ASIAN JOURNAL OF CIVIL ENGINEERING (BUILDING AND HOUSING) VOL. 9, NO. 6 (2008)
PAGES 549-561

DAMAGE COUPLED TO YIELD FUNCTION FOR THE ELASTOPLASTIC ANALYSIS OF FRAMED STRUCTURES

E. Chica[*a], J.M.G. Teran[b], A.L. Iban[b] and P. López[c]
[a]*Department of Mechanical Engineering, University of Antioquia, C/ 67 No 53-108, AA 1226, Medellín-Colombia*
[b]*Solid Mechanics and Structures Group. E.T.S.I.I. (Industrial Engineering), University of Valladolid, C/Paseo del Cause s/n, CP 47011, Valladolid-Spain*
[c]*Structural Design Area, CARTIF. Technological Park of Boecillo, CP 47151 Valladolid-Spain*

Abstract

A method for coupling the variable damage to the yield function of a 2D beam element is presented. The damage is represented by a scalar internal variable which expresses the loss of strength of the material during ductile or fatigue processes and it is concentrated at the ends of the element.

Yield surfaces, considering the interaction of bending moment, axial force, shear force and damage of material are also given. And the yield function obtained can be used to determine the elastoplastic stiffness matrix of beam element used for the structural analysis.

Keywords: Elastoplasticity; damage; yield function; yield surface

1. Introduction

It is well known that the usual numerical way for the determination of the plastic collapse load of framed structures is the use of one-dimensional finite element models (2D beam element) together with the plastic hinge concept and an incremental procedure. We present an approach that takes into account the extended plastic cross section concept (that includes the plastic hinge concept) and Continuum Damage Mechanics (CDM) concepts for coupling the variable damage to the yield function of the cross section. We try to apply this approach to determine an explicit form of the tangent stiffness matrix called "elastoplastic degradation stiffness matrix" and also to determine more exactly the collapse load of the frame.

The yield function Z(F) includes the effect of the stress components F(axial *N*, shear *V* and bending moment *M*) acting in the system to predict the yielding of the material. This can be graphically represented as the place of the points of space of stress (yield surface) that constitute the limit for a given state of the material [1, 2]. To define the yield function Z (F, D) for damaged material, it has been necessary take into account: the Navier hypothesis for

[*] E-mail address of the corresponding author: echica@udea.edu.co (E. Chica)

beams, the Von Mises yield criterion and the hypothesis of strain equivalence of CDM formulated by Lemaitre [3] that assumes that the strain associated with a damage state under the applied stress is equivalent to the strain associated with its undamaged state under the effective stress [4]. The plasticity is supposed to be concentrated only in the cross section of the ends of the beams and it is in plastic state by the combination of stress that satisfies the yielding condition Z(N,V,M,D) at damage D. The evolution of damage can be determined using the kinetic law of damage evolution, where the damage rate and the effective accumulated plastic strain rate are coupled by mean of the definition of the plastic multiplier.

In the next section a description of an analytical procedure used for determine the yield function of elastoplastic 2D beam element of rectangular cross section is presented. The function obtained can be used in framework of plastic analysis of structures to calculate the loss of rigidity of the material due to its deterioration and its influence on the collapse load of the structure.

2. Materials and Methods

An analytical yield surface equation of 2D beam element (Figure 1), based on CDM and the classical hypothesis of Solid Mechanics is presented, taking into account the combined action of axial and shear forces, bending moment and the effects of damage of material. Basic assumptions, such as the following, have been taken into account [1,2,3,4]:

- Material nonlinearity is simulated by the formation of plastic zones of zero length at the ends of the each beam element.
- The effect of strain hardening is not considered.
- For the plastic behavior, Von Mises yield criterion and associated flow rule are adopted.
- Damage (D) is isotropic and, like plasticity, it is supposed to be concentrated at the beam ends.
- For simplicity, all expressions are only applicable for rectangular cross section of base (b) and heigh (h).

Under combined forces, the elastic limit is defined mathematically by a certain yield criterion or yield condition. The initial yield criterion depends only on the stress, and can be generally expressed as:

$$f = (\sigma_{eq} - R - \sigma_f) \le 0 \tag{1}$$

Where f is the yield function, σ_{eq} is the Von Mises equivalent stress, R is the isotropic hardening (although it is not considered) and σ_f is the yield stress. The equivalent stress σ_{eq} associated with a damaged state has to be replaced by $\overline{\sigma_{eq}}$ (damage effective equivalent stress) given according to the CDM concept of effective stress as:

$$\overline{\sigma_{eq}} = \frac{\sigma_{eq}}{1-D} \tag{2}$$

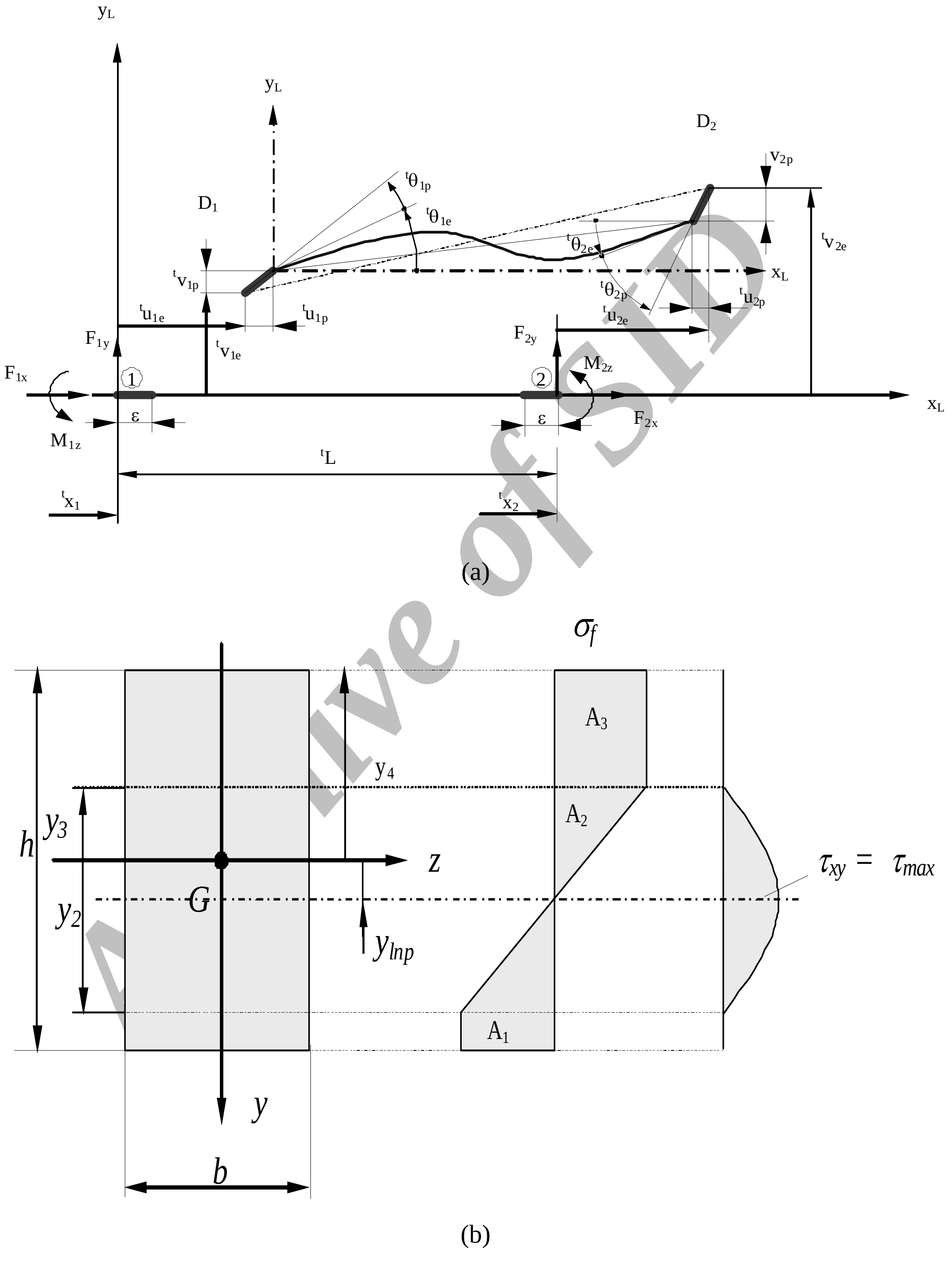

Figure 1. (a) Beam element with elastoplastic displacement at the end of the element and damage. (b) Stress distributions in a rectangular cross section

Substituting Eq. (2) into Eq. (1) we obtain the constitutive yield function for damaged material.

$$f = (\frac{\sigma_{eq}}{1-D} - \sigma_f) \leq 0 \tag{3}$$

In 2D beam elements σ_{eq} is given by:

$$\sigma_{eq} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\,\tau_{xy}^2} \tag{4}$$

and if the normal stress σ_y is not considered, then we can write:

$$\sigma_{eq} = \sqrt{\sigma_x^2 + 3\,\tau_{xy}^2} \tag{5}$$

where σ_x is the normal stress in the beam due to axial force and bending moment and τ_{xy} is the shear stress. In the points of the cross section of the beam where the normal stress is null (neutral axis $\sigma_x = 0$), yielding is only due to the effects of the shear stress (τ_{xy}). Therefore, in order to achieve yielding in the neutral fibre, the shear stress would have to be equal to the yield shear stress (σ_f), given as:

$$\tau_f = \frac{\sigma_{eq}}{\sqrt{3}} \tag{6}$$

and

$$f = (\frac{\sqrt{3}\tau_f}{1-D} - \sigma_f) \leq 0 \ \text{ or } \ f = \tau_f \leq \frac{(1-D)\sigma_f}{\sqrt{3}} \tag{7}$$

In the case of a section subjected to bending moment (M_z), axial stress (N_x) and shear stress (V_y) simultaneously, when the yielding of section takes place, the elastic area disminishes (y_2-y_3) and simultaneously the position of the neutral axis of the section varies (y_{lnp}, Figure 2) i.e when the loading process continues, yielding starts at the top or the bottom fibres, and the plastic zone propagates to the interior of the cross section. During the elastoplastic stage, the cross section has an elastic zone with linear stress variation, and one or two plastic zones with constant stress equal to the positive or negative yield stress. This process is continuum until the total yielding of the cross section that appear when the shear stress (τ_{xy}) is equal to the yield shear stress (τ_f).

The normal stress (σ_x) in the elastic zone of the material of the cross section of Figure 1(b) is given by

$$\frac{\sigma_f}{y_2 - y_{\text{lnp}}} = \frac{\sigma_{xp}}{y - y_{\text{lnp}}} \qquad \Rightarrow \qquad \sigma_{xp} = \frac{\sigma_f}{y_2 - y_{\text{lnp}}}\left(y - y_{\text{lnp}}\right) \tag{8}$$

And considering that the resultant normal force on the cross section must be cero we can write that the axial force is equal to

$$N_x(x) = \int_A \sigma_x(x) dA \tag{9}$$

Considering the Eq. (9) for each one of the areas of stress distributions shown in the Figure 1(b) corresponding to the plastic behavior (A_1, A_3) and elastic (A_2) we write the following expression for the resultant normal force:

$$N_x = -\int_{A_1} \sigma_f \, dA - \int_{A_2} \sigma_{xp} \, dA + \int_{A_3} \sigma_f \, dA \tag{10}$$

Now, taking into account the laws of variation of the normal stress in the elastic and plastic domain on the cross section according to Eq. (8) and substituting in Eq. (10) we get

$$N_x = \sigma_f \left[-\int_{y_2}^{y_1} z(y)\, dy - \frac{1}{y_2 - y_{\text{lnp}}} \int_{y_3}^{y_2} \left(y - y_{\text{lnp}}\right) z(y)\, dy + \int_{y_4}^{y_3} z(y)\, dy \right] \tag{11}$$

Using the same procedure described for the normal force, we can write the equilibrium equation for bending moment considering the elastic or plastic behavior of section:

$$M_z = \sigma_f \left[\int_{y_2}^{y_1} y\, z(y)\, dy + \frac{1}{y_2 - y_{\text{lnp}}} \int_{y_3}^{y_2} \left(y - y_{\text{lnp}}\right) y\, z(y)\, dy - \int_{y_4}^{y_3} y\, z(y)\, dy \right] \tag{12}$$

In the Eqs. (11) and (12) the limits of the section (y_1 and y_4) are known for a rectangular cross section, however the penetration of yielding (y_2 and y_3) and the position of the neutral axis (y_{lnp}) are dependent variables of the bending moment (M_z) and the axial stress (N_x) in each instant of the loading process. Since the number of equations is two and the number of unknown variables is three, it is necessary to introduce an additional equation to solve the system. This equation will depend on the geometry of the section, for what an application for the case of a concrete section is carried out.

In this way for the case of a beam of a constant rectangular section of width b and heigh h, with an elastoplastic behaviour of the material (Figure 1(b) the position of the neutral axis will be defined as

$$y_{\text{lnp}} = \frac{y_2 + y_3}{2} \tag{13}$$

Using the equilibrium equation Eq. (11) and Eq. (12) and the position of the neutral axis Eq. (13) we can obtain the penetration of the yielding of the section (y_2 and y_3) in function of the bending moment (Mz) and axial stress (Nx) and behavior of the material (characterized by the yield stress σ_f)

$$y_2 = \frac{N_x + \sqrt{-3N_x^2 - +3b^2\sigma_f{}^2h^2 - 12\sigma_f bM_z}}{2b\sigma_f} \tag{14}$$

$$y_3 = \frac{N_x - \sqrt{-3N_x^2 + 3b^2\sigma_f{}^2h^2 - 12\sigma_f bM_z}}{2b\sigma_f} \tag{15}$$

$$y_{\mathrm{lnp}} = \frac{N_x}{2b\sigma_f} \tag{16}$$

In the different domains of behavior of the section (elastoplastic), the normal stress doesn't follow the law of Navier, as it has been indicated in the Eq. (8), but can be expressed as

$$\begin{array}{ll} h/2 > y > y_2 & \sigma_x = \sigma_f \\ y_2 > y > y_3 & \sigma_x = \dfrac{\sigma_f}{y_2 - y_{lnp}}\left(y - y_{lnp}\right) \\ y_3 > y > -h/2 & \sigma_x = -\sigma_f \end{array} \tag{17}$$

From the position of the neutral axis (y_{lnp}) and of the penetration of the yielding (y_2, y_3) given by the Eqs. (16), (14) and (15) respectively, the normal stress is obtained in the elastic domain of the section with behavior elastoplastic

$$\sigma_x(M_z, N_x) = \frac{\sigma_f\left(2y\sigma_f b + N_x\right)\sqrt{3}}{3\sqrt{-N_x^2 + b^2\sigma y^2 h^2 - 4\sigma_f bM_z}} \tag{18}$$

The distribution of the shear stress on the section with elastoplastic behavior must comply the equations of internal balance. Therefore for the case of a two-dimensional study with null forces of mass we can write:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{19}$$

The variation of the shear stress can be written as:

$$\tau_{xy} = \int_{y_2}^{y} \frac{\partial \sigma_x}{\partial x} dy \tag{20}$$

To determine the distribution of shear stress (τ_{xy}) it is necessary to evaluate the normal stress (σ_x) deriving with respect to X. The normal stress (σ_x) Eq. (18) doesn't depend explicitly on that variable but if does on the bending moment (M_z) and axial stress (N_x). Therefore is possible to write the normal stress derived with respect to X as

$$\frac{\partial \sigma_x (M_z, N_x)}{\partial x} = \frac{\partial \sigma_x}{\partial M_z} \frac{dM_z}{dx} + \frac{\partial \sigma_x}{\partial N_x} \frac{dN_x}{dx} \tag{21}$$

Where, considering only the axial force (N_x) due to a punctual force we can rewrite the Eq. (21) as:

$$\frac{dN_x}{dx} = 0 \qquad \Rightarrow \qquad \frac{\partial \sigma_x (M_z, N_x)}{\partial x} = \frac{\partial \sigma_x}{\partial M_z} \frac{dM_z}{dx} \tag{22}$$

Substituting Eq. (22) in Eq. (18) we get

$$\begin{gathered} \frac{\partial \sigma_x}{\partial M_z} = -\frac{2\sqrt{3}\sigma_f{}^2 b (N_x + 2yb\sigma f)}{\sqrt{\left(-N_x^2 + b^2 \sigma_f{}^2 h^2 - 4\sigma_f b M_z\right)^3}} \\ \frac{dM_z}{dx} = V_y \end{gathered} \tag{23}$$

Therefore the distribution of shear stress in the elastic area of the elastoplastic behavior of the cross section Eq. (20) can be expressed by:

$$\tau_{xyp}(y) = -\frac{\sqrt{3}\sigma_f V_y \left(-4N_x{}^2 + 3b^2 \sigma_f{}^2 h^2 - 12b\sigma_f Mz - 4y^2 b^2 \sigma_f{}^2 + 4N_x yb\sigma f\right)}{\sqrt{\left(-N_x^2 + b^2 \sigma_f{}^2 h^2 - 4\sigma_f b M_z\right)^3}} \tag{24}$$

The variation of the shear stress of the cross section of the Figure 1(b) for each one of its domains is defined by:

$$\begin{array}{ll} h/2 > y > y_2 & \tau_{xy} = 0 \\ y_2 > y > y_3 & \tau_{xy}(y) = \tau_{xyp}(y) \\ y_3 > y > -h/2 & \tau_{xy} = 0 \end{array} \tag{25}$$

The distribution of the shear stress in the elastic domain of the material allows to determine the combination of values of the bending moment (M_z), axial stress (N_x) and shear stress (V_y) that makes the section reach its limit state of complete yielding, and starting from them to obtain the yield surface of the section. This occurs if the maximum value of the shear stress of the elastic area (τ_{xypmax}) is equal to the shear yield stress (τ_f).

The maximum shear stress (τ_{xypmax}) appears in the neutral axis of the yielding section, therefore substituting the position of the neutral axis of the section ($y = y_{lnp}$) Eq. (16) in the law of variation of the shear stress τ_{xyp} (24), we get

$$\tau_{xyp_{max}}(y) = -\frac{\sqrt{3}\sigma_f V_y\left(-3N_x{}^2 + 3b^2\sigma_f{}^2h^2 - 12b\sigma_f Mz\right)}{6\sqrt{\left(-N_x^2 + b^2\sigma_f{}^2h^2 - 4\sigma_f bM_z\right)^3}} \tag{26}$$

As the yielding begins when $\tau_{x\,yp\max} = \tau_f$ it is possible to write:

$$M_z = \frac{\sigma_f bh^2}{4} - \frac{N_x^2}{4b\sigma_f} - \frac{9}{16}\frac{V_y^2}{b\sigma_f(1-D)^2} \tag{27}$$

If the stress that causes the yielding is considered independently, it is possible to write the value of the plastic bending moment (Mp), plastic axial force (Np) and the value of the plastic shear force (Vp) that cause the full yielding of the cross section of the beam.[9,10,11,12] $M_p = \frac{\sigma_f bh^2}{4}$ $V_p = \frac{2\sigma_f bh}{3\sqrt{3}}$ $N_p = \sigma_f bh$. Now, under the hypothesis of strain equivalence [2,22] these expressions are modified to consider the possibility of damage, so we can write $M_p = \frac{\sigma_f bh^2}{4(1-D)}$ $V_p = \frac{2\sigma_f bh}{3\sqrt{3}(1-D)}$ $N_p = \frac{\sigma_f bh}{(1-D)}$. Substituting this formulas in the expression of M_z, we can obtain the yield function (Z_{MNVd}) for the 2D beam element, taking into account the effects of damage of material and also the stress due to axial force, shear force and bending moment. [9]

$$Z_{MNVd} = \frac{|M_z|}{M_p} + \left(\frac{N_x}{N_p}\right)^2 \frac{1}{(1-D)} + \frac{1}{3}\left(\frac{V_y}{V_p}\right)^2 \frac{1}{(1-D)^3} - (1-D) = 0 \tag{28}$$

3. Results and Result Analysis

If we represent the expression $|M_z|/M_p$ graphically in the normalized domain $(0 \le N_x/N_p \le 1)$ and $(0 \le V_y/V_p \le 1)$ for D=0, we obtain the yield function and yield surface without considering the damage of the material.

$$Z_{MNV} = \frac{|M_z|}{M_p} + \left(\frac{N_x}{N_p}\right)^2 + \frac{1}{3}\left(\frac{V_y}{V_p}\right)^2 - 1 = 0 \tag{29}$$

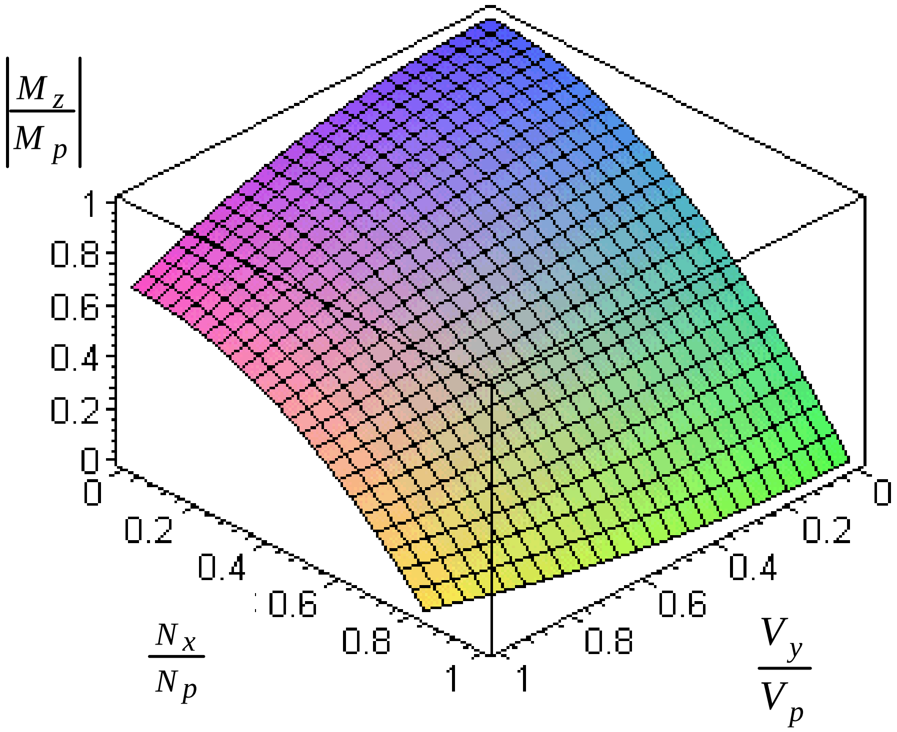

Figure 2. Yield function Z_{MNV} for rectangular cross section

The yield function associated to the rectangular section Eq. (29) has a curve of contour expressed by Eq. (30).

$$\begin{cases} For \quad N_x = 0 \quad d = 0 \quad \Rightarrow \quad Z_{MNV} = Z_{MV} \\ For \quad V_y = 0 \quad d = 0 \quad \Rightarrow \quad Z_{MNV} = Z_{MN} \end{cases} \tag{30}$$

$$Z_{MV} = \frac{|M_z|}{M_p} + \frac{1}{3}\left(\frac{V_y}{V_p}\right)^2 - 1 = 0 \tag{31}$$

$$Z_{MN} = \frac{|M_z|}{M_p} + \left(\frac{N_x}{N_p}\right)^2 - 1 = 0 \tag{32}$$

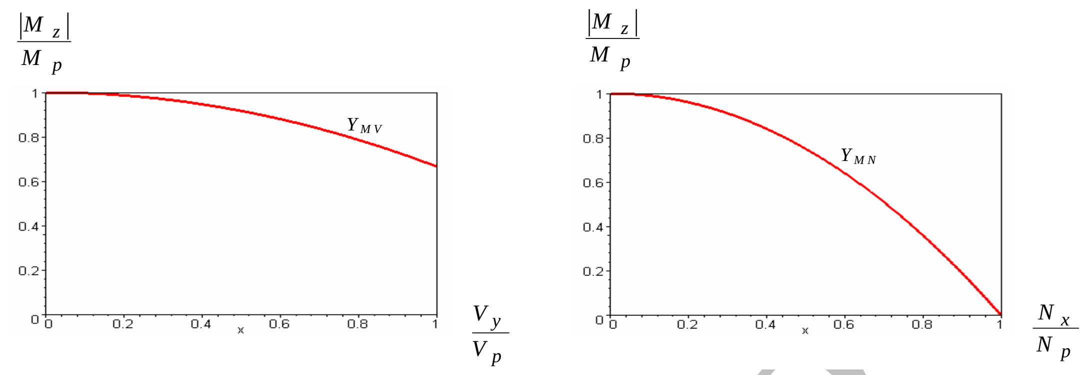

Figure 3. Yield function Z_{MV} and Z_{MN}.

The yield surfaces for a damaged material are shown in Figures 4, 5, 6 and the yield functions associated to the section and each Figure areas follows:

$$\begin{cases} \text{For} \quad N_x = 0 \quad D = 0-1 \quad \Rightarrow \quad Z_{MNVd} = Z_{MVd} \\ For \quad V_y = 0 \quad D = 0-1 \quad \Rightarrow \quad Z_{MNVd} = Z_{MNd} \\ For \quad \mathrm{N_x} = 0 \; V_y = 0 \quad D = 0-1 \quad \Rightarrow \quad Z_{MNVd} = Z_{Md} \end{cases} \tag{33}$$

$$Z_{MVd} = \frac{|M_z|}{M_p} + \frac{1}{3}\left(\frac{V_y}{V_p}\right)^2 \frac{1}{(1-D)^3} - (1-D) = 0 \tag{34}$$

$$Z_{MNd} = \frac{|M_z|}{M_p} + \left(\frac{N_x}{N_p}\right)^2 \frac{1}{(1-D)} - (1-D) = 0 \tag{35}$$

$$Z_{Md} = \frac{|M_z|}{M_p} - (1-D) = 0 \tag{36}$$

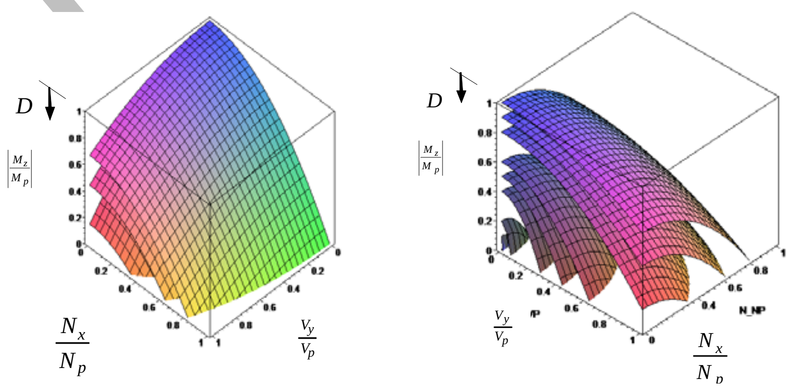

Figure 4. Yield surface for rectangular cross section depending on the stresses and damage Z_{MNVd}

In the Figure 4, there are graphically represented several yield surfaces for ten values of the variable damage D, which varies between 0 and 1. Notice that the yield surface decreases as damage of the cross section increases because the accumulation of the damage of the section implies a decrease of its capacity of load.

The expression (28) can be used for determining the elastoplastic stiffness matrix of the beam element and therefore, it will be possible to consider damage material in the structural analysis of frame i.e. $dF = K^{ep} du^{ep}$ where $d\boldsymbol{F}$ is the stress vector at each beam end, K^{ep} is the elasticplastic degradation stiffness matrix and du^{ep} is the elastoplastic displacement vector at the ends of the element. [9,10]

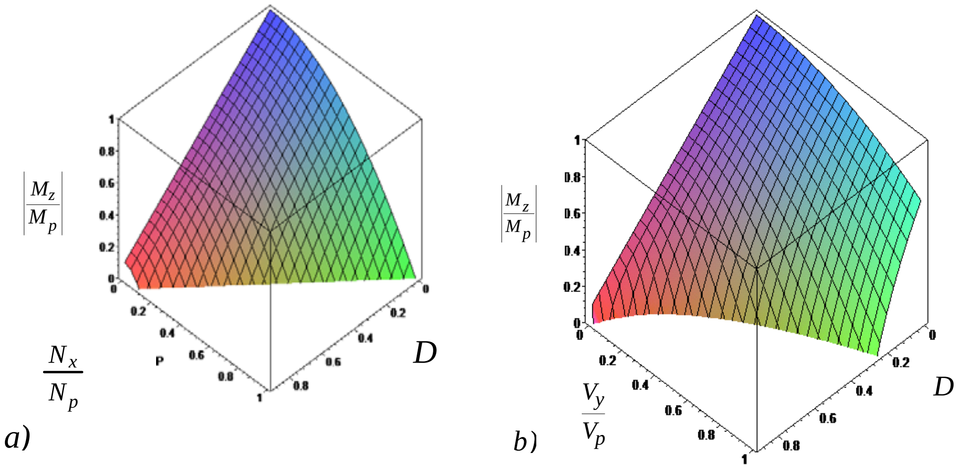

Figure 5. a) Yield surface Z_{MNd}. b) Yield surface Z_{MVd}.

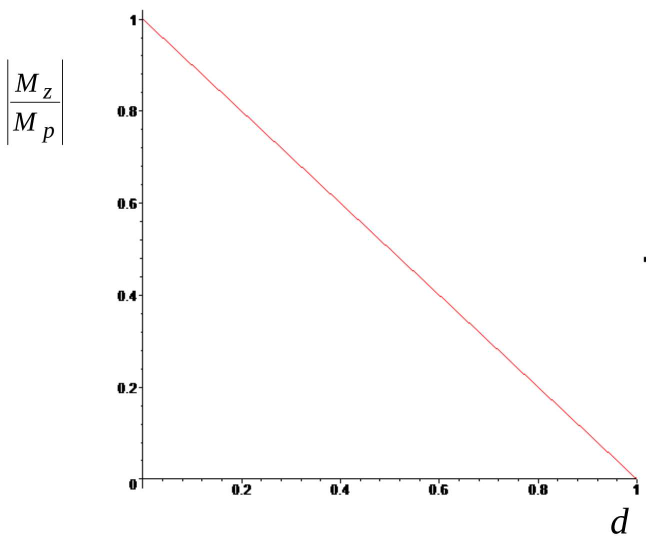

Figure 6. Yield function Z_{Md}

$$dF = K^{ep} du^{ep}$$
$$K^{ep} = K\left[1 - \frac{\frac{\partial Z}{\partial F} K \frac{\partial Z}{\partial F}}{\frac{\partial Z}{\partial F} K \frac{\partial Z}{\partial F} + \frac{\partial Z}{\partial D}\frac{\partial \phi^*}{\partial Y}}\right] \tag{37}$$

Where, K is the tangent stiffness matrix of each beam, $\phi*$ is the damage dissipation potential (that depends on the internal variable associated to damage, the Y-Damage energy release rate and the accumulated effective plastic strain. An incremental and iterative algorithm can be used for the analysis of frame [11,13,14].

4. Conclusions

In this work, the bases for the numeric study of the beam 2D element in regime elastoplastic have been presented, considering the different stresses that can take part in the yielding of each cross section and the damage that it the material can suffer. The variable damage was introduced considering the theory of the Continuous Damage Mechanics, specifically the principle of equivalent deformation formulated by Lemaitre. The yield function considered here can be used to obtain the degradable elastoplastic stiffness matrix of the 2D beam element.

Depending on the stresses considered in the study of the yielding of sections different yield surfaces are showed. When the degradation of the material is considered, there is a yield surface for each value of the variable damage that defines the combination of the stresses that cause the yielding of the cross section of the beam (Figure 4). The existence of damage modifies the characteristics of the rigidity of the beam causing a smaller load capacity of the structure in general.

References

1. Timoshenko S, Goodier JN. *Theory of Elasticity*, Mc Graw-Hill , New York, 1951.
2. García-Terán JM. Formulación y análisis del comportamiento elasto-plástico acoplado y no lineal geométrico de sistemas estructurales de barras, Tesis Doctoral, Dpto. Resistencia de Materiales y Estructuras, Universidad de Valladolid, Valladolid, España 2002.
3. Lemaitre J. *A Course in Damage Mechanics*, Springer, New York, 1992.
4. Rabotnov Y. Creep Rupture in: Applied Mechanics, *Proceedings of the 12th Internacional congress of Applied Mechanics*, Stranford Springer-Verlag, Bberlin, 1968, pp. 342-49.
5. Krenk S, Vissing-J C, Thesbjerg L. Efficient collapse analysis techniques for framed structures, *Computers and Structures*, No. 4,**72**(1999)81-96.
6. Neal BG. *The Plastic Methods of Structural Analysis*, Science Paperbacks, 1985.

7. Olsen PC. Rigid plastic analysis of plane frame structures, *Computer Methods in Applied Mechanics and Engineering*, **179**(1999)19-30.
8. Ibijola EA. On some fundamental concepts of Continuum damage mechanics, *Computer Methods in Applied Mechanics and Engineering*, **191**(2002)1505-20.
9. Chica E, Teran JMG, Iban AL. Yield surface for elastoplastic beam 2D element considering damage material, *8th World Congress on Computational Mechanics (WCCM8), 5th European Congress on Computational Methods in Applied Sciences and Engineering (Eccomas 2008)*,Venice, Italy, 2008.
10. Chica E, Teran JMG, Iban AL. Lopez P. Analysis of frame involving damage material, geometrical and materal nonlinearities, *Fourth International Conference on Advanced Computational Methods in Engineering*, University of Liege, Liege, Belgium, 2008.
11. Bonora N. A non-linear CDM damage model for ductile failure, *Engineering Fracture Mechanics*, **58**(1997)11-28.
12. Ibijola EA. On some fundamental concepts of continuum damage mechanics, *Computer Methods in Applied Mechanics and Engineering*, **191**(2002)1505-20.
13. Yingchun X. A multi-mechanism damage coupling model, *International Journal of Fatigue*, **26**(2004)1241-50.
14. Faleiro J, Oller S, Barbat A. Plastic-damage seismic model for reinforced concrete frames, *Computers and Structures*, Nos. 7-8, 86(2008)581-97.