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NUMERICAL APPROACH FOR THE ANALYSIS OF ANISOTROPIC RECTANGULAR PLATES USING DISCRETE SINGULAR CONVOLUTION

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ABSTRACT

In the present study, free vibration, deflection and buckling analyses of rectangular composite plates via discrete singular convolution has been presented. In the proposed approach, the derivatives in both the governing equations and the boundary conditions are discretized by the method of DSC. The obtained results are compared with those of other numerical methods available in the literature. Numerical calculations showed that accurate results can be achieved. It has been also shown that the DSC method yields efficient and convergence solutions and these results are in excellent agreement with the analytical solutions and other sources of numerical solutions.

Keywords: Discrete singular convolution method, numerical methods, anisotropic plate, free vibration, buckling

1. INTRODUCTION

A variety of numerical methods are avaliable today for engineering analysis. Generally, there are a few numerical methods, which are finite difference, finite element, differential quadrature and boundary element method. These numerical approachs have been used extensively for solving differential equations. Discrete singular convolution (DSC) method is a new method that was introduced by Wei [1]. Several researchers have applied the DSC method to solve a variety of problems in different fields of science and engineering [3-12]. The pioneer work for the application of the DSC method to the general area of solid mechanics was carried out by Wei [6,7,8], Wei et. al. [6,7,8], Zhao et al. [11], Lim et al. [12,13] and Civalek [15,16,17]. New developments, such as the new way to apply the boundary conditions [4] to increase the solution accuracy, have been made on the DSC approach to make the method more attractive for engineering practice. Details on the development of the DSC method and its applications to structural mechanics problems may be found in a recent paper by Wei [6]. The unique properties of advanced composite

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materials have resulted in extensive applications of laminated plates to aerospace, automobile, mechanical, shipbuding, and nuclear industries. Because of the increasingly wide application of composite structural elements, especially laminated plates, the analysis of such structures has been receiving much interest in the past [19-26]. The primary objective of this study is to explore the application of the DSC method to the static, buckling, and vibration analysis of anisotropic rectangular plates. The given results are verified by comparison against results available in the open literature. To the author knowledge, this is the first instance in which the DSC method has been adopted for free vibration and buckling analysis of anisotropic rectangular plates.

2. DISCRETE SINGULAR CONVOLUTION (DSC)

Accurate and efficient numerical approaches for differential equations are of great importance in both engineering and physical sciences. The method of discrete singular convolutions (DSC) has emerged as a new approach for numerical solutions of differential equations. This new method has a potential approach for computer realization as a wavelet collocation scheme [2,3]. By using the appropriate realizations of a singular convolution kernel, this method can be efficient, accurate and reliable approach for numerical solutions [5-13]. In this paper, details of the DSC method are not given; interested readers may refer to the works of [1-8]. In the method of DSC, weighted linear combination of the function values in the direction of space variable is used to approximate the any order derivative of a given function with respect to a space variable at a discrete point. Consider a distribution, T and $\eta(t)$ as an element of the space of the test function. A singular convolution can be defined by [6]

$$F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t - x)\eta(x)dx \tag{1}$$

where $T(t-x)$ is a singular kernel. For example, singular kernels of delta type [7]

$$T(x) = \delta^{(n)}(x)\,; \quad (n = 0,1,2,...,) \tag{2}$$

Kernel $T(x)=\delta(x)$ is important for interpolation of surfaces and curves, and $T(x)=\delta^{(n)}(x)$ for n>1 is essential for numerically solving differential equations. The DSC algorithm can be realized by using many approximation kernels. However, it was shown [6,7,8] that for many problems, the use of the regularized Shannon kernel (RSK) is very efficient. The RSK is given by [6]

$$\delta_{\Delta,\sigma}(x - x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right];\ \sigma > 0 \tag{3}$$

where $\Delta=\pi/(N-1)$ is the grid spacing and N is the number of grid points. Thus, it is suitable to say that in the DSC method, the function $f(x)$ and its derivatives with respect to the x coordinate at a grid point x_i are approximated by a linear sum of discrete values $f(x_k)$ in a narrow bandwidth $[x-x_M, x+x_M]$. For example mth order derivative of a function $g(x)$ at the ith point is given by

$$f^{(m)}(x_i) \approx \sum_{j=-M}^{M} \delta_{\Delta,\sigma}^{(m)}(x_i - x_j) g(x_j); \quad (m=0,1,2,...) \tag{4}$$

where superscript m denotes the mth-order derivative with respect to x.

3. APPLICATIONS

The general governing differential equations of rectangular composite plates are given as [24]

$$D_{11}\frac{\partial^4 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y}$$

$$+ 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} = q_0 + \rho h \omega^2 w - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} \tag{5}$$

where D_{ij} are the coefficients of the bending rigidity for plate, h is the plate thickness, N_x and N_y are the applied compressive loads in the respective x and y directions, q_0 is the pressure, w is the deflection, ρ is the density, ω is the natural frequency, x and y are the midplane Cartesian coordinate. The boundary conditions applied in the numerical analysis are [24]

i) Simply supported edges

$$w = 0 \text{ and } -D_{11}\frac{\partial^2 w}{\partial x^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} - D_{12}\frac{\partial^2 w}{\partial y^2} = 0 \text{ at } x = 0, a \tag{6a}$$

$$w = 0 \text{ and } -D_{12}\frac{\partial^2 w}{\partial x^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} - D_{22}\frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0, b \tag{6b}$$

ii) Clamped edges

$$w = 0 \text{ and } \frac{\partial w}{\partial x} = 0 \text{ at } x = 0, a \tag{7a}$$

$$w = 0 \text{ and } \frac{\partial w}{\partial y} = 0 \text{ at } y = 0, b \tag{7b}$$

Governing equations can be written in DSC form as

$$D_{11}\sum_{k=-M}^{M}\delta_{\Delta,\sigma}^{(4)}(k\Delta x)W_{kj} + 4D_{16}\sum_{l=-M}^{M}\delta_{\Delta,\sigma}^{(1)}(k\Delta x)\sum_{k=-M}^{M}\delta_{\Delta,\sigma}^{(3)}(k\Delta x)W_{kl}$$

$$+2(D_{12}+2D_{66})\sum_{l=-M}^{M}\delta_{\Delta,\sigma}^{(2)}(k\Delta x)\sum_{k=-M}^{M}\delta_{\Delta,\sigma}^{(2)}(k\Delta x)W_{kl} + 4D_{26}\sum_{k=-M}^{M}\delta_{\Delta,\sigma}^{(1)}(k\Delta x)\sum_{l=-M}^{M}\delta_{\Delta,\sigma}^{(3)}(k\Delta x)W_{kl}$$

$$+4D_{22}\sum_{k=-M}^{M}\delta_{\Delta,\sigma}^{(4)}(k\Delta x)W_{ik} = \gamma h\omega^{2}W_{ij} \tag{8}$$

where $\delta_{\Delta,\sigma}(x)$, $\delta^{(1)}{}_{\Delta,\sigma}(x)$, $\delta^{(3)}{}_{\Delta,\sigma}(x)$, and $\delta^{(4)}{}_{\Delta,\sigma}(x)$ are the first-, second-, third-and fourth-order derivatives of the regularized Shannon's delta kernel. These are given in Appendix in detail.

3.1 Implementation of boundary conditions

Simply supported and clamped boundary conditions are applied in the analysis. It is known that, to obtain a unique solution for a differential equation, appropriate boundary conditions must be satisfied. In applying the DSC method Wei et al. [9,10] and Zhao et al. [11] proposed a practical method in applying the simply supported and clamped boundary conditions. In the present study, same procedure proposed by Wei et al. [9,10] and Zhao et al. [11] are used. More detailed formulation about the implementation of boundary conditions in DSC can be found in these references [9,10,11,16]. Consider a uniform grid having following form

$$0 = X_0 < X_1 < ... < X_{N_x} = 1 \tag{9a}$$

$$0 = Y_0 < Y_1 < ... < Y_{N_y} = 1\,. \tag{9b}$$

Consider a column vector **W** given as

$$\mathbf{W} = (W_{0,0},...W_{0,N}, W_{1,0},...W_{N,N})^{T} \tag{10}$$

with $(N_x+1)(N_y+1)$ entries $W_{i,j} = W(X_i, Y_j)$; $(i = 0,1,...,N_x;\ j = 0,1,...,N_y)$. Let us define the $(N_x+1)(N_y+1)$ differentiation matrices $\mathbf{D}_r^n (r = X, Y; n = 1,2,...)$, with their elements given by

$$[\mathrm{D}_{\mathrm{x}}^{(\mathrm{n})}]_{\mathrm{i,j}} = \delta_{\sigma,\Delta}^{(\mathrm{n})}(\mathrm{x}_{\mathrm{i}} - \mathrm{x}_{\mathrm{j}}) \tag{11a}$$

$$[\mathbf{D}_y^{(n)}]_{i,j} = \delta_{\sigma,\Delta}^{(n)}(y_i - y_j) \tag{11b}$$

where $\delta_{\sigma,\Delta}^{(n)}(r_i - r_j),\ (r = x, y)$ is a DSC kernel of delta type. In this stage, we consider the following relation between the inner nodes and outer nodes on the left boundary [9,11]:

$$W(X_{-i}) - W(X_0) = W(X_0)\left(\sum_{j=0}^{J} a_i X_{-i}\right)[W(X_i) - W(X_0)] \tag{12}$$

where parameter $a_i, (i = 1,2,..., M)$ are to be determined by the boundary conditions. Thus, the first and second order derivatives of W on the left boundary are approximated by

$$W'(X_0) = \left(\delta_{\sigma,\Delta}^{(1)}(X_i - X_0) - \sum_{j=0}^{J}(1-a_i)\delta_{\sigma,\Delta}^{(1)}(X_i - X_j)\right)W(X_0)$$

$$+\sum_{j=0}^{J}(1-a_i)\delta_{\sigma,\Delta}^{(1)}(X_i - X_j)W(X_i) \tag{13}$$

$$W''(X_0) = \left(\delta_{\sigma,\Delta}^{(2)}(X_i - X_0) + \sum_{j=0}^{J}(1-a_i)\delta_{\sigma,\Delta}^{(2)}(X_i - X_j)\right)W(X_0)$$

$$+\sum_{j=0}^{J}(1+a_i)\delta_{\sigma,\Delta}^{(2)}(X_i - X_j)W(X_i) \tag{14}$$

Similarly, the first and second order derivatives of f on the right boundary (at X_{N-1}) are approximated by

$$W(X_{N-1+i}) - W(X_{N-1}) = W(X_{N-1-i})\left(\sum_{j=0}^{J} a_i X_{-i}\right)[W(X_i) - W(X_N)] \tag{15}$$

Consequently, we obtain the following relation

$$W(X_{N-1+i}) = a_i W(X_{N-1-i}) + W(X_{N-1})[1 - a_i]. \tag{16}$$

Hence, the first and second order derivatives of f on the right boundary are given by

$$W'(X_{N-1}) = \left(\delta^{(1)}_{\sigma,\Delta}(X_i - X_{N-1}) - \sum_{j=0}^{J} (1-a_i)\delta^{(1)}_{\sigma,\Delta}(X_i - X_j) \right) W(X_{N-1})$$

$$+ \sum_{j=0}^{J} (1-a_i)\delta^{(1)}_{\sigma,\Delta}(X_i - X_j) W(X_i) \quad (17)$$

$$W''(X_{N-1}) = \left(\delta^{(2)}_{\sigma,\Delta}(X_i - X_{N-1}) + \sum_{j=0}^{J} (1-a_i)\delta^{(2)}_{\sigma,\Delta}(X_i - X_j) \right) W(X_{N-1})$$

$$+ \sum_{j=0}^{J} (1+a_i)\delta^{(2)}_{\sigma,\Delta}(X_i - X_j) W(X_i) \quad (18)$$

As a result the DSC forms of given boundary conditions can be easily written using above procedure. For example, DSC form of simply supported boundary conditions can be given for right boundary by

$$w_{ij} = 0 \quad (19)$$

$$\left(\delta^{(1)}_{\sigma,\Delta}(X_i - X_{N-1}) - \sum_{j=0}^{J} (1-a_i)\delta^{(1)}_{\sigma,\Delta}(X_i - X_j) \right) W(X_{N-1})$$

$$+ \sum_{j=0}^{J} (1-a_i)\delta^{(1)}_{\sigma,\Delta}(X_i - X_j) W(X_i) \quad (20)$$

4. NUMERICAL RESULTS

In this section some numerical results are presented. As a first example, consider the free vibration problem of isotropic plate. The results are listed in Table 1 and Table 2 for CCCC and SSSS boundary conditions. Comparisons are made with the analytical solutions provided by Leissa [23]. It can be seen that good accuracy is achieved by the present DSC method with N=16. Non-dimensional fundamental frequency of CCCC rectangular anisotropic plates are given in Table 2 for two different aspect ratios.

Table 1. Non-dimensional frequency parameters of CCCC isotropic plates $\Omega = (\omega a^2)\sqrt{\rho h / D}$

N	Present DSC results			
	Mode sequence			
	1	**2**	**3**	**4**
12	36.005	73.433	73.421	108.278
16	35.993	73.415	73.415	108.273
18	35.992	73.413	73.413	108.273
20	35.992	73.413	73.413	108.270
22	35.992	73.413	73.413	108.270
Leissa [23]	35.992	73.413	73.413	108.270

Table 2. Non-dimensional frequency parameters of SSSS isotropic plates $\Omega = (\omega a^2)\sqrt{\rho h / D}$

N	Present DSC results			
	Mode sequence			
	1	**2**	**3**	**4**
12	19.7401	49.3493	49.3493	78.9578
16	19.7398	49.3488	49.3488	78.9573
18	19.7393	49.3482	49.3482	78.9573
20	19.7392	49.3480	49.3480	78.9571
22	19.7392	49.3480	49.3480	78.9570
Leissa [23]	19.7392	49.3480	49.3480	78.9568

From Table 3, it can be seen that the results compare well with data obtained by the Rayleigh-Ritz [18] and differential quadrature [24] method. Other results of a convergence study of frequencies are presented in Table 4. Results from the present DSC method are compared with the results of the method of differential quadrature (DQ) and harmonic differential quadrature (HDQ) by Bert et al. [24]. From the results presented in this table, it is clear that the present DSC results are in excellent agreement with those obtained using a variety of numerical methods using N=16. Non-dimensional frequency parameters of CCCC anisotropic rectangular plates are analysed and results are listed in Tables 5-7.

Table 3. Non-dimensional fundamental frequency of CCCC rectangular anisotropic plates ($\Omega = (\omega b^2)\sqrt{\rho h / D}$; $D = E_1 h^3 /[12(1 - \upsilon_{12}^2 E_r]$; $E_r = E_2 / E_1$; θ=15)

a/b	Bert et al. [24]	Whitney [18]	DSC
1	23.09	23.10	23.12
2	9.67	9.68	9.70

Table 4. Non-dimensional frequencies ($\Omega = (\omega a^2 / \pi^2)\sqrt{\rho h / D_{11}}$) of SSSS square anisotropic plates (D22/D11=1; D12+ 2D66/ D11=1)

N	Bert et al. [24] DQ	Bert et al. [24] HDQ	Present study HDQ	Present study DSC
9	2.00	2.00	-	2.008
13	-	2.00	2.00	2.000
16	-	-	2.00	2.000

Table 5. Non-dimensional frequency parameters of CCCC anisotropic plates $\Omega = (\omega a^2)\sqrt{\rho h / D}$ b/a=0.5; E1/E2 = 10; G12/E2=0.25; $\upsilon_{12} = 0.3$; $D = E_1 h^3 /[12(1 - \upsilon_{12}\upsilon_{21})]$)

Orientation β	Whitney [Ref.18]			Present DSC results		
	Mode sequence			Mode sequence		
	1	2	3	1	2	3
0°	9.34	17.61	20.83	9.34	17.61	20.84
15°	9.68	17.19	22.02	9.68	17.20	22.03
45°	13.88	17.73	23.85	13.86	17.66	23.68
60°	17.87	19.86	23.75	17.88	19.78	23.58
90°	22.57	23.38	25.30	22.57	23.38	25.31

The geometric and material properties are b/a=0.5 and b/a=1; E_1/E_2=10, G_{12}/E_2=0.25, $\upsilon_{12} = 0.3$. The present results are compared with the results of Whitney [18]. In comparison with the results of Whitney [18], the DSC results provide satisfactory accuracy. Buckling coefficients ($K = (Nb^2 / Q_{22}h^3)$ of SSSS square anisotropic plates under biaxial compression (N_x=N_y=N) are listed in Table 8 for the value of Q_{11}/Q_{22}=25, Q_{12}/Q_{22}=0.25, Q_{66}/Q_{22}=0.5. Results from the present DSC method are compared with the results of Bert et al. [24] and Whitney [18]. The results by DSC are close to the numerical solution by Bert et al. [18] using the differential quadrature method. In Table 9, non-dimensional deflections ($W = wD_{11} / qa^4$) of SSSS square anisotropic plates for different stiffness ratio are given. Consequently, by comparing the computed results with those available in published works, the present analysis by the DSC method is examined and a very good agreement is observed.

Table 6. Non-dimensional frequency ($\Omega = (\omega a^2)\sqrt{\rho h / D}$) parameters of CCCC anisotropic square plates (b/a=1; E_1/E_2 = 10; G_{12}/E_2=0.25; $\upsilon_{12} = 0.3$; $D = E_1 h^3 /[12(1-\upsilon_{12}\upsilon_{21})]$)

Orientation β	Whitney [Ref.18]			Present DSC results		
	Mode sequence			Mode sequence		
	1	2	3	1	2	3
0°	23.97	31.15	46.41	23.97	31.13	46.40
15°	23.10	31.52	47.65	23.08	31.50	47.61
30°	21.35	33.18	50.72	21.33	33.16	50.64
45°	20.51	35.01	47.07	20.50	34.98	46.88

Table 7. Non-dimensional frequencies ($\Omega = (\omega a^2 / \pi^2)\sqrt{\rho h / D_{11}}$) of SSSS square anisotropic plates for different stiffness ratio

Material parameters	Present study DSC		
	7×7	16×16	18×18
D_{22}/D_{11}=1, D_{12}+ $2D_{66}$/ D_{11}=1, D_{16}/D_{11}=0	2.098	2.00	2.00
D_{22}/D_{11}=1, D_{12}+ $2D_{66}$/ D_{11}=1.061, D_{16}/D_{11}=-0.174	2.012	1.986	1.987

D_{22}/D_{11}=1, D_{12}+ 2D_{66}/ D_{11}=1.500, D_{16}/D_{11}=-0.500	1.919	1.884	1.886

Table 8. Buckling coefficients ($K = (Nb^2 / Q_{22}h^3)$ of SSSS square anisotropic plates under biaxial compression (Nx=Ny=N; Q11/Q22=25; Q12/Q22=0.25; Q66/Q22=0.5)

N	**Bert et al. [24] DQ**	**Whiney [18]**	**Ashton [20]**	**Present study DSC**
7	8.740	8.418	11.060	9.113
9	8.574	-	-	8.585
16	-	-	-	8.578

Table 9. Non-dimensional deflections ($W = wD_{11} / qa^4$) of SSSS square anisotropic plates for different stiffness ratio

Material parameters	**Ashton [20]**	**DQM Bert et al. [24]**	**Present DSC solution**
D_{22}/D_{11}=1, D_{12}+ 2D_{66}/ D_{11}=1, D_{16}/D_{11}=0	0.00406	0.00406	0.00406
D_{22}/D_{11}=1, D_{12}+ 2D_{66}/ D_{11}=1.061, D_{16}/D_{11}=-0.174	0.00411	0.00411	0.00410
D_{22}/D_{11}=1, D_{12}+ 2D_{66}/ D_{11}=1.500, D_{16}/D_{11}=-0.500	0.00452	0.00455	0.00454
D_{22}/D_{11}=1, D_{12}+ 2D_{66}/ D_{11}=1.690, D_{16}/D_{11}=-0.587	0.00476	0.00478	0.00476

5. CONCLUSIONS

Free vibration, deflection and buckling analyses of rectangular composite plates via discrete singular convolution have been presented. In the proposed approach, the derivatives in both the governing equations and the boundary conditions are discretized by the method of DSC.

The obtained results are compared with those of other numerical methods available in the literature. Numerical results indicate that the discrete singular convolution is a simple, accurate and reliable algorithm for vibration and buckling analyses of anisotropic composite plates. In addition, the new numerical technique DSC algorithm has been examined and found to be simple, accurate and efficient. Some different applications of the present method to analysis of solid mechanic problems are currently under investigation.

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