ASIAN JOURNAL OF CIVIL ENGINEERING (BUILDING AND HOUSING) VOL. 13, NO. 2 (2012)
PAGES 257-273

A NEW BLOCK TRI-DIAGONAL STIFFNESS MATRIX FOR EFFICIENT FINITE ELEMENT ANALYSIS OF STRUCTURES

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Received: 5 April 2011; **Accepted:** 17 September 2011

ABSTRACT

In this paper, a new canonical form is introduced for efficient analysis of structures with special geometric properties. Using the properties of this matrix, the number of operations needed for the matrix inversion is considerably reduced employing the decomposition of the block stiffness matrices. The condition for applicability of the presented method is also discussed. For the previously developed canonical forms, the Kronecker products and the corresponding theorems could be used for certain class of repeated structures. Here this class is extended to the stiffness matrices having more general block tri-diagonal form where the diagonal blocks are not necessarily identical, requiring a different treatment. Two examples of finite element models are analyzed to illustrate the efficiency of the presented method.

Keywords: Canonical forms; block tri-diagonal matrix; regular structures; finite element models; stiffness matrix; matrix inversion

1. INTRODUCTION

In the previous researches, different canonical forms for the stiffness matrices have been studied and methods were presented for efficient eigensolution of such matrices [1-3]. Having the eigenvalues the calculation of the inverse becomes feasible. In the previous studies, the matrices had repeated patterns and we could express them in the form of the sum of the Kronecker product of some matrices.

These relations can be used for the analysis of structures which are modular. As an example of these structures one can refer to shell structures formed from many finite elements. Analysis of such structures can be time consuming and requires considerable storage for mathematical operations. Another example of such structures is plate structures.

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In this paper we use mathematical approaches for the analysis of plates having high number of elements. The simplification is achieved by operating on small matrices in place of large matrices of the plate structures. For this analysis we use finite element method which is briefly described in the following section. In Section 2 the geometric model of the plates and the characteristics of the stiffness matrices are described. Then the basic concepts, shape functions, degrees of freedom and at the end the stiffness matrix are presented. The formation of the stiffness matrix considering the geometric properties and nodal numbering to obtain the special pattern are the important parts of this section.

Mathematical methods for inverting block diagonal matrices are then presented and by comparing the matrices involved in these methods with the matrices of the plates studied in this paper, simple equations are derived for the analysis of the plates. It should be mentioned that in these operations one tries to use the geometry to form stiffness matrices of special form. Using these properties the number of operations for inverting the stiffness matrices is reduced. Based on the presented method using other special geometric forms, one can obtain different canonical forms and using these forms the equations for simple inversion of the matrices can be obtained. Finally these relationships are used for the efficient analysis of space trusses.

In brief it should be mentioned that the present method is applicable to those structures for which the stiffness matrix in block tri-diagonal, however, these blocks need not be identical.

2. ANALYSIS OF PLATES BY FINITE ELEMENT METHOD

Plate elements are the most well-known plane elements having both rotational and translational degrees of freedom (DOFs). These elements are categorized as plate and shell elements. The main difference of these elements is in the number of their DOFs, resulting in difference performance under identical loading. Plates show bending behavior and the only out of plane action is defined by the displacement vertical to the plane of the plate, while the membranes have in-plane behavior and have no stiffness under the out of plane loading. In this study only the behavior of plates will be studied.

Unlike the finite difference method which uses the differential equation of the deformation of the plate, the numerical methods employ the interpolation functions for relating the deformation of each point of the plate to the nodal displacements of the model. Therefore having these functions, one can easily obtain all the characteristics of the elements and structures such as mass matrix, stiffness matrix, load vector etc.

The main aim of this paper is to present an efficient approach for matrix inversion and plate elements described are only employed as examples. Here, the assumptions and the steps of the analytical characteristics of a plate element are briefly description, and for details the reader may refer to Refs. [4-6].

2.1 MZC and BFS plate elements

As mentioned above the plates have both bending and out-of-plane stiffness. This means that for each element in each node at least 3 degrees of freedom (two rotational and one vertical displacement) should be considered.

The element MZC is developed by Melosh-Zienkiewicz-Cheung is a non-conforming bending element for plates. This is because in bending the normal slope for this element is not compatible. The general form of this element is a trapezoidal element as shown in Figure 1. The element forces and displacements are as follows:

$$q_i = \{q_{i1}, q_{i2}, q_{i3}\} = \left\{w_i, \frac{\partial w_i}{\partial y}, \frac{\partial w_i}{\partial x}\right\} \quad , \quad i = 1:4 \tag{1}$$

$$P_i = \{p_{i1}, p_{i2}, p_{i3}\} = \{w_{zi}, M_{xi}, M_{yi}\} \; , \;\; i = 1:4 \tag{2}$$

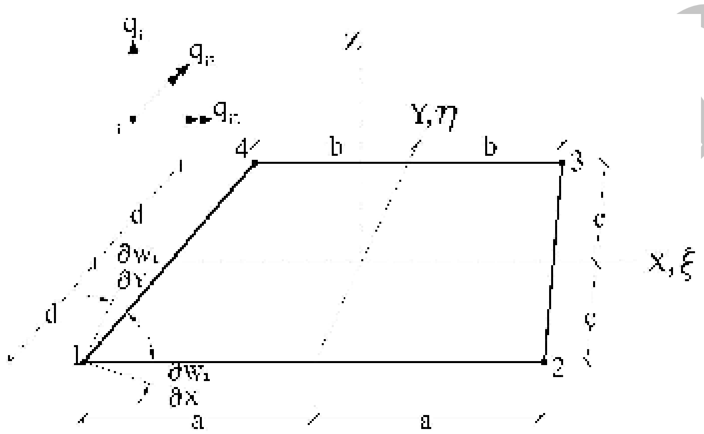


Figure 1. Trapezoidal element of MZC

Choosing a complete polynomial function of order 3, the shape functions of a rectangular element can be obtained. Obviously for any other four-sided element like trapezoidal element, the integration should be performed in the lengths corresponding to the element. The shape functions are defined as follows:

$$f_i = \{f_{i1}, f_{i2}, f_{i3}\} \quad , \quad i = 1:4$$

$$f_{i1} = \frac{1}{8}(1+\xi_0)(1+\eta_0)\left(2+\xi_0+\eta_0-\xi^2-\eta^2\right) \quad , \quad i = 1:4$$

$$f_{i2} = -\frac{1}{8} b\eta_i (1+\xi_0)(1-\eta_0)(1+\eta_0)^2 \quad , \quad i = 1:4$$

$$f_{i3} = \frac{1}{8} ax_i (1-\xi_0)(1+\eta_0)(1+\xi_0)^2 \quad , \quad i = 1:4$$

$$\xi_0 = \xi_i \xi \qquad \eta_0 = \eta_i \eta \quad , \quad i = 1:4$$

Considering the definition of the general linear operator $\overline{d}$ which is given by Eq. (3) the strain-displacement of B matrix can be obtained as provided by Eq. (4).

$$\overline{d} = \left\{ \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{2\partial^2}{\partial x \partial y} \right\} \tag{3}$$

$$\overline{B}_i = \overline{d}\, f_i = \begin{bmatrix} f_{i1,xx} & f_{i2,xx} & f_{i3,xx} \\ f_{i1,yy} & f_{i2,yy} & f_{i3,yy} \\ 2f_{i1,xy} & 2f_{i2,xy} & 2f_{i3,xy} \end{bmatrix} \tag{4}$$

Based on this matrix one can find the stiffness matrix and the nodal forces of the element using Eq. (5) and Eq. (6), respectively.

$$K = \int_A \overline{B}^T . \overline{E} . \overline{B}\, dA \tag{5}$$

$$P_0 = \int_A \overline{B}^T . \overline{E} . j_0\, dA \tag{6}$$

In these relationships j_0 is the initial strain. After analysis one can find the initial stress of the elements.

The BFS element was developed by Bogner-Fox-Schmit. In this element apart from the DOFs of MZC some warping is also included. In this case, the slopes of the edges in two directions are compatible. The displacements and the DOFs of this element are illustrated in Figure 2. According to the definition, the nodal displacements can be expressed as in Eq. (7).

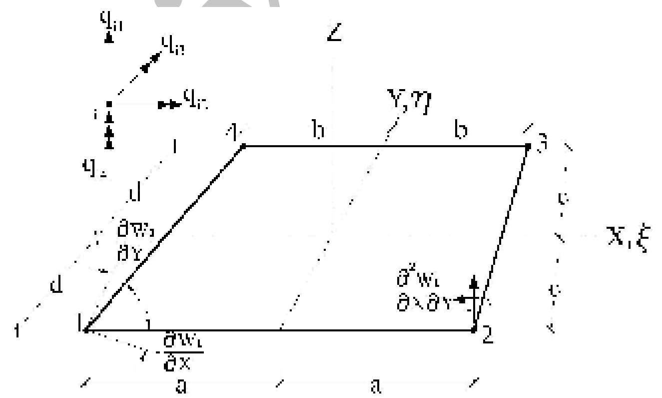


Figure 2. Trapezoidal element of BFS

$$q_i = \{q_{i1}, q_{i2}, q_{i3}, q_{i4}\} = \left\{ w_i, \frac{\partial w_i}{\partial y}, \frac{\partial w_i}{\partial x}, \frac{\partial^2 w_i}{\partial x \partial y} \right\} \quad , \quad i = 1:4 \tag{7}$$

For this element the displacemnet function should be selected such that apart from 3 nodal displacements, linear variation and second order vertical slope at the edges in both direction should also be fulfilled. Therefore, apart from a complete polynomial of order 3 with 10 terms, 6 additional terms should be considered as the following:

$$\begin{aligned} w = a_1x^3y + a_2x + a_3y + a_4x^2 + a_5y^2 + a_6xy + a_7x^3 + a_8y^3 + a_9x^2y + \ldots \\ \ldots a_{10}xy^2 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}x^3y^2 + a_{14}xy^3 + a_{15}x^2y^3 + a_{16}x^3y^3 \end{aligned} \quad (8)$$

In fact this function is an incomplete polynomial of order 4. Utilizing this function one can find the shape functions of the element, or using Eqs. (5 and 6), the stiffness matrix and the load vector can be found and the analysis can be performed.

As it can be seen from the number of DOFs of the elements, one can find the stiffness matrix of each MZC element as 12×12 and for the BFS element this matrix is 16×16 .

2.2 Analysis of plates with regular mesh

In order to increase the accuracy of the analysis of plates, usually the elements are refined to smaller elements of different shapes. Since the analysis of these models is performed in an identical manner, thus one can analyze the entire model simpler using smaller number of elements. In other words instead of a plate with 3n or 4n degrees of freedom (n being the number of nodes) one can analyse the plate with a fraction of these DOFs as many times as needed depending on the geometry of the model. The method utilized in this section can be used for the analysis of all those plates which fulfill the following two requirements:

1. The elements constituting the entire nodel should be decomposible in two direction and all the elements along each axis should be identical. This means that the graph corresponding to the finite element model should be in the form of the Cartesian product of two paths P and P.
2. The boundary conditions in each edge should be in a continuous form for all the nodes.

These two conditions permit the selection of a repeated row in the model and then order the nodes row by row for the entire model. Obviously the stiffness matrix of this model will be a block tri-diagonal matrix.

In Ref. [4] the detrminants of such matrices are calculated. In Ref. [5] an iterative algorithm is provided for block tri-diagonal matrices and using the constituting blocks, the matrix is inverted. Here we use an LU factorization.

For investgation of the model and the way it works, a plate is considered as shown in Figure 3. This plate is decretized into trapezoidal elements.

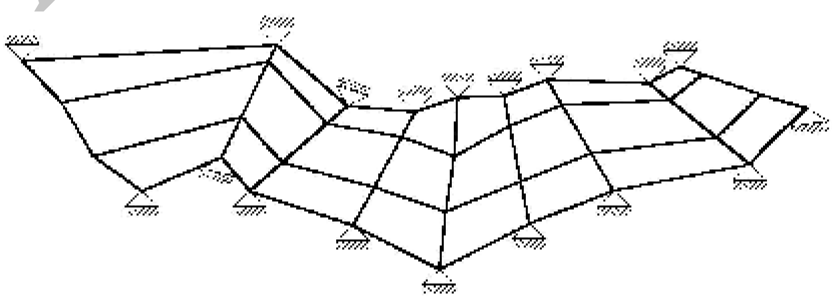


Figure 3. Trapezoidal elements with regular pattern

As can be seen the bands in both directions are obvious, however, there is no similarity among the elements from positioning and the geometry point of view. The supports are in the lognitudnal edges in a continuous manner. Considering the element MZC described previously, the stiffness matrix of the plate can be obtained in the form of Eq. (9). In this example we have want to illustrate the general canonical form of the stiffness matrix, while as will be described in the following, special cases will also be inverstigated.

$$K = \begin{bmatrix} K_1 & K_{10}^T & 0 & 0 & 0 \\ K_{10} & K_2 & . & 0 & 0 \\ 0 & . & . & K_{16}^T & 0 \\ 0 & 0 & K_{16} & K_8 & K_{17}^T \\ 0 & 0 & 0 & K_{17} & K_9 \end{bmatrix} \tag{9}$$

The stiffness matrix of the plate for the free DOFs is a 90×90 matrix, and in this example it is a block tri-diagonal matrix. Each block is a 10×10 matrix. Thus one can perform all the opertions with 10×10 matrices which is less time consuming compared to operting with the 90×90 matrix.

Now acoording to what is explained the coming section, for the models with regular geometry the following pattern can be obtained for the stiffness matrices:

$$K = \begin{bmatrix} B_1 & -A^T & 0 & 0 & 0 \\ -A & B_2 & -A^T & 0 & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & -A & B_2 & -A^T \\ 0 & 0 & 0 & -A & B_3 \end{bmatrix} \tag{10}$$

In this case, the above matrix can be decomposed into the product of two matrices [L] and [U] as shown in the following:

$$K = L_j U_j = \begin{bmatrix} B_1 & & & & & & & & \\ -L_1 & H_2 & & & & & 0 & & \\ & . & . & & & & & & \\ & & . & . & & & & & \\ & & & . & . & & & & \\ & & & -L_1 & H_j & -L_2 & & & \\ & & & & & . & . & & \\ & & & & & & . & . & \\ & 0 & & & & & & H_{k-1} & -L_2 \\ & & & & & & & & B_3 \end{bmatrix} \times \begin{bmatrix} I & -U_1 & & & & & & & \\ & . & . & & & & & 0 & \\ & & . & . & & & & & \\ & & & . & . & & & & \\ & & & & I & -U_{j-1} & & & \\ & & & & & I & & & \\ & & & & & -U_j & I & & \\ & & & & & & . & . & \\ & 0 & & & & & & . & . \\ & & & & & & & . & . \\ & & & & & & & -U_{k-1} & I \end{bmatrix} \tag{11}$$

In this relation [L] and [U] are obtained using the following iterative approach:

$$i < j\,; L_1 = A, H_1 = B_1, \begin{cases} U_i = H_1^{-1} B_1 (i = 1) \\ U_i = H_i^{-1}\left((A')\right),\ H_i = B_2 - L_1 U_{i-1} (i = 2 : j-1) \end{cases}$$

$$i > j\,; L_2 = A', H_K = B_3, U_i = H_{i+1}^{-1} A, H_i = B_2 - L_2 U_i (i = k-1 : j+1)$$

$$i = j\,; L_1 = A, U_i = H_{i+1}^{-1} A, \begin{cases} H_i = B_1 - L_2 U_i \quad , i = 1 \\ H_i = B_2 - L_1 U_{i-1} - L_2 U_i \quad , 1 < i < K \\ H_i = B_3 - L_1 U_{i-1} \quad , i = K \end{cases}$$

Now using [U] and [H] uutilizing Eq. (13) the inverse of the stiffness matrix can be obtained.

$$\begin{cases} K_{jj}^{-1} = H_j^{-1} \\ K_{ij}^{-1} = U_i K_{i+1\,j}^{-1} (i = j-1 : 1) \\ K_{ij}^{-1} = U_{i-1} K_{i-1\,j}^{-1} (i = j+1 : k) \end{cases} \tag{13}$$

As it can be seen all the operations are performed on the blocks matrices. Only 4 fidderent block exist in the main stiffness matrix. And the operations are performed on three blocks. Atb the end having the inverse matrix, the complete analysis of the plate will be performed and all the displacements can be obtained. An interesting point is that in this method one does not need to have a symmetric loading and any general loading can be handled. In what follows the approach for transforming the general matrix of Eq. (9) to the stiffness matrix of Eq. (10) will be discussed.

4. SPECIAL CONDITIONS FOR THE FINITE ELEMENT ANALYSIS OF PLATES WITH REGULAR GEOMETRY

As mentioned in the previous section there is no restriction in positioning orthe shape of the elements. If special conditiond hold fot the geometry of the plate, the relations can be simplified and the time and storage required will be reduced, and the special form of the stiffness matrix will be attained.

(a) The most common descretiztion of a plate for its analysis is when the elements form some bands as shown in Figure 4.

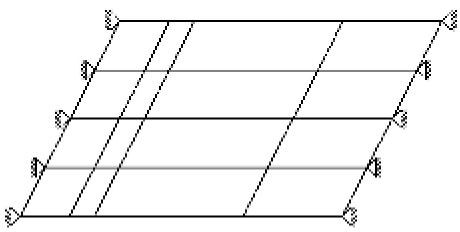


Figure 4. Discretiztion of a plate with regular geometry having similar bands in each row

As it can be seen, the trapeziodal elements are in both vertical and horizontal dirctions and as an example all the elements in vertical direction are completely identical. In such a case the stiffness matrix will be in the form of Eq. (14). As it is obvious from this relationship due to the repetition in each row one can observe similar K2 matrices on the main diagonal. In such a case using the relationships for inverting the matrix , the inversion of some blocks can be avoided.

$$K = \begin{bmatrix} K_1 & K_4^T & 0 & 0 & 0 \\ K_4 & K_2 & K_5^T & 0 & 0 \\ 0 & K_5 & K_2 & K_5^T & 0 \\ 0 & 0 & K_5 & K_2 & K_6^T \\ 0 & 0 & 0 & K_6 & K_3 \end{bmatrix} \tag{14}$$

(b) the second case occurs when all the constituting elements of the plate are identical as shown in Figure 5. In this case the stiffness matrix will be in the form of Eq. (15).

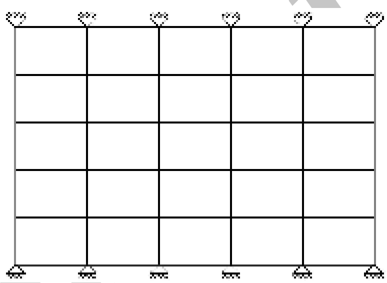


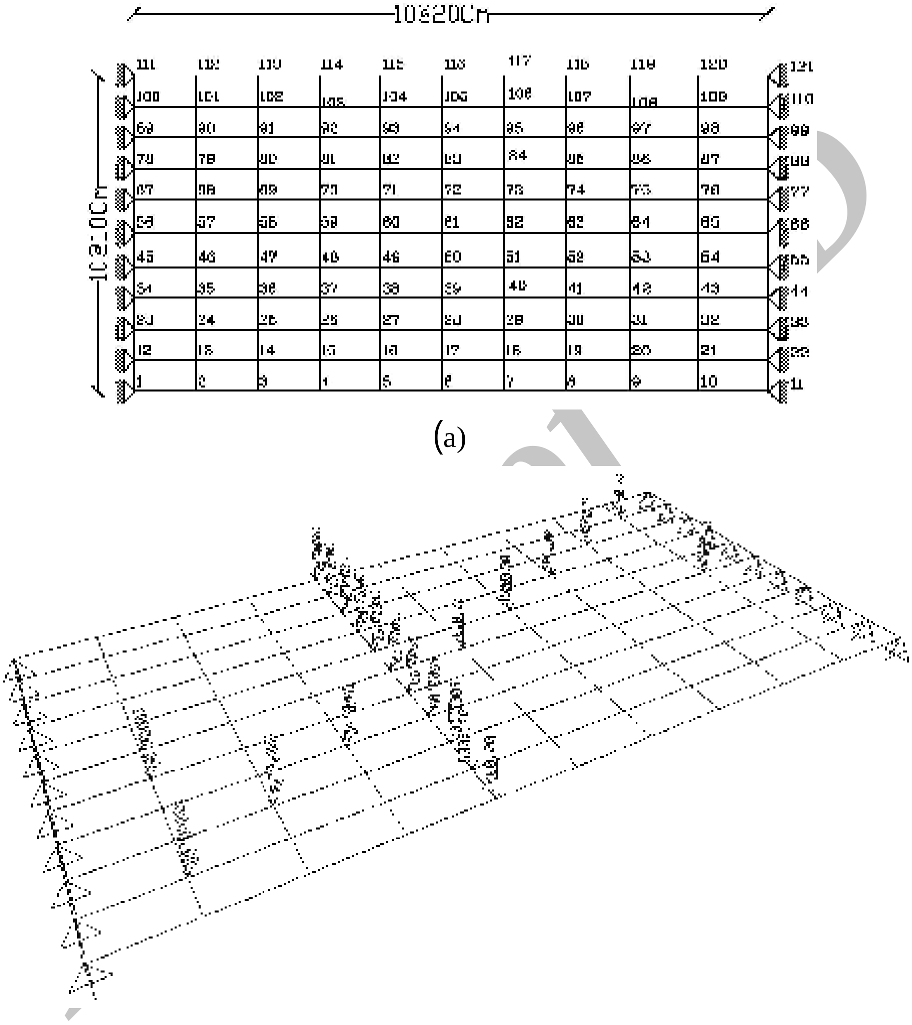
Figure 5. A finite element model wit regular geometry and identical elements

$$K = \begin{bmatrix} K_1 & K_4^T & 0 & 0 & 0 \\ K_4 & K_2 & K_4^T & 0 & 0 \\ 0 & K_4 & K_2 & K_4^T & 0 \\ 0 & 0 & K_4 & K_2 & K_4^T \\ 0 & 0 & 0 & K_4 & K_3 \end{bmatrix} \tag{15}$$

In fact all the relations in Eq. (12) can be calculated based on 4 above matrices, and this reduces the number of operation for inverting the stifness matrix drastically. On the other hand it avoids working with large scale matrices and uses far less computational storage. In the following an example is provided to illustarte the use of the presented method.

5. A SOLVED EXAMPLE OF PLATE USING THE SIMPLIFIED INVERTING METHOD

In this section the analysis of a steel plate is explained using the present method. For avoiding the complexities of the stiffness matrix, simple rectangular elements are utilized



(a)

(b)

Figure 6. (a) The geometric model of a plate with BFS (b) Loading of the plate

A steel plate of 2.00m by 1.00m with thickness of 0.03m is considered as shown in Figure 6(a). This plate has a loading as shown in Figure 6(b) and magnitudes provided in Table 1. The modulus of elasticity of the plate is taken as $1.99*10^{11}N/m^2$. For the analysis BFS elements are used due to its high accuracy. Descretization of the plate is 10×20. The nodal numbering is shown on each element.

Table 1: Nodal loading of the plate

Node	**5**	**13**	**16**	**25**	**27**	**37**	**38**	**46**	**49**	**60**	**61**	**65**	**71**	**73**	**82**	**85**	**93**	**96**	**104**	**109**	**115**
F_z(kN)	5	10	5	12	5	5	5	20	5	20	7	20	20	15	20	9	20	10	20	20	20

For the analysis of this plate first we form its stiffness matrix. Having 121 nodes for this plate each having 4 DOFs, the stiffness matrix will be a 484×484 matrix. After imposing the boundary conditions this matrix becomes 462×462 . Thus according to the considered nodal numbering, the stiffness matrix will be obtained in the following form:

$$K_{462\times 462} = \frac{Et^3}{12(1-n^2)} \begin{bmatrix} B_1 & A^T & 0 & 0 & 0 & 0 \\ A & B_2 & A^T & 0 & 0 & 0 \\ 0 & A & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & B_2 & A^T \\ 0 & 0 & 0 & 0 & A & B_3 \end{bmatrix}_{11\times 11}$$

As it can be observed the dimensions of the blocks are 42×42. Here part of these 4 matrices are illustrated.

$$B_1 = 10^6 \begin{bmatrix} 1.55 & -0.84 & 0.045 & 5.73 & . & . & 0 & 0 & 0 & 0 \\ -0.84 & 1.13 & -0.034 & . & . & . & . & 0 & 0 & 0 \\ 0.045 & -0.034 & . & . & . & . & . & . & 0 & 0 \\ 5.73 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & -0.001 \\ 0 & 0 & . & . & . & . & . & . & 0.84 & -0.045 \\ 0 & 0 & 0 & . & . & . & . & 0.84 & 1.13 & -0.034 \\ 0 & 0 & 0 & 0 & . & . & -0.001 & -0.045 & -0.034 & 0.002 \end{bmatrix}_{42\times 42}$$

$$B_2 = 10^6 \begin{bmatrix} 3.10 & 0 & 0.09 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 2.26 & 0 & . & . & . & . & 0 & 0 & 0 \\ 0.09 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & -0.002 \\ 0 & 0 & . & . & . & . & . & . & 0 & -0.09 \\ 0 & 0 & 0 & . & . & . & . & 0 & 2.26 & 0 \\ 0 & 0 & 0 & 0 & . & . & -0.002 & -0.04 & 0 & 0.004 \end{bmatrix}_{42\times 42}$$

$$B_3 = 10^6 \begin{bmatrix} 1.55 & 0.84 & 0.045 & -5.73 & . & . & 0 & 0 & 0 & 0 \\ 0.84 & 1.13 & 0.034 & . & . & . & . & 0 & 0 & 0 \\ 0.045 & 0.034 & . & . & . & . & . & . & 0 & 0 \\ -5.73 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & -0.001 \\ 0 & 0 & . & . & . & . & . & . & -0.84 & -0.045 \\ 0 & 0 & 0 & . & . & . & . & -0.84 & 1.13 & 0.034 \\ 0 & 0 & 0 & 0 & . & . & -0.001 & -0.045 & 4.19 & 0.002 \end{bmatrix}_{42\times42}$$

$$A = 10^6 \begin{bmatrix} -0.71 & 0.62 & -0.019 & -7.23 & . & . & 0 & 0 & 0 & 0 \\ -0.62 & 0.64 & -0.022 & . & . & . & . & 0 & 0 & 0 \\ -0.019 & 0.022 & . & . & . & . & . & . & 0 & 0 \\ 7.23 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0.0006 \\ 0 & 0 & . & . & . & . & . & . & -0.62 & 0.019 \\ 0 & 0 & 0 & . & . & . & . & 0.62 & 0.64 & -0.022 \\ 0 & 0 & 0 & 0 & . & . & 0.0006 & 0.019 & 0.022 & -0.0006 \end{bmatrix}_{42\times42}$$

It can be seen that in this example according to Eq. (10) we at at most four different matrices, and thus one can use Eq. (12) to obtain [*L*], [*U*] and [*H*]. As an example, after performing the required operqtions, the matrices [U_{k-1}]and [H_k] in the last step (j=k=11) are illustrated.

$$U_{10} = \begin{bmatrix} -0.264 & -0.115 & -0.0057 & -0.68 & . & . & 0 & 0 & 0 & 0 \\ 0.305 & 0.304 & 0.0114 & . & . & . & . & 0 & 0 & 0 \\ 2.60 & -1.21 & . & . & . & . & . & . & 0 & 0 \\ -0.0068 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0.040 \\ 0 & 0 & . & . & . & . & . & . & 0.115 & 0.0057 \\ 0 & 0 & 0 & . & . & . & . & -0.305 & 0.304 & 0.0114 \\ 0 & 0 & 0 & 0 & . & . & 0.058 & -2.60 & -1.21 & 0.070 \end{bmatrix}_{42\times42}$$

$$H_{11} = 10^6 \begin{bmatrix} 1.15 & 0.46 & 3.12 & 2.39 & . & . & 2.06e\text{-}6 & -1.6e\text{-}5 & 0 & 0 \\ 0.46 & 0.74 & 0.021 & . & . & . & . & 0 & 0 & 0 \\ 0.031 & 0.021 & . & . & . & . & . & . & 0 & 0 \\ 0.029 & . & . & . & . & . & . & . & . & \text{-}1.5E\text{-}6 \\ . & . & . & . & . & . & . & . & . & . \\ 2.06E\text{-}6 & . & . & . & . & . & . & . & . & \text{-}8.60 \\ \text{-}1.61E\text{-}5 & 0 & . & . & . & . & . & . & \text{-}0.46 & \text{-}3.82 \\ 0 & 0 & 0 & . & . & . & . & \text{-}0.46 & 0.73 & 2.47 \\ 0 & 0 & 0 & 0 & . & . & \text{-}7E\text{-}3 & \text{-}0.031 & 0.020 & 0.0001 \end{bmatrix}_{42\times42}$$

Having the [*H*] and [*U*] matrices and using Eq. (13), the inverse of [*K*] can be calculated.

At the end having the equation of force, the displacements of the structure can be calculated. For simplicity the displacements of the nodes 5 to 115 are shown in Table 2. The dimension of the length is meter and that of rotation is Radian.

Table 2: The displacements of the fifth column

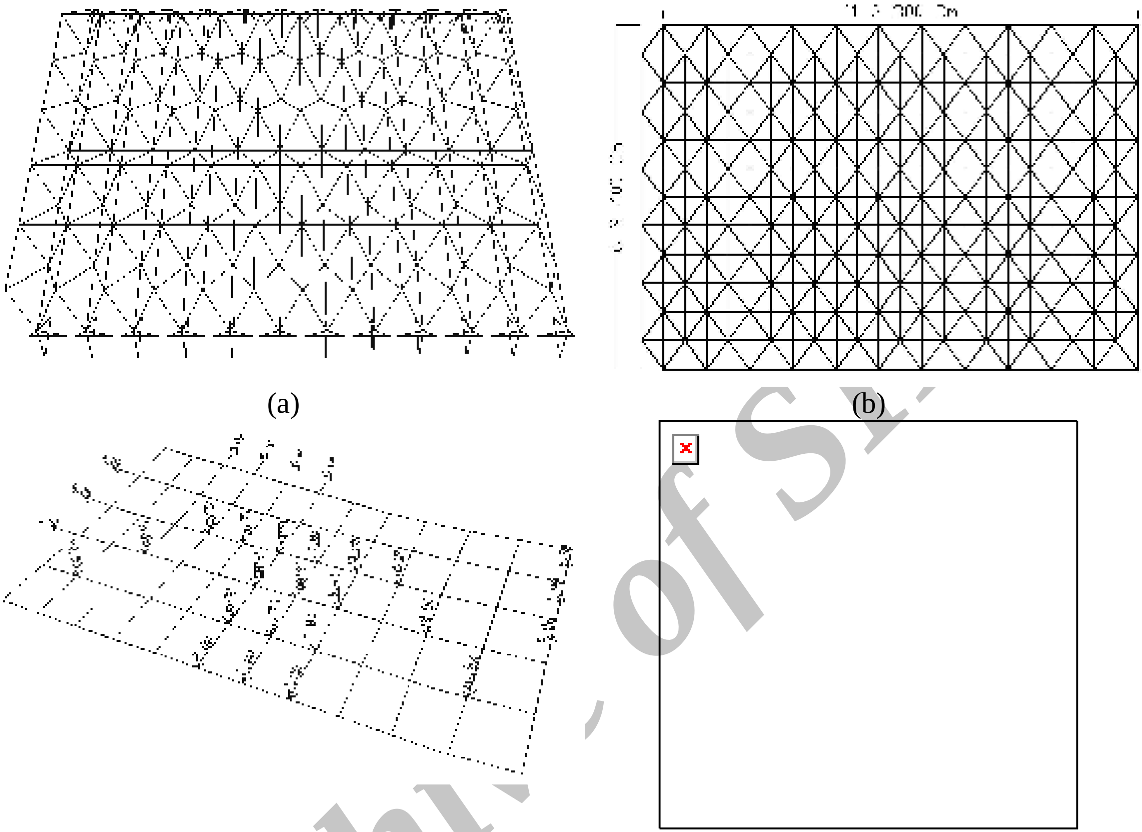
Node / Defl.	5	16	27	38	49	60	71	82	93	104	115
w	-0.069	-0.068	-0.068	-0.069	-0.069	-0.070	-0.072	-0.074	-0.076	-0.078	-0.082
$\frac{\partial w}{\partial x}$	0.010	0.004	-0.001	-0.005	-0.009	-0.013	-0.016	-0.019	-0.023	-0.028	-0.035
$\frac{\partial w}{\partial y}$	0.030	0.030	0.030	0.031	0.031	0.032	0.033	0.034	0.035	0.036	0.038
$\frac{\partial^2 w}{\partial x \partial y}$	0.00	-0.002	-0.004	-0.006	-0.008	-0.009	-0.009	-0.010	-0.011	-0.013	-0.015

As it can be observed, for the analysis of this plate using the present method instead of inverting a matrix of dimension 462×462 matrix, we need only the inverse of matrices of dimension 42×42.

6. AN EXAMPLE OF 3D TRUSS ANAYIZED BY THE SIMPLE INVERTING METHOD

In the following the capability of the present method in efficient analysis of 3D trusses is illustrated. This model consists of two horizonal planes connected to each other by inclined members. The horizontal planes has moved with respect to each other with 2.00m in X-direction and 1.50m in the vertical direction. The height of two planes is 2.00 m. The lower plane has 6 bays each being 4.00m in the X-direction and 11 bays of 3.00m in the Y-direction. The upper plane is a similar one, with the only difference of having 5 bays in the

X-direction. The geometrical properties of the members are provided in Table 3 and the loading is as in Table 4.



(a) (b) (c) (d)

Figure 7. (a) Three dimensional model of the structure (b) The span lengths in the plan (c) Loading at the nodes of the upper horizontal plane (d) Nodal numbering for the two sectors of the structure

The elsatic modulus of the material is $2*10^{11}$ N/m^2. The structure is supported in a pinned form along the two rows parallel to the Y-axis.

Table 3: The sections used in the structure of Figure 7

Properties Element	**A(m^2) E-4**	**I_x(m^4) E-8**	**I_y(m^4) E-8**	**J(m^4) E-8**
Bot plane, X Dir.	23.90	1317.00	101.00	4.73
Top plane, X Dir.	13.20	318.00	27.70	1.69
Bot. & top plane, Y Dir.	20.36	604.40	62.60	5.37
Diagonals	11.02	105.80	19.40	2.00

Table 4: Loading of the structure of Figure 7

Node **Loading**	**63**	**73**	**74**	**75**	**76**	**85**	**86**	**87**	**88**	**89**	**98**	**102**	**111**	**115**	**124**	**127**	**139**	**153**	**154**	**155**
Load (kN)	-10	-2	-15	-15	-10	4	-2	-15	-15	-10	4	-10	4	-10	4	-10	-10	8	8	8
Direction	Z	X	Z	Z	Z	X	X	Z	Z	Z	X	Z	X	Z	X	Z	Z	Y	Y	Y

For the analysis of this structure first we form its stiffness matrix. Having 156 nodes for this plate each having 6 DOFs, the stiffness matrix will be a 936×936 matrix. After imposing the boundary conditions this matrix becomes 864×864. Thus according to the nodal numbering shown in Figure 7(d), the stiffness matrix will be obtained in the following form.

$$K_{864\times864}=\begin{bmatrix} B_1 & A^T & 0 & 0 & 0 & 0 \\ A & B_2 & A^T & 0 & 0 & 0 \\ 0 & A & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & B_2 & A^T \\ 0 & 0 & 0 & 0 & A & B_1 \end{bmatrix}_{11\times11}$$

As it can be seen from the dimensions of this matrix, the blocks are 72×72 matrices. Here parts of these four matrices are provided.

$$B_1=10^4\begin{bmatrix} 30.55 & -1.40 & -12.67 & 0 & . & . & 0 & 0 & 0 & 0 \\ -1.40 & 24.13 & -1.40 & . & . & . & . & 0 & 0 & 0 \\ -12.67 & -1.40 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & 1.40 & 12.67 \\ 0 & 0 & 0 & . & . & . & . & 1.40 & 24.13 & -1.40 \\ 0 & 0 & 0 & 0 & . & . & 0 & 12.67 & -1.40 & 438.34 \end{bmatrix}_{72\times72}$$

$$B_2 = 10^4 \begin{bmatrix} 61.01 & 0 & -25.33 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 28.06 & 0 & . & . & . & . & 0 & 0 & 0 \\ -25.33 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & 0 & 25.33 \\ 0 & 0 & 0 & . & . & . & . & 0 & 28.06 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 25.33 & 0 & 613.3 \end{bmatrix}_{72\times 72}$$

$$B_3 = 10^4 \begin{bmatrix} 44.3 & 0 & -25.33 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 27.93 & 0 & . & . & . & . & 0 & 0 & 0 \\ -25.33 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & 0 & 25.33 \\ 0 & 0 & 0 & . & . & . & . & 0 & 27.93 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 25.33 & 0 & 452.11 \end{bmatrix}_{72\times 72}$$

$$A = 10^4 \begin{bmatrix} 8.35 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & -0.14 & 0 & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & . & 0 & -0.14 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 80.59 \end{bmatrix}_{72\times 72}$$

It can be seen that in this example according to Eq. (12) we can obtain the matrices [*L*], [*U*] and [*H*]. The matrices [U_{k-1}] and [H_k] in the last step ($j = k = 12$) are as follows:

$$U_{11}=\begin{bmatrix} -0.200 & 0 & -0.109 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0.002 & 0.005 & 0 & . & . & . & . & 0 & 0 & 0 \\ -0.011 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & 0 & 0.109 \\ 0 & 0 & 0 & . & . & . & . & -0.002 & 0.005 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0.011 & 0 & -0.195 \end{bmatrix}_{72\times 72}$$

$$H_{12}=10^{4}\begin{bmatrix} 44.30 & 0 & -25.33 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 27.93 & 0 & . & . & . & . & . & 0 & 0 & 0 \\ -25.33 & 0 & . & . & . & . & . & . & . & 0 & 0 \\ 0 & . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & . & . & . & . & 0 & 25.33 \\ 0 & 0 & 0 & . & . & . & . & . & 0 & 27.93 & 0 \\ 0 & 0 & 0 & . & . & . & . & . & 25.33 & 0 & 452.11 \end{bmatrix}_{72\times 72}$$

Having the [*H*] and [*U*] matrices and using Eq. (13), the inverse of [*K*] can be calculated.

At the end having the equation of force, the displacements of the structure can be calculated. For simplicity the displacements of the nodes 66 to 78 are shown in Table 5. The dimension of the length is meter and that of rotation is radian.

Table 5: The displacements of the fifth column (E-3)

D \ N	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**
Δx	0	-0.178	-0.080	0.118	0.251	0.203	-1.234	1.505	1.087	0.346	-0.424	-0.977	0
Δy	0	0.001	0.012	0.036	0.053	0.041	0.044	0.033	0.98	0.153	0.167	0.127	0
Δz	0	-3.692	-6.486	-7.168	-5.701	-2.960	-1.393	-1.780	-5.341	-7.152	-6.674	-4.357	0
θx	-0.009	-0.062	-0.108	-0.110	-0.079	-0.037	-0.001	-0.001	-0.003	-0.008	-0.012	-0.007	-0.008
θy	0.905	0.859	0.452	-0.114	-0.563	-0.734	-0.718	0.889	0.692	0.166	-0.362	-0.677	-0.709
θz	0	0.004	0.003	0.001	-0.002	-0.007	-0.008	0.003	0.004	0.001	-0.002	-0.005	-0.005

As it can be seen for the analysis of this plate instead of inverting a matrix of dimension 936×936, matrices of dimension 72×72 are inverted. This results in considerable reduction of commputational time, and it is especially suitable for the analysis of problems with high number of nodes.

7. CONCLUSIONS

In this paper a simple approach is presented for the analysis of those structures for which the stiffness matrices are in a genaralised canonical form. It is show that if the graph model of a structure or a finite element model has repeated form, then using a suitable nodal numbering, one can perform the analysis using the constituting blocks of its stiffness matrix. Here the pattern of the loading does not need to be repeated.

First we investigated the plate using two four sided elements in a general form. The choice of these elements results in stiffness matrices which are block tri-diagonal for those structures having repeated forms.

The presented method is also applied to the analysis of space structures with high number of nodes. As it is shown, with a suitable nodal numbering of the repeated sector, the stiffness matrix of such a structure can be decomposed into smaller blocks. The important point about this model is that one can use any type of element having its own cross section, and it is not necessary to have identical cross sections.

Acknowledgement: The second author is grateful to the Faculty of Engineering, University of Tehran for the Support.

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