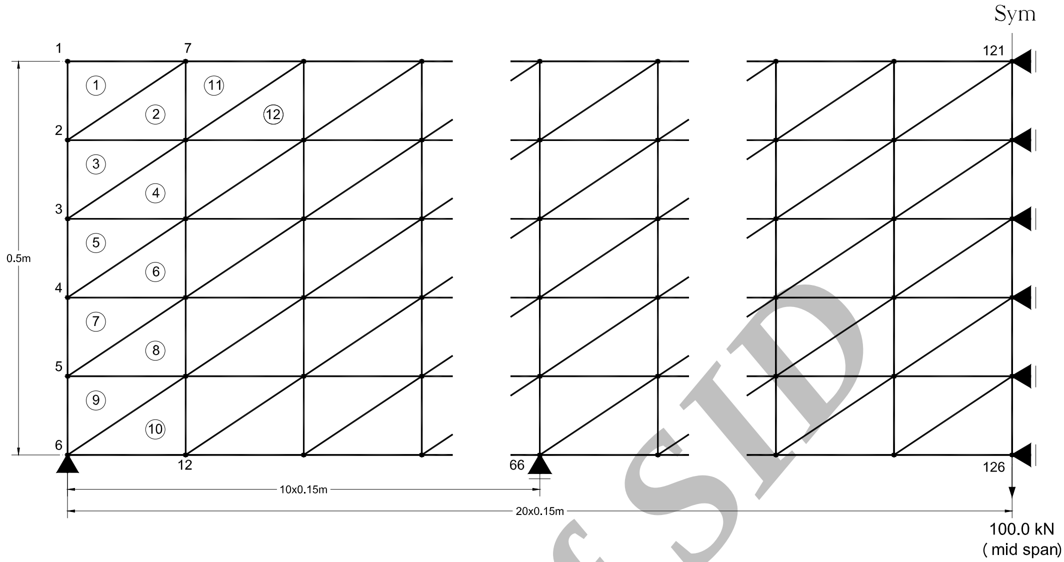
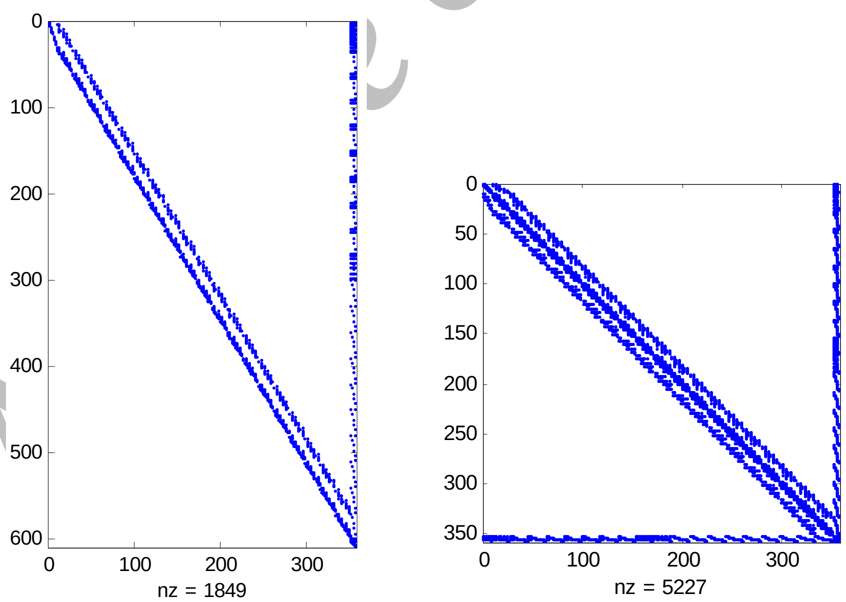


Figure 4. Finite element model with node and element numbering, and indeterminate support conditions



(a) (b)

Figure 5. (a) Pattern of null basis matrix $\mathbf{B}_1$; (b) Pattern of flexibility matrix $\mathbf{G}$

Example 2: A plate is shown in Figure 6. This plate has roller supports in one end and is clamped in the other end. The plate is discritized using rectangular plate bending elements,

Figure 7. Each nodes with clamped condition (nodes 1~7) is modeled using additional three members, and each roller support (nodes 85~91) is modeled as a single member. The specifications of the model are as follows:

Number of rectangular elements = 72, Thickness = 0.1m, Elastic modulus = E = 2e+8kN/m^2, Poisson's ratio= v = 0.2, Number of nodes = 91, DSI=403 (Internal DSI=378, External DSI=25)

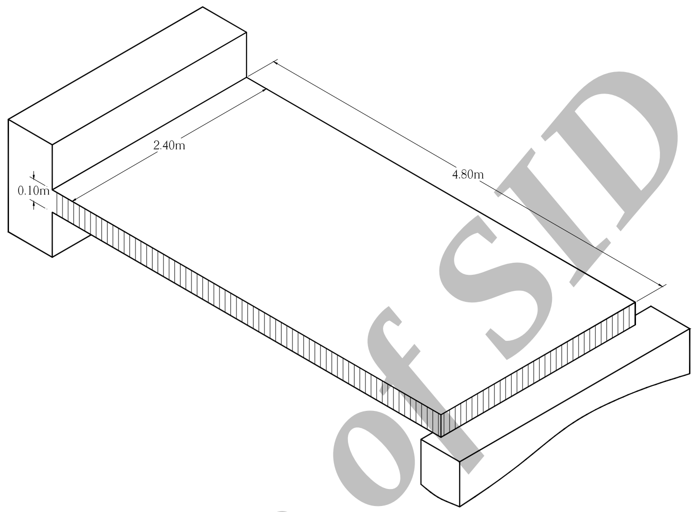


Figure 6. A plate with different support conditions

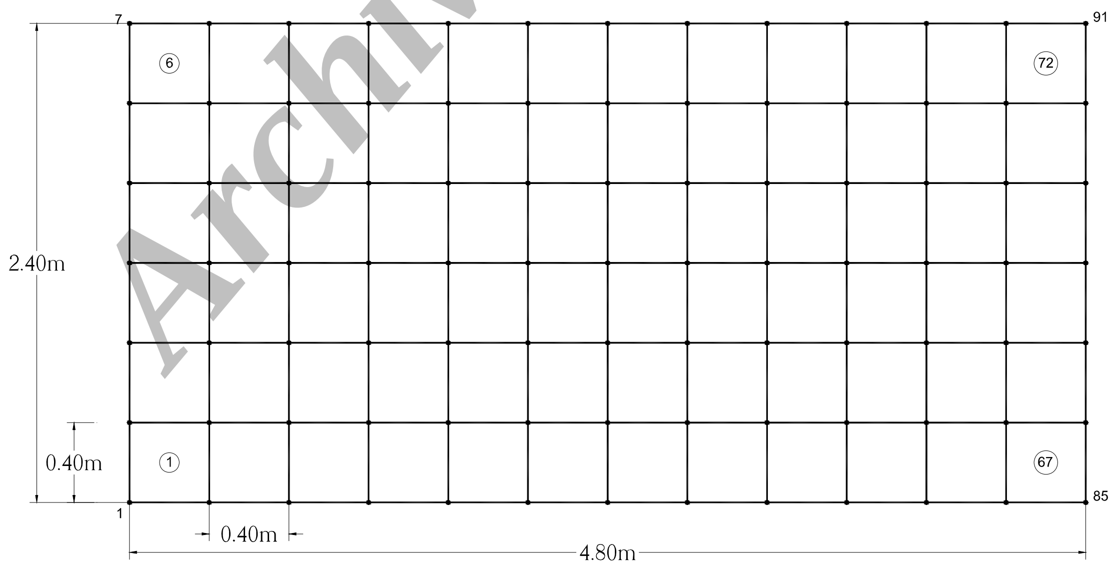
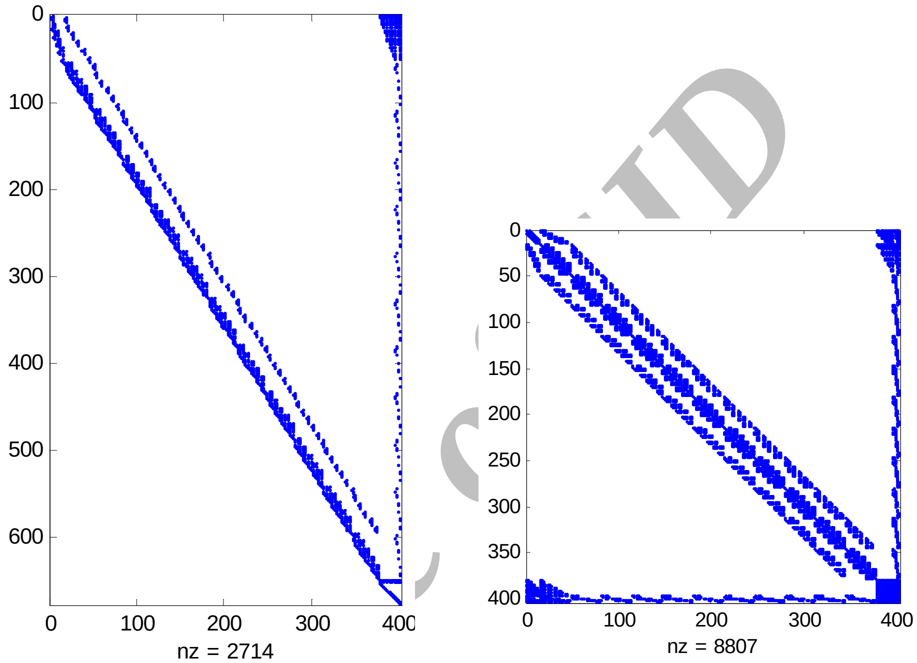


Figure 7. The finite element mesh of a plate with node and element numberings

In plate bending problems, usually the number of supports and external redundant forces are significantly more than plane stress/strain problems. This will lead to partially large width of non-sparse part, however, this width is still far less than the total width (total degree of static indeterminacy).

In Figure 8, the patterns of matrices which are completely identical to those shown in Figure 3 are depicted.



(a) (b)

Figure 8. (a) Pattern of null basis matrix $\mathbf{B}_1$ (b) Pattern of flexibility matrix **G**

Example 3: A cantilever beam which is discretized using 96 tetrahedron elements is shown in Figure 9. Support conditions are also indeterminate and include five simple supports which are applied to nodes 1~5. The specifications of the model are as follows:

Number of tetrahedron elements = 96 , E = 2e+7 kN/m^2, v = 0.2, Number of nodes = 4, DSI=456 (Internal DSI=447 and External DSI=9).

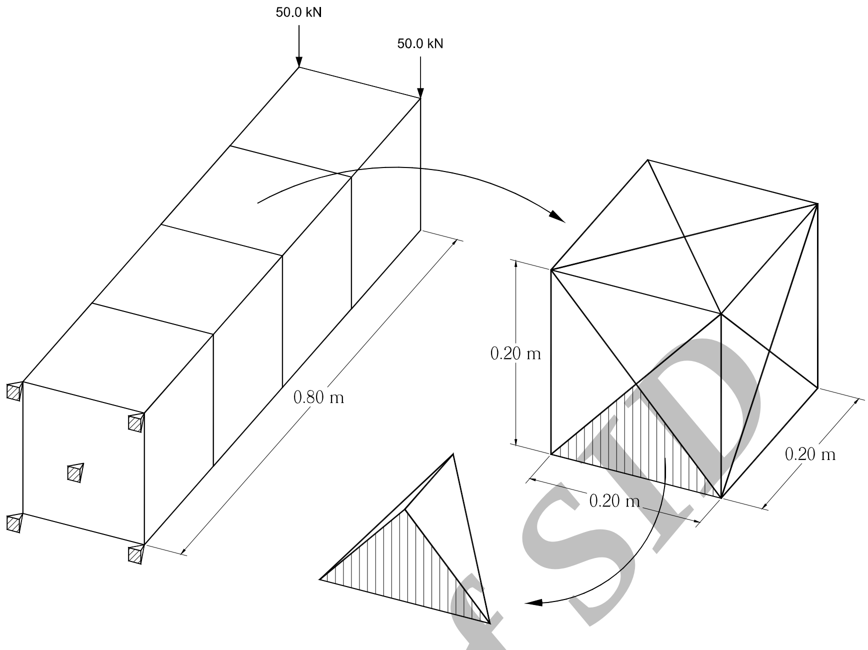
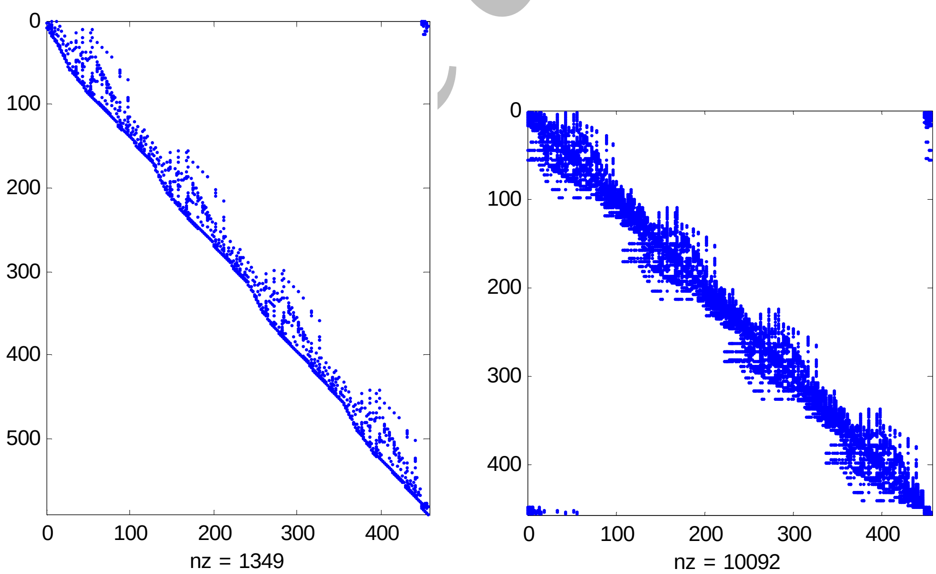


Figure 9. A cantilever beam and its finite element model



(a) (b)

Figure 10. (a) Pattern of null basis matrix $\mathbf{B}_1$ (b) Pattern of flexibility matrix $\mathbf{G}$

Again patterns of the matrices are quite reasonable and the width of non-sparse part is insignificant.

6. CONCLUDING REMARKS

The following conclusions are derived considering the results of the examples studied in this paper:

- Using the present method, the effect of different support conditions can easily be included with no considerable decrease in the sparsity and computational time. This is performed utilizing combinations of algebraic procedures after the calculation of null basis of externally determinate structure.
- The methods presented are also suitable for applying elastic support conditions.
- The present method is completely general and can be used for any finite element models.

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