ISME

An Exact Solution for Quasi-Static Poro-Thermoelasticity in Spherical Coordinates

M. Jabbari*
Assistant Professor

H. Dehbani†
postgraduate

In this paper the Quasi-Static poro-thermoelasticity model of a hollow and solid sphere under radial symmetric loading condition (r, t) is‡ considered. A full analytical method is used and an exact unique solution of the Quasi-Static equations is presented.
The thermal, mechanical and pressure boundary conditions, the body force, the heat source and the injected volume rate per unit volume of a distribute water source are considered in the most general forms where no limiting assumption is used. This generality allows to simulate variety of applicable problems.

Keywords: Quasi-Static Poro-Thermoelasticity, Hollow sphere, Exact solution

1 Introduction

Quasi-Static thermal and poro-mechanical processes play an important role in a number of problems of interest in geomechanics such as stability of boreholes and permeability enhancement in geothermal reservoirs or high temperature petroleum bearing formations. A poro-thermoelastic approach combines the theory of heat conduction with poroelastic constitutive equations coupling the temperature field with the stresses and pore pressure.
There are a limited number of papers that present the closed-form or analytical solution for the quasi-static porothermoelasticity problems. Bai [1] investigated the response of saturated porous media subjected to local thermal loading on the surface of semi-infinite space. He used the numerical integral methods for calculating the unsteady temperature, pore pressure and displacement fields. This author also studied the fluctuation responses of saturated porous media subjected to cyclic thermal loading [2]. In mentioned paper an analytical solution was deduced by using the Laplace transform and the Gauss-Legendre method of Laplace transform inversion. Droujinine [3] investigated dispersion and attenuation of body waves in a wide range of materials representing realistic rock structures. He used the time-domain asymptotic ray theory to a new generalized coordinate-free wave equation with an arbitrary tensor relaxation function. Bai and Li [4] found solution for cylindrical cavety in saturated thermoporoelastic medium by using Laplace transform and numerical Laplace transform inversion.
Also the numbers of papers that present the closed-form or analytical solutions for the quasi-static thermoelasticity problems are limited. Hetnarski [5] found the solution of quasi-static thermoelasticity in the form of series function. Hetnarski and Ignaczak presented a study of

* Corresponding author , Assistant Professor, Postgraduate School, South Tehran Branch, Islamic Azad University, Tehran, Iran, m_jabbari@azad.ac.ir
† postgraduate School, Sama Technical and Vocational Training College, Islamic Azad University, Varamin-Pishva Branch, Varamin, Iran, h.dehbani60@gmail.com

the one-dimensional thermoelastic waves produced by an instantaneous plane source of heat in homogeneous isotropic infinite and semi-infinite bodies of the Green-Lindsay type [6]. Also, these authors presented an analysis of laser-induced waves propagating in an absorbing thermoelastic semi-space of the Green-Lindsay type [7]. Georgiadis and Lykotrafitis obtained a three-dimensional transient thermoelastic solution for Rayleigh-type disturbances propagating on the surface of a half-space [8]. Wagner [9] presented the fundamental matrix of a system of partial differential operators that governs the diffusion of heat and the strains in elastic media. This method can used to predict the temperature distribution and the strains by an instantaneous point heat, point source of heat, or by a suddenly applied delta force.
Smith and Booker presented [10] a direct method (Green's functions) for porothermoelasticity in the Laplace transform domain. The boundary element techniques have been successfully applied to many geological engineering the boundary element method (BEM) or the boundary integral equation formulation also has proven effective for the poroelastic and thermoelastic problems.
Koshelev and Ghassemi [11] is suggested a stationary boundary element method for thermo and poroelasticity based on the complex variable hypersingular boundary integral equation. Zhou and Ghassemi [12] developed a finite element method to implement the fully coupled chemo-poro-thermoelastic linear and non-linear problems.
In this work a full analytical method is used to obtain the response of the governing equations, where an exact solution is presented. The method of solution is based on the Fourier expansion and eigenfunction methods, which is a traditional and routine method in solving the partial differential equations. Since the coefficients of equations are not functions of the time variable (t), an exponential form is considered for the general solution matched with the physical wave properties of thermal, mechanical and pressure waves. For the particular solution, that is the response to mechanical and thermal shocks, the eigenfuncion method and Laplace transformation is used.

2 Governing equations

A hollow cylinder with inner and outer radius r_i and r_o, respectively, made of isotropic material subjected to radial-symmetric mechanical, thermal and pressure shocks is considered. The theory of porothermoelasticity for wave propagation is considered to allow coupling between deformation, thermal energy and pressure fields and to describe the physical behavior of the elastic domain to mechanical, thermal and pressure shock loads. Navier equation in term of the displacement components is obtained as [4]

$$u,_{rr}+\frac{2}{r}u,_{r}-\frac{2}{r^2}u-\alpha\frac{(1+\nu)(1-2\nu)}{(1-\nu)E}p,_{r}-\beta\frac{(1+\nu)(1-2\nu)}{(1-\nu)E}T,_{r}-\rho\frac{(1+\nu)(1-2\nu)}{(1-\nu)E}\ddot{u}=-\frac{(1+\nu)(1-2\nu)}{(1-\nu)E}F(r,t) \tag{1}$$

Heat conduction equation in radial-symmetric direction with the mechanical coupling term is

$$T,_{rr}+\frac{2}{r}T,_{r}-Z\frac{T_\circ}{K}\dot{T}+Y\frac{T_\circ}{K}\dot{p}-\beta\frac{T_\circ}{K}(\dot{u},_{r}+\frac{2}{r}\dot{u})=-\frac{1}{K}Q(r,t) \tag{2}$$

According to Darcy's law and continuity condition of seepage, the equation of mass conservation can be written as

$$p,_{rr}+\frac{2}{r}p,_{r}-\alpha_p\frac{\gamma_w}{k}\dot{p}-Y\frac{\gamma_w}{k}\dot{T}-\alpha\frac{\gamma_w}{k}(\dot{u},_{r}+\frac{2}{r}\dot{u})=-\frac{\gamma_w}{k}W(r,t) \tag{3}$$

where (,) denotes partial derivative, u is the displacement component in the radial direction, p the pore pressure and $\alpha = 1 - C_s / C$ is Biot's coefficient, $C_s = 3(1-2\nu_s)E_s$ the is coefficient of volumetric compression of solid grains, with E_s and ν_s being the elastic modulus and Poission's ratio of solid grains and $C = 3(1-2\nu)E$ is the coefficient of volumetric compression of solid skeleton, with E and ν being the elastic modulus and Poission's ratio of solid skeleton, $T_\circ$ is the initial reference temperature, $\beta = 3\alpha_s / C$ the thermal expansion factor, α_s is the coefficient of linear thermal expansion of solid grains, $Y = 3(n\alpha_w + (\alpha - n)\alpha_s)$ and $\alpha_p = n(C_w - C_s) + \alpha C_s$ are coupling parameters, α_w and C_w are the coefficients of linear thermal expansion and volumetric compression of pure water, n is the porosity, k is the hydraulic conductivity, γ_w is the unit of pore water and $Z = ((1-n)\rho_s c_s + n\rho_w c_w)/T_\circ - 3\beta\alpha_s$ is coupling parameter, ρ_w and ρ_s are the densities of pore water and solid grains , c_w and c_s are the heat capacities of pore water and solid grains and K is the coefficient of heat conductivity. Here, F(r, t), Q(r, t) and W(r, t) are the body force, heat generation source and the injected volume rate per unit volume of a distribute water source, respectively. The mechanical, thermal and pressure boundary conditions are

$$\begin{aligned}
&C_{11}u(r_i,t) + C_{12}u_{,r}(r_i,t) + C_{13}T(r_i,t) + C_{14}p(r_i,t) = f_1(t) \\
&C_{21}u(r_o,t) + C_{22}u_{,r}(r_o,t) + C_{23}T(r_o,t) + C_{24}p(r_o,t) = f_2(t) \\
&C_{31}T(r_i,t) + C_{32}T_{,r}(r_i,t) = f_3(t) \\
&C_{41}T(r_o,t) + C_{42}T_{,r}(r_o,t) = f_4(t) \\
&C_{51}p(r_i,t) = f_5(t) \\
&C_{61}p(r_o,t) = f_6(t)
\end{aligned} \tag{4}$$

Where C_{ij} are the mechanical, thermal and pressure coefficients, which by assigning different values for them different types of mechanical, thermal and pressure boundary conditions may be obtained. These boundary conditions include the displacement, strain, stress, specified temperature, convection, pressure, and heat flux. The initial boundary conditions are assumed in general form

$$\begin{aligned}
&u(r,0) = f_7(r) \\
&u_{,t}(r,0) = f_8(r) \\
&T(r,0) = f_9(r) \\
&p(r,0) = f_{10}(r)
\end{aligned} \tag{5}$$

3 Solution

Equations (1) to (3) are the system of non-homogeneous partial differential equations with non-constant coefficients (functions of radius variable r only) has general and particular solutions.

3-1 General solution with homogeneous boundary conditions

Since the coefficients of these equations are independent of time variable (t), the exponential function form of time variable may be assumed for the general solution as

$$
\begin{aligned}
u(r,t) &= [U(r)]e^{\lambda t} \\
T(r,t) &= [\theta(r)]e^{\lambda t} \\
p(r,t) &= [P(r)]e^{\lambda t}
\end{aligned} \tag{6}
$$

Substituting Eqs.(6) into homogeneous parts of Eqs. (1) to (3), yields

$$
\begin{aligned}
&U'' + \frac{1}{r}U' - \frac{1}{r^2}U + d_1 P' + d_2\theta' = 0 \\
&\theta'' + \frac{1}{r}\theta' + d_4\lambda\theta + d_5\lambda P + d_6\lambda(U' + \frac{1}{r}U) = 0 \\
&P'' + \frac{1}{r}P' + d_7\lambda P + d_8\lambda\theta + d_9\lambda(U' + \frac{1}{r}U) = 0
\end{aligned} \tag{7}
$$

Equations (7) are a system of ordinary differential equations, where the prime symbol (') shows differentiation with respect to the radius variable ® and d_1 to d_9 are constant parameters given in the appendix.

3.2 Change in Dependent Variables

To obtain a solution for Eqs.(7) the dependent variables are changed as

$$
\begin{aligned}
U^*(r) &= r^{-\frac{1}{2}}U(r) \\
\theta^*(r) &= r^{-\frac{1}{2}}\theta(r) \\
P^*(r) &= r^{-\frac{1}{2}}P(r)
\end{aligned} \tag{8}
$$

Substituting Eqs.(8) into Eqs.(7) gives

$$
\begin{aligned}
&U'' + \frac{1}{r}U' - \frac{9}{4}\frac{1}{r^2}U + d_3\lambda^2 U - d_2\frac{1}{2}\frac{1}{r}\theta + d_2\theta' - d_1\frac{1}{2}\frac{1}{r}P + d_1 P' = 0 \\
&\theta'' + \frac{1}{r}\theta' - \frac{1}{4}\frac{1}{r^2}\theta + d_4\lambda\theta + d\lambda\frac{3}{2}\frac{1}{r}U + d_6\lambda U' + d_5\lambda P = 0 \\
&P'' + \frac{1}{r}P' - \frac{1}{4}\frac{1}{r^2}P + d_7\lambda P + d_8\lambda\theta + d_9\lambda\frac{3}{2}\frac{1}{r}U + d_9\lambda U' = 0
\end{aligned} \tag{9}
$$

3.3 Solution

The first solutions of U_1 , θ_1 and P_1 are considered as

$$
\begin{aligned}
U_1(r) &= A_1 J_{\frac{3}{2}}(\beta r) \\
\theta_1(r) &= B_1 J_{\frac{1}{2}}(\beta r) \\
P_1(r) &= C_1 J_{\frac{1}{2}}(\beta r)
\end{aligned} \tag{10}
$$

Substituting Eqs.(10) into Eqs. (9) yields

$$\begin{aligned}
&\{(-\beta^2+\lambda^2 d_3)A_1 - d_2\beta B_1 - d_1\beta C_1\}J_{\frac{3}{2}}(\beta r)=0\\
&\{\lambda d_6\beta A_1 + \left(-\beta^2+\lambda d_4\right)B_1 + \lambda d_5 C_1\}J_{\frac{1}{2}}(\beta r)=0\\
&\{\lambda d_9\beta A_1 + \lambda d_8 B_1 + (-\beta^2+\lambda d_7)C_1\}J_{\frac{1}{2}}(\beta r)=0
\end{aligned} \tag{11}$$

Equations (11) show that U_1 , θ_1 and P_1 can be the solutions of Eqs. (9), if and only if

$$\begin{bmatrix} -\beta^2+\lambda^2 d_3 & -d_2\beta & -d_1\beta \\ \lambda d_6\beta & -\beta^2+\lambda d_4 & \lambda d_5 \\ \lambda d_9\beta & \lambda d_8 & -\beta^2+\lambda d_7 \end{bmatrix}\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

The non-trivial solution of Eq. (12) is obtained by equating the determinant of this equation to zero as

$$\begin{aligned}
&d_3d_4d_7\lambda^4 - d_3d_5d_8\lambda^4 - \beta^2 d_3d_4\lambda^3 - \beta^2 d_3d_7\lambda^3 - \beta^2 d_4d_7\lambda^2 + \beta^2 d_5d_8\lambda^2\\
&+\beta^2 d_1d_4d_9\lambda^2 - \beta^2 d_1d_6d_8\lambda^2 + \beta^2 d_2d_6d_7\lambda^2 - \beta^2 d_2d_5d_9\lambda^2 + \beta^4 d_3\lambda^2\\
&+\beta^4 d_4\lambda + \beta^4 d_7\lambda - \beta^4 d_2d_6\lambda - \beta^4 d_1d_9\lambda - \beta^6 = 0
\end{aligned} \tag{13}$$

Equation (13) is the first characteristic equation. Thus, it is concluded that U_1 , θ_1 and P_1 satisfy the system of equations (9) and they are the first solution of the system. The second solutions of U_2 , θ_2 and P_2 are considered as

$$\begin{aligned}
U_2(r) &= [A_2 J_{\frac{3}{2}}(\beta r) + A_3 r J_{\frac{5}{2}}(\beta r)]\\
\theta_2(r) &= [B_2 J_{\frac{1}{2}}(\beta r) + B_3 r J_{\frac{3}{2}}(\beta r)]\\
P_2(r) &= [C_2 J_{\frac{1}{2}}(\beta r) + C_3 r J_{\frac{3}{2}}(\beta r)]
\end{aligned} \tag{14}$$

Substituting Eqs. (14) to Eqs. (9) yield

$$\begin{aligned}
&\left\{\left(\beta^2 - d_3\lambda^2\right)A_3 + C_3 d_1\beta + B_3 d_2\beta\right\} r J_{\frac{1}{2}}(\beta r) \\
&+\left\{-A_3\beta + A_2 d_3\lambda^2 - A_2\beta^2 - C_3 d_1 - B_3 d_2 - B_2 d_2\beta - C_2 d_1\beta + A_3 d_3\lambda^2 \frac{3}{\beta}\right\} J_{\frac{3}{2}}(\beta r) = 0 \\
&\left\{-B_2\beta^2 + 2B_3\beta + B_2 d_4\lambda + A_2 d_6\beta\lambda + C_2 d_5\lambda\right\} J_{\frac{1}{2}}(\beta r) \\
&+\left\{A_3 d_6\beta\lambda + \left(-\beta^2 + d_4\lambda\right)B_3 + C_3 d_5\lambda\right\} r J_{\frac{3}{2}}(\beta r) = 0 \\
&\left\{2C_3\beta + C_2 d_7\lambda + B_2 d_8\lambda + A_2 d_9\lambda\beta - C_2\beta^2\right\} J_{\frac{1}{2}}(\beta r) \\
&+\left\{+A_3 d_9\lambda\beta + B_3 d_8\lambda + \left(-\beta^2 + d_7\lambda\right)C_3\right\} r J_{\frac{3}{2}}(\beta r) = 0
\end{aligned} \tag{15}$$

The expressions for U_2 , θ_2 and P_2 can be the solutions of Eqs. (9), if and only if

$$\begin{bmatrix} -\beta^2 + \lambda^2 d_3 & -d_2\beta & -d_1\beta \\ \lambda d_6\beta & -\beta^2 + \lambda d_4 & \lambda d_5 \\ \lambda d_9\beta & \lambda d_8 & -\beta^2 + \lambda d_7 \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

$$\left(d_3\lambda^2 - \beta^2\right)A_2 + \left(d_3\lambda^2 \frac{3}{\beta} - \beta\right)A_3 - d_2\beta B_2 - d_2 B_3 - d_1\beta C_2 - d_1 C_3 = 0 \tag{17}$$

$$d_6\lambda\beta A_2 + \left(-\beta^2 + d_4\lambda\right)B_2 + 2\beta B_3 + d_5\lambda C_2 = 0 \tag{18}$$

$$d_9\lambda\beta A_2 + d_8\lambda B_2 + \left(d_7\lambda - \beta^2\right)C_2 + 2\beta C_3 = 0 \tag{19}$$

The non-trivial solution of Eqs. (16) is obtained by equating the determinant to zero as

$$\begin{aligned}
&d_3 d_4 d_7\lambda^4 - d_3 d_5 d_8\lambda^4 - \beta^2 d_3 d_4\lambda^3 - \beta^2 d_3 d_7\lambda^3 - \beta^2 d_4 d_7\lambda^2 + \beta^2 d_5 d_8\lambda^2 \\
&+\beta^2 d_1 d_4 d_9\lambda^2 - \beta^2 d_1 d_6 d_8\lambda^2 + \beta^2 d_2 d_6 d_7\lambda^2 - \beta^2 d_2 d_5 d_9\lambda^2 + \beta^4 d_3\lambda^2 \\
&+\beta^4 d_4\lambda + \beta^4 d_7\lambda - \beta^4 d_2 d_6\lambda - \beta^4 d_1 d_9\lambda - \beta^6 = 0
\end{aligned} \tag{20}$$

Equations (17) to (19) give the relations between A_2, A_3, B_2, B_3, C_2 and C_3 and they play as the balancing ratios that make Eq. (14) to be the second solution of the system of equations (9). The third solution of the system of the ordinary differential equations with non-constant coefficients (9) must be considered as

$$\begin{aligned}
U_3(r) &= [A_4 J_{\frac{3}{2}}(\beta r) + A_5 r J_{\frac{5}{2}}(\beta r) + A_6 r^2 J_{\frac{7}{2}}(\beta r)] \\
\theta_3(r) &= [B_4 J_{\frac{1}{2}}\beta r) + B_5 r J_{\frac{3}{2}}(\beta r) + B_6 r^2 J_{\frac{5}{2}}(\beta r)] \\
P_3(r) &= [C_4 J_{\frac{1}{2}}(\beta r) + C_5 r J_{\frac{3}{2}}(\beta r) + C_6 r^2 J_{\frac{5}{2}}(\beta r)]
\end{aligned} \tag{21}$$

Substituting Eqs. (21) into Eq. (9) yield

$$\left\{\begin{array}{l}\left(-\beta^2+d_3\lambda^2\right)A_4+\left(-\beta+d_3\lambda^2\frac{3}{\beta}\right)A_5+\left(-3+d_3\lambda^2\frac{15}{\beta^2}\right)A_6\\ -B_4d_2\beta-d_2B_5-3d_2\frac{1}{\beta}B_6-C_4\beta d_1-\frac{3}{2}C_5d_1-C_6\frac{3}{\beta}d_1\end{array}\right\}J_{\frac{3}{2}}(\beta r)=0$$

$$\left\{\left(\beta^2-d_3\lambda^2\right)A_6+C_6\beta d_1+B_6d_2\beta\right\}r^2J_{\frac{3}{2}}(\beta r)=0$$

$$\left\{\begin{array}{l}\left(\beta^2-d_3\lambda^2\right)A_5+\left(\beta-\frac{5}{\beta}d_3\lambda^2\right)A_6+B_5d_2\beta+d_2B_6+C_5\beta d_1\\ +C_6\frac{3}{2}d_1-\frac{1}{2}C_6d_1\end{array}\right\}rJ_{\frac{1}{2}}(\beta r)=0$$

$$\left\{d_6\lambda\beta A_4+\left(-\beta^2+d_4\lambda\right)B_4+2\beta B_5+d_5\lambda C_4\right\}J_{\frac{1}{2}}(\beta r)=0$$

$$\left\{-d_6\lambda\beta A_6+\left(\beta^2-d_4\lambda\right)B_6-d_5\lambda C_6\right\}r^2J_{\frac{1}{2}}(\beta r)=0$$

$$\left\{d_6\lambda\beta A_5+3d_6\lambda A_6+\left(d_4\lambda-\beta^2\right)B_5+\left(d_4\frac{3}{\beta}\lambda+\beta\right)B_6+d_5\lambda C_5+d_5\frac{3}{\beta}\lambda C_6\right\}rJ_{\frac{3}{2}}(\beta r)=0$$

$$\left\{+d_9\lambda A_4\beta+d_8\lambda B_4+\left(-\beta^2+d_7\lambda\right)C_4+2C_5\beta\right\}J_{\frac{1}{2}}(\beta r)=0$$

$$\left\{\left(-d_7\lambda+\beta^2\right)C_6-d_8\lambda B_6-d_9\lambda A_6\beta\right\}r^2J_{\frac{1}{2}}(\beta r)=0$$

$$\left\{\left(-\beta^2+d_7\lambda\right)C_5+\left(\beta+d_7\lambda\frac{3}{\beta}\right)C_6+d_8\lambda B_5+d_8\frac{3}{\beta}\lambda B_6+d_9\lambda\beta A_5+3d_9\lambda A_6\right\}rJ_{\frac{3}{2}}(\beta r)=0 \quad (22)$$

The expressions for U_3, θ_3 and P_3 can be solutions of Eq. (9), if and only if

$$\begin{bmatrix}-\beta^2+\lambda^2d_3 & -d_2\beta & -d_1\beta\\ \lambda d_6\beta & -\beta^2+\lambda d_4 & \lambda d_5\\ \lambda d_9\beta & \lambda d_8 & -\beta^2+\lambda d_7\end{bmatrix}\begin{Bmatrix}A_6\\ B_6\\ C_6\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad (23)$$

$$\begin{array}{c}\left(-\beta^2+d_3\lambda^2\right)A_4+\left(-\beta+d_3\lambda^2\frac{3}{\beta}\right)A_5+\left(-3+d_3\lambda^2\frac{15}{\beta^2}\right)A_6\\ -B_4d_2\beta-d_2B_5-3d_2\frac{1}{\beta}B_6-C_4\beta d_1-\frac{3}{2}C_5d_1-C_6\frac{3}{\beta}d_1=0\end{array} \quad (24)$$

$$\left(\beta^2-d_3\lambda^2\right)A_5+\left(\beta-\frac{5}{\beta}d_3\lambda^2\right)A_6+B_5d_2\beta+d_2B_6+C_5\beta d_1+C_6\frac{3}{2}d_1-\frac{1}{2}C_6d_1=0 \quad (25)$$

$$d_6\lambda\beta A_4+\left(-\beta^2+d_4\lambda\right)B_4+2\beta B_5+d_5\lambda C_4=0 \quad (26)$$

$$d_6\lambda\beta A_5 + 3d_6\lambda A_6 + \left(d_4\lambda - \beta^2\right)B_5 + \left(d_4\frac{3}{\beta}\lambda + \beta\right)B_6 + d_5\lambda C_5 + d_5\frac{3}{\beta}\lambda C_6 = 0 \tag{27}$$

$$d_9\lambda A_4\beta + d_8\lambda B_4 + \left(-\beta^2 + d_7\lambda\right)C_4 + 2C_5\beta = 0 \tag{28}$$

$$\left(-\beta^2 + d_7\lambda\right)C_5 + \left(\beta + d_7\lambda\frac{3}{\beta}\right)C_6 + d_8\lambda B_5 + d_8\frac{3}{\beta}\lambda B_6 + d_9\lambda\beta A_5 + 3d_9\lambda A_6 = 0 \tag{29}$$

The non-trivial solution of Eq. (23) is obtained by equating the determinant of this equation to zero as

$$\begin{aligned} &d_3d_4d_7\lambda^4 - d_3d_5d_8\lambda^4 - \beta^2 d_3d_4\lambda^3 - \beta^2 d_3d_7\lambda^3 - \beta^2 d_4d_7\lambda^2 + \beta^2 d_5d_8\lambda^2 \\ &+ \beta^2 d_1d_4d_9\lambda^2 - \beta^2 d_1d_6d_8\lambda^2 + \beta^2 d_2d_6d_7\lambda^2 - \beta^2 d_2d_5d_9\lambda^2 + \beta^4 d_3\lambda^2 \\ &+ \beta^4 d_4\lambda + \beta^4 d_7\lambda - \beta^4 d_2d_6\lambda - \beta^4 d_1d_9\lambda - \beta^6 = 0 \end{aligned} \tag{30}$$

The characteristic equation (30) is the same as the characteristic equations (13) and (20). This equality is interesting as it prevents mathematical dilemma and complexity and a single value for the eigenvalue β simultaneously satisfies three characteristic equations (13), (20) and (30). Equations (24) to (29) gives the relations between $A_4, A_5, A_6, B_4, B_5, B_6, C_4, C_5$ and C_6. These relations play as the balancing ratios that help Eq. (21) to be the third solution of the system of equations (9). The complete general solutions for the solid sphere are

$$\begin{aligned} U^g(r) &= A_1 J_{\frac{3}{2}}(\beta r) + A_3[\zeta_1 J_{\frac{3}{2}}(\beta r) + rJ_{\frac{5}{2}}(\beta r)] + A_6[\zeta_2 J_{\frac{3}{2}}(\beta r) + \zeta_3 rJ_{\frac{5}{2}}(\beta r) + r^2 J_{\frac{7}{2}}(\beta r)] \\ \theta^g(r) &= A_1\zeta_4 J_{\frac{1}{2}}(\beta r) + A_3[\zeta_5 J_{\frac{1}{2}}(\beta r) + \zeta_6 rJ_{\frac{3}{2}}(\beta r)] + A_6[\zeta_7 J_{\frac{1}{2}}(\beta r) + \zeta_8 rJ_{\frac{3}{2}}(\beta r) + \zeta_9 r^2 J_{\frac{5}{2}}(\beta r)] \\ P^g(r) &= A_1\zeta_{10} J_{\frac{1}{2}}(\beta r) + A_3[\zeta_{11} J_{\frac{1}{2}}(\beta r) + \zeta_{12} rJ_{\frac{3}{2}}(\beta r)] + A_6[\zeta_{13} J_{\frac{1}{2}}(\beta r) + \zeta_{14} rJ_{\frac{3}{2}}(\beta r) + \zeta_{15} r^2 J_{\frac{5}{2}}(\beta r)] \end{aligned} \tag{31}$$

and for hollow sphere are

$$\begin{aligned} U^g(r) &= A_1 J_{\frac{3}{2}}(\beta r) + A_3[\zeta_1 J_{\frac{3}{2}}(\beta r) + rJ_{\frac{5}{2}}(\beta r)] + A_6[\zeta_2 J_{\frac{3}{2}}(\beta r) + \zeta_3 rJ_{\frac{5}{2}}(\beta r) + r^2 J_{\frac{7}{2}}(\beta r)] \\ &+ \hat{A}_1 Y_{\frac{3}{2}}(\beta r) + \hat{A}_3[\zeta_1 Y_{\frac{3}{2}}(\beta r) + rY_{\frac{5}{2}}(\beta r)] + \hat{A}_6[\zeta_2 Y_{\frac{3}{2}}(\beta r) + \zeta_3 rY_{\frac{5}{2}}(\beta r) + r^2 Y_{\frac{7}{2}}(\beta r)] \\ \theta^g(r) &= A_1\zeta_4 J_{\frac{1}{2}}(\beta r) + A_3[\zeta_5 J_{\frac{1}{2}}(\beta r) + \zeta_6 rJ_{\frac{3}{2}}(\beta r)] + A_6[\zeta_7 J_{\frac{1}{2}}(\beta r) + \zeta_8 rJ_{\frac{3}{2}}(\beta r) + \zeta_9 r^2 J_{\frac{5}{2}}(\beta r)] \\ &+ \hat{A}_1\zeta_4 Y_{\frac{1}{2}}(\beta r) + \hat{A}_3[\zeta_5 Y_{\frac{1}{2}}(\beta r) + \zeta_6 rY_{\frac{3}{2}}(\beta r)] + \hat{A}_6[\zeta_7 Y_{\frac{1}{2}}(\beta r) + \zeta_8 rY_{\frac{3}{2}}(\beta r) + \zeta_9 r^2 Y_{\frac{5}{2}}(\beta r)] \\ P^g(r) &= A_1\zeta_{10} J_{\frac{1}{2}}(\beta r) + A_3[\zeta_{11} J_{\frac{1}{2}}(\beta r) + \zeta_{12} rJ_{\frac{3}{2}}(\beta r)] + A_6[\zeta_{13} J_{\frac{1}{2}}(\beta r) + \zeta_{14} rJ_{\frac{3}{2}}(\beta r) + \zeta_{15} r^2 J_{\frac{5}{2}}(\beta r)] \\ &+ \hat{A}_1\zeta_{10} Y_{\frac{1}{2}}(\beta r) + \hat{A}_3[\zeta_{11} Y_{\frac{1}{2}}(\beta r) + \zeta_{12} rY_{\frac{3}{2}}(\beta r)] + \hat{A}_6[\zeta_{13} Y_{\frac{1}{2}}(\beta r) + \zeta_{14} rY_{\frac{3}{2}}(\beta r) + \zeta_{15} r^2 Y_{\frac{5}{2}}(\beta r)] \end{aligned} \tag{۳۲}$$

Where ζ_1 to ζ_{15} are ratios obtained from Eqs. (23) to (29), (16) to (19) and (12) and are given in the appendix. Substituting U^g , θ^g and P^g in the homogeneous form of the boundary conditions (4), three linear algebraic equations are obtained. They are the coefficients depending on λ and β. Setting the determinant of the coefficients equal to zero, the second characteristic equation is obtained. Simultaneous solution of this equation and Eq. (11), results into infinite number of two eigenvalues β_n and λ_n. Therefore U^g , θ^g and P^g for solid sphere are rewritten as

$$\begin{aligned} U^g(r) &= A_1\left[J_{\frac{3}{2}}(\beta r)+\zeta_{16}[\zeta_1 J_{\frac{3}{2}}(\beta r)+rJ_{\frac{5}{2}}(\beta r)]+\zeta_{17}[\zeta_2 J_{\frac{3}{2}}(\beta r)+\zeta_3 rJ_{\frac{5}{2}}(\beta r)+r^2 J_{\frac{7}{2}}(\beta r)]\right] \\ \theta^g(r) &= A_1\left[\zeta_4 J_{\frac{1}{2}}(\beta r)+\zeta_{16}[\zeta_5 J_{\frac{1}{2}}(\beta r)+\zeta_6 rJ_{\frac{3}{2}}(\beta r)]+\zeta_{17}[\zeta_7 J_{\frac{1}{2}}(\beta r)+\zeta_8 rJ_{\frac{3}{2}}(\beta r)+\zeta_9 r^2 J_{\frac{5}{2}}(\beta r)]\right] \\ P^g(r) &= A_1\left[\zeta_{10} J_{\frac{1}{2}}(\beta r)+\zeta_{16}[\zeta_{11} J_{\frac{1}{2}}(\beta r)+\zeta_{12} rJ_{\frac{3}{2}}(\beta r)]+\zeta_{17}[\zeta_{13} J_{\frac{1}{2}}(\beta r)+\zeta_{14} rJ_{\frac{3}{2}}(\beta r)+\zeta_{15} r^2 J_{\frac{5}{2}}(\beta r)]\right] \end{aligned} \quad (33)$$

Where ζ_{16} and ζ_{17} are presented in the appendix. Let us show the functions in the brackets of Eq. (33) by functions $H_\circ, H_1$ and H_2 as

$$\begin{aligned} H_0 &= J_{\frac{3}{2}}(\beta r)+\zeta_{16}[\zeta_1 J_{\frac{3}{2}}(\beta r)+rJ_{\frac{5}{2}}(\beta r)]+\zeta_{17}[\zeta_2 J_{\frac{3}{2}}(\beta r)+\zeta_3 rJ_{\frac{5}{2}}(\beta r)+r^2 J_{\frac{7}{2}}(\beta r)] \\ H_1 &= \zeta_4 J_{\frac{1}{2}}(\beta r)+\zeta_{16}[\zeta_5 J_{\frac{1}{2}}(\beta r)+\zeta_6 rJ_{\frac{3}{2}}(\beta r)]+\zeta_{17}[\zeta_7 J_{\frac{1}{2}}(\beta r)+\zeta_8 rJ_{\frac{3}{2}}(\beta r)+\zeta_9 r^2 J_{\frac{5}{2}}(\beta r)] \\ H_2 &= \zeta_{10} J_{\frac{1}{2}}(\beta r)+\zeta_{16}[\zeta_{11} J_{\frac{1}{2}}(\beta r)+\zeta_{12} rJ_{\frac{3}{2}}(\beta r)]+\zeta_{17}[\zeta_{13} J_{\frac{1}{2}}(\beta r)+\zeta_{14} rJ_{\frac{3}{2}}(\beta r)+\zeta_{15} r^2 J_{\frac{5}{2}}(\beta r)] \end{aligned} \quad (34)$$

According to the Sturm-Liouville theorem, these functions are orthogonal with respect to the weight function p(r) = r such as

$$\int_{r_i}^{r_o} H(\beta_n r)H(\beta_m r) r dr = \begin{cases} 0 & n \neq m \\ \|H(\beta_n r)\|^2 & n = m \end{cases} \quad (35)$$

where $\|H(\beta_n r)\|$ is norm of the H function and equals

$$\|H(\beta_n r)\| = \left[\int_{r_i}^{r_o} rH^2(\beta_n r)dr\right]^{\frac{1}{2}} \quad (٣٦)$$

Due to the orthogonality of function H, every piece-wise continuous function, such as f(r), can be expanded in terms of the function H (either $H_\circ$, H_1 or H_2), and is called the H-Fourier series as

$$f(r)=\sum_{n=1}^{\infty} e_n \quad H(\beta_n r) \tag{37}$$

where e_n equals

$$e_n=\frac{1}{\left\|H(\beta_n r)\right\|^2}\int_{r_i}^{r_o} f(r)\, H(r)\, r\, dr \tag{38}$$

Using Eqs. (6), (33) and (34) the displacement and temperature distributions due to the general solution become

$$\begin{aligned} u^g(r,t) &= \sum_{n=1}^{\infty}\ \{\sum_{m=1}^{4}\ a_{nm}e^{\lambda_{nm}t}\}H_0(\beta_n r)\\ T^g(r,t) &= \sum_{n=1}^{\infty}\ \{\sum_{m=1}^{4}\ N_{nm}a_{nm}e^{\lambda_{nm}t}\}H_1(\beta_n r)\\ p^g(r,t) &= \sum_{n=1}^{\infty}\ \{\sum_{m=1}^{4}\ M_{nm}a_{nm}e^{\lambda_{nm}t}\}H_2(\beta_n r) \end{aligned} \tag{39}$$

where N_{mn} and M_{nm} are ratios obtained by substituting Eqs. (39) to Eq. (1) to (3). Using the initial conditions (5) and with the help of Eqs. (36), (37) and (38), four unknown constants are obtained.

3-4 Particular solution with non-homogeneous boundary conditions

The general solutions may be used as proper functions for guessing the particular solution adopted to the non-homogeneous parts of the Eqs.(1) to (3) and the non-homogeneous boundary conditions (4) as

$$\begin{aligned} u^p(r,t) &= \sum_{n=1}^{\infty}\{\left[G_{1n}(t)J_{\frac{3}{2}}(\beta_n r)+G_{2n}(t)rJ_{\frac{5}{2}}(\beta_n r)+G_{3n}(t)r^2J_{\frac{7}{2}}(\beta_n r)\right]+r^2G_{4n}(t)\}\\ T^p(r,t) &= \sum_{n=1}^{\infty}\{\left[G_{5n}(t)J_{\frac{1}{2}}(\beta_n r)+G_{6n}(t)rJ_{\frac{3}{2}}\beta_n r)+G_{7n}(t)r^2J_{\frac{5}{2}}(\beta_n r)\right]+r^2G_{8n}(t)\}\\ p^p(r,t) &= \sum_{n=1}^{\infty}\{\left[G_{9n}(t)J_{\frac{1}{2}}(\beta_n r)+G_{10n}(t)rJ_{\frac{3}{2}}(\beta_n r)+G_{11n}(t)r^2J_{\frac{5}{2}}\beta_n r)\right]+r^2G_{12n}(t)\} \end{aligned} \tag{40}$$

For the solid sphere, the second type of Bessel function Y is excluded. It is necessary and suitable to expand the body force F(r, t), heat source Q(r, t) and porosity function W(r, t) in H-Fourier expansion form as

$$\begin{aligned} F(r,t) &= \sum_{n=1}^{\infty}\ F_n(t)H_0(\beta_n r)\\ Q(r,t) &= \sum_{n=1}^{\infty}\ Q_n(t)H_1(\beta_n r)\\ P(r,t) &= \sum_{n=1}^{\infty}\ P_n(t)H_2(\beta_n r) \end{aligned} \tag{41}$$

where $F_n(t)$, $Q_n(t)$ and $P_n(t)$ are

$$F_n(t) = \frac{1}{\left\| H_0(\beta_n r) \right\|^2} \int_{r_i}^{r_o} F(r,t) H_0(\beta_n r) r dr$$
$$Q_n(t) = \frac{1}{\left\| H_1(\beta_n r) \right\|^2} \int_{r_i}^{r_o} Q(r,t) H_1(\beta_n r) r dr \tag{42}$$
$$P_n(t) = \frac{1}{\left\| H_2(\beta_n r) \right\|^2} \int_{r_i}^{r_o} P(r,t) H_2(\beta_n r) r dr$$

Substituting Eqs. (40) and (41) into non-homogeneous form of equations (1) into (3) yield

$$\left\{\begin{array}{l} -G_1(t)\beta^2 + d_3\ddot{G}_1(t) - G_2(t)\beta + d_3\ddot{G}_2(t)\frac{3}{\beta} - 3G_3(t) + d_3\ddot{G}_3(t)\frac{15}{\beta^2} + \left\{C_0 + C_1\frac{3}{\beta} + C_2\frac{15}{\beta^2}\right\} d_{16}G_4(t) \\ + \left\{C_0 + C_1\frac{3}{\beta} + C_2\frac{15}{\beta^2}\right\} d_{19}\ddot{G}_4(t) - G_5(t)d_2\beta - d_2G_6(t) - 3d_2G_7(t)\frac{1}{\beta} + \left\{C_0 + C_1\frac{3}{\beta} + C_2\frac{15}{\beta^2}\right\} d_{17}G_8(t) \\ - d_1G_9(t)\beta - d_1G_{10}(t) - 3d_1G_{11}(t)\frac{1}{\beta} + \left\{C_0 + C_1\frac{3}{\beta} + C_2\frac{15}{\beta^2}\right\} d_{18}G_{12}(t) - \left\{C_0 + C_1\frac{3}{\beta} + C_2\frac{15}{\beta^2}\right\} d_{10}d_{13}F_n(t) \end{array}\right\} = 0 \tag{43a}$$

$$\left\{\begin{array}{l} G_3(t)\beta^2 - d_3\ddot{G}_3(t) - d_{16}C_2G_4(t) - d_{19}C_2\ddot{G}_4(t) + G_7(t)d_2\beta - d_{17}C_2G_8(t) - d_{18}C_2G_{12}(t) \\ + d_1G_{11}(t)\beta + d_{10}d_{13}C_2F_n(t) \end{array}\right\} = 0 \tag{43b}$$

$$\left\{\begin{array}{l} G_2(t)\beta^2 - d_3\ddot{G}_2(t) + G_3(t)\beta - d_3\ddot{G}_3(t)\frac{5}{\beta} + \left\{-C_1 - C_2\frac{5}{\beta}\right\} d_{16}G_4(t) + \left\{-C_1 - C_2\frac{5}{\beta}\right\} d_{19}\ddot{G}_4(t) \\ + d_2G_6(t)\beta + d_2G_7(t) - \left\{C_1 + C_2\frac{5}{\beta}\right\} d_{17}G_8(t) + d_1G_{10}(t)\beta + d_1G_{11}(t) - \left\{C_1 + C_2\frac{5}{\beta}\right\} d_{18}G_{12}(t) \\ + \left\{C_1F_n(t) + C_2\frac{5}{\beta}\right\} d_{10}d_{13}F_n(t) \end{array}\right\} = 0 \tag{43c}$$

$$\left\{\begin{array}{l} + d_6\dot{G}_1(t)\beta + d_{21}E_0\dot{G}_4(t) - G_5(t)\beta^2 + d_4\dot{G}_5(t) + 2G_6(t)\beta + d_{20}E_0G_8(t) + d_{22}E_0\dot{G}_8(t) \\ + d_5\dot{G}_9(t)d_{23}E_0\dot{G}_{12}(t) - d_{11}d_{14}E_0Q_n(t) \end{array}\right\} = 0 \tag{43d}$$

$$\left\{\begin{array}{l} - d_6\dot{G}_3(t)\beta - d_{21}E_2\dot{G}_4(t) + G_7(t)\beta^2 - d_4\dot{G}_7(t) - d_{20}E_2G_8(t) - d_{22}E_2\dot{G}_8(t) - d_5\dot{G}_{11}(t) \\ - d_{23}E_2\dot{G}_{12}(t) + d_{11}d_{14}E_2Q_n(t) \end{array}\right\} = 0 \tag{43e}$$

$$\left\{\begin{array}{l} d_6\dot{G}_2(t)\beta + 3d_6\dot{G}_3(t) + \left\{E_1 + E_2\frac{3}{\beta}\right\} d_{21}\dot{G}_4(t) - G_6(t)\beta^2 + d_4\dot{G}_6(t) + G_7(t)\beta + d_4\dot{G}_7(t)\frac{3}{\beta} \\ + \left\{E_1 + E_2\frac{3}{\beta}\right\} d_{20}G_8(t) + \left\{E_1 + E_2\frac{3}{\beta}\right\} d_{22}\dot{G}_8(t) + d_5\dot{G}_{10}(t) + d_5\dot{G}_{11}(t)\frac{3}{\beta} + \left\{E_1 + E_2\frac{3}{\beta}\right\} d_{23}\dot{G}_{12}(t) \\ - \left\{E_1 + E_2\frac{3}{\beta}\right\} d_{11}d_{14}Q_n(t) \end{array}\right\} = 0 \tag{43f}$$

$$\left\{\begin{array}{l} +\dot{G}_1(t)d_9\beta + D_0 d_{25}\dot{G}_4(t) + d_8\dot{G}_5(t) - G_9(t)\beta^2 + d_7\dot{G}_9(t) + D_0 d_{26}\dot{G}_8(t) + 2G_{10}(t)\beta \\ + D_0 d_{24} G_{12}(t) + D_0 d_{27}\dot{G}_{12}(t) - d_{12}d_{15}D_0 W_n(t) \end{array}\right\} = 0 \tag{43g}$$

$$\left\{\begin{array}{l} G_{11}(t)\beta^2 - d_7\dot{G}_{11}(t) - d_8\dot{G}_7(t) - d_9\dot{G}_3(t)\beta - D_2 r^2 d_{25}\dot{G}_4(t) - D_2 d_{26}\dot{G}_8(t) - D_2 d_{24} G_{12}(t) \\ - D_2 d_{27}\dot{G}_{12}(t) + d_{12}d_{15}D_2 W_n(t) \end{array}\right\} = 0 \tag{43h}$$

$$\left\{\begin{array}{l} +\dot{G}_2(t)\beta d_9 + 3\dot{G}_3(t)d_9 + \left\{D_1 + D_2\dfrac{3}{\beta}\right\}d_{25}\dot{G}_4(t) + d_8\dot{G}_6(t) \\ + d_8\dot{G}_7(t)\dfrac{3}{\beta}\left\{D_1\dot{G}_8(t) + D_2\dfrac{3}{\beta}\right\}\dot{G}_8(t)d_{26} - G_{10}(t)\beta^2 + d_7\dot{G}_{10}(t) + G_{11}(t)\beta + d_7\dot{G}_{11}(t)\dfrac{3}{\beta} \\ + \left\{D_1 + D_2\dfrac{3}{\beta}\right\}d_{27}\dot{G}_{12}(t) + \left\{D_1 + D_2\dfrac{3}{\beta}\right\}d_{24}G_{12}(t) - \left\{+D_1 + D_2\dfrac{3}{\beta}\right\}d_{12}d_{15}W_n(t) \end{array}\right\} = 0 \tag{43i}$$

where d_{10} to d_{27} are the coefficients of the H-expansion and constant parameters presented in the appendix. By taking Laplace transform of Eq. (43) and using three boundary conditions of Eq. (4) (for solid sphere only second, fourth and sixth boundary conditions are applicable), a system of algebraic equations is obtained and solved by Cramer's methods in the Laplace domain, where by the inverse Laplace transform the functions are transformed into the real time domain and finally $G_{1n}(t)$ to $G_{12n}(t)$ are calculated.

In this process it is necessary to consider the following points

1- The initial conditions (5) are considered only for the general solutions and the initial conditions of $G_{1n}(t)$ to $G_{12n}(t)$ for the particular solutions are considered equal to zero.

2- Laplace transform of Eqs. (43) is in terms of polynomial function form of the Laplace parameter s (not the Bessel functions form of s). Therefore, the exact inverse Laplace transform is possible and somehow simple.

3- For the hollow Sphere it is enough to include the second type of Bessel function Y (r) in a sequence of the particular solution as

$$u^p(r,t)=\sum_{n=1}^{\infty}\{\left[G_{1n}(t)J_{\frac{3}{2}}(\beta nr)+G_{2n}(t)rJ_{\frac{5}{2}}(\beta nr)+G_{3n}(t)r^2J_{\frac{7}{2}}(\beta nr)\right]$$

$$+\left[G_{4n}(t)Y_{\frac{3}{2}}(\beta nr)+G_{5n}(t)rY_{\frac{5}{2}}(\beta nr)+G_{6n}(t)r^2Y_{\frac{7}{2}}(\beta nr)\right]+rG_{7n}(t)+r^2G_{8n}(t)\}$$

$$T^p(r,t)=\sum_{n=1}^{\infty}\{\left[G_{9n}(t)J_{\frac{1}{2}}(\beta nr)+G_{10n}(t)rJ_{\frac{3}{2}}\beta nr)+G_{11n}(t)r^2J_{\frac{5}{2}}(\beta nr)\right]$$

$$\left[G_{12n}(t)Y_{\frac{1}{2}}(\beta nr)+G_{13n}(t)rY_{\frac{3}{2}}\beta nr)+G_{14n}(t)r^2Y_{\frac{5}{2}}(\beta nr)\right]+rG_{15n}(t)+r^2G_{16n}(t)\} \quad (44)$$

$$p^p(r,t)=\sum_{n=1}^{\infty}\{\left[G_{17n}(t)J_{\frac{1}{2}}(\beta nr)+G_{18n}(t)rJ_{\frac{3}{2}}(\beta nr)+G_{19n}(t)r^2J_{\frac{5}{2}}\beta nr)\right]$$

$$+\left[G_{20n}(t)J_{\frac{1}{2}}(\beta nr)+G_{21n}(t)rJ_{\frac{3}{2}}(\beta nr)+G_{23n}(t)r^2J_{\frac{5}{2}}\beta nr)\right]+r^2G_{24n}(t)\}$$

By substituting Eqs. (44) in Eqs. (1) to (3), eighteen equations are obtained, where using the six boundary conditions (4) twenty four functions $G_{1n}(t)$ to $G_{24n}(t)$ are obtained for the hollow sphere.

4 Results and discussions

In this section, the responses of a solid cylinder with radius 1m under different thermal boundary are considered .the result obtained from the present formulation for a solid cylinder are shown for several case. These include investigating the wave propagation in solid cylinder on variation of the temperature, displacement. The initial temperature $T_\circ$ is considered to be $293^\circ K$ The material properties are:

Table1

Material Parameters					
Parameter	Value	Unit	Parameter	Value	Unit
n	0.4	-	α_s	1.5×10^{-5}	1/°C
E	٦×١٠^{٥}	Pa	α_w	٢×10^{-4}	1/°C
ν	0.3	-	c_s	0.8	J/g°C
T_0	٢٩٣		c_w	4.2	J/g°C
K_s	٢×١٠^{١٠}	Pa	ρ_s	2.6×١٠^{٦}	g/m^3
K_w	٥×١٠^{٩}	Pa	ρ_w	١×١٠^{٦}	g/m^3
K	0.3	w/m°C	α	١	-

For the first example, an instantaneous hot spot $T(1,t)=10^{-3}T_0\delta(t)$, where δ(t) is a unit Dirac function, is considered and the outside radius of the cylinder is assumed to be fixed (u(1,t))=0 and p(1,t)=0).Figures 1a to 1c show the wave fronts for the displacement , temperature and pressure at three times(0.0001,0.00005,0.00001 second).

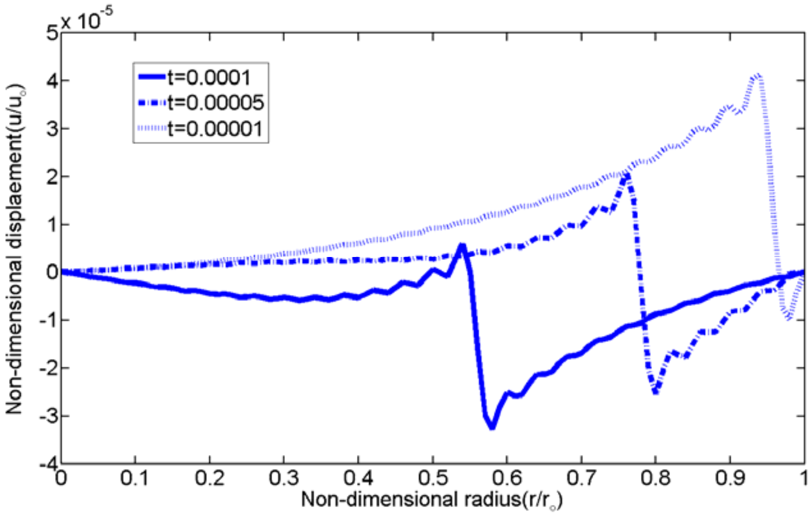

Figure 1.a Non-dimensional displacement distribution due to input $T(1,t)=10^{-3}\,T_0\delta(t)$

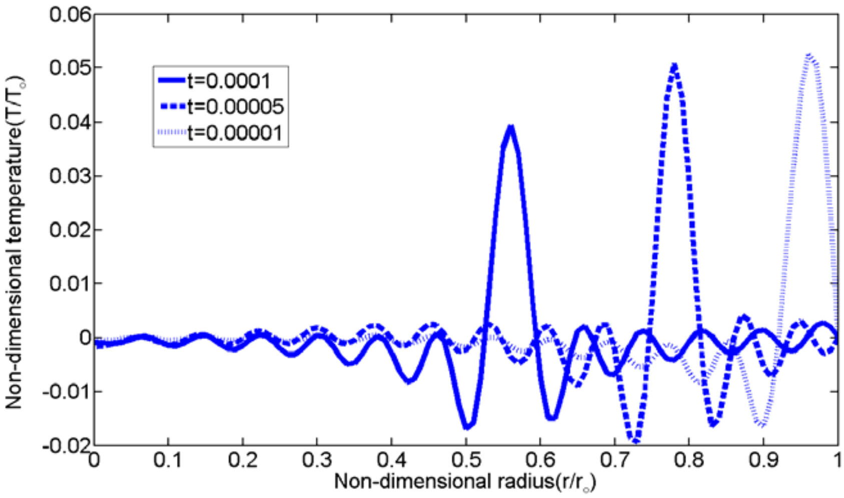

Figure 1.b Non-dimensional temperature distribution due to input $T(1,t)=10^{-3}\,T_0\delta(t)$

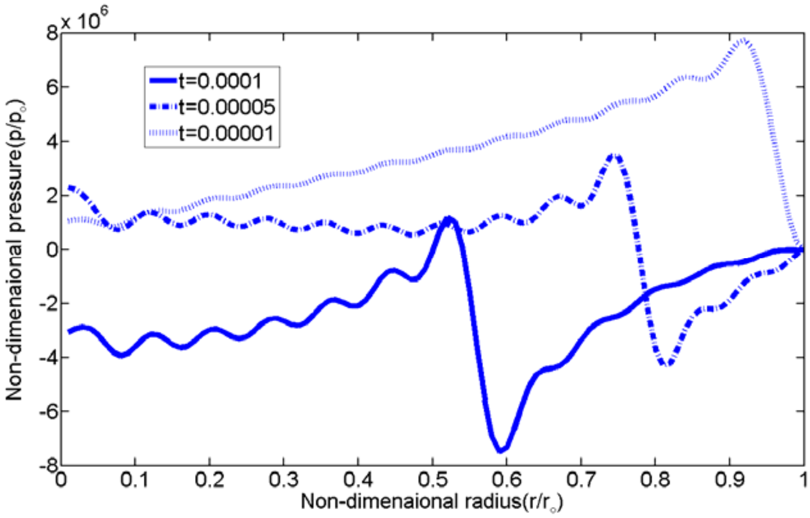

Figure 1.c Non-dimensional pressure distribution due to input $T(1,t)=10^{-3}\,T_0\delta(t)$

For the second example, a Heaviside function is applied to the outside surface of the cylinder given as T(1,t)=10^{-3} H(t), and the surface is assumed to be at zero temperature (u(1,t)=0 and p(1,t)=0). Figures 2a to 2c show variations for the displacement, temperature and pressure at three times (0.0001, 0.00005, 0.00001 second).

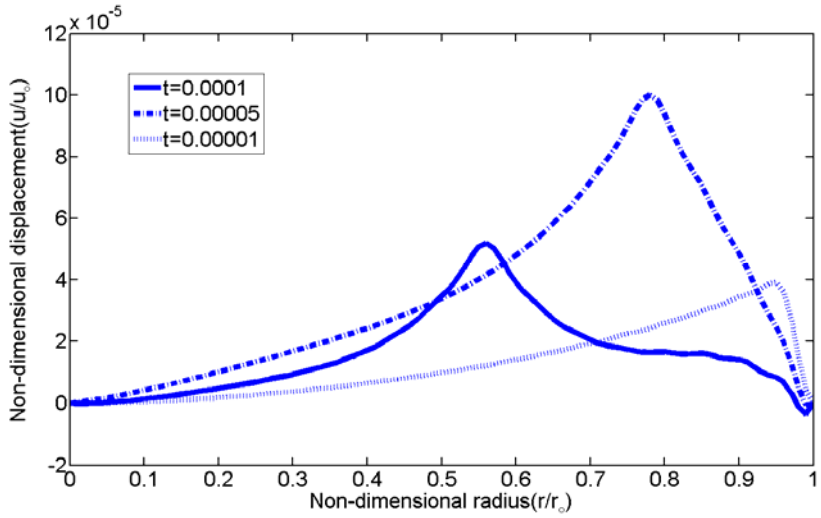

Figure 2.a Non-dimensional displacement distribution due to input $T(1,t)=10^{-3}H(t)$

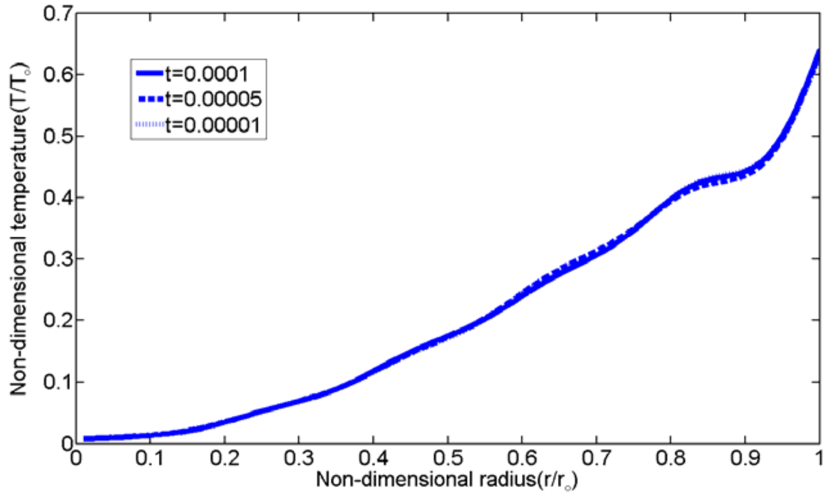

Figure2.b Non-dimensional temperature distribution due to input $T(1,t)=10^{-3}H(t)$

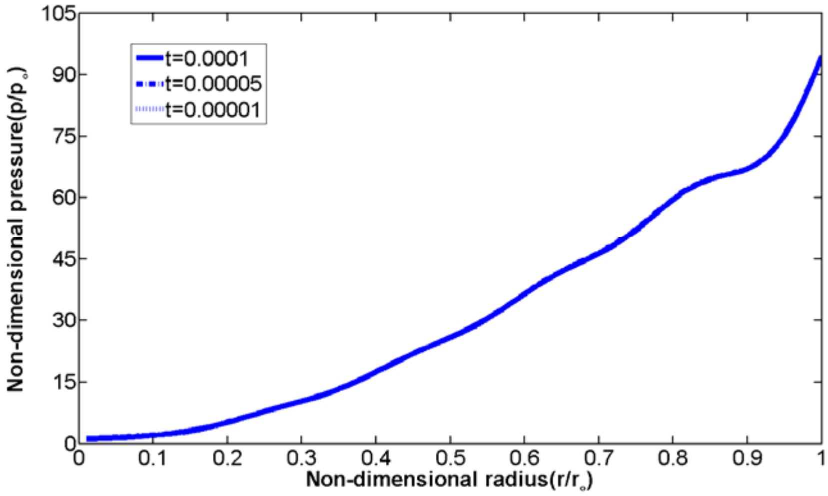

Figure2.c Non-dimensional pressure distribution due to input $T(1,t)=10^{-3}H(t)$

For the third example, a sinosial function is applied to the outside surface of the cylinder given as $T(1,t)=\sin(10^{-3}t)$, and the surface is assumed to be at zero temperature (u(1,t)=0 and p(1,t)=0). Figures 3a to 3c show variations for the displacement, temperature and pressure at three times (0.0001, 0.00005, 0.00001 second).

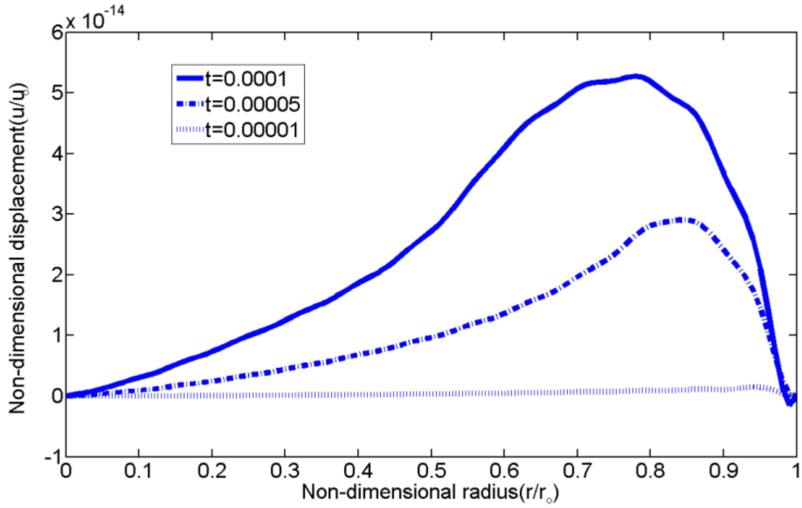

Figure3.a Non-dimensional displacement distribution due to input $T(1,t)=\sin(10^{-3}t)$

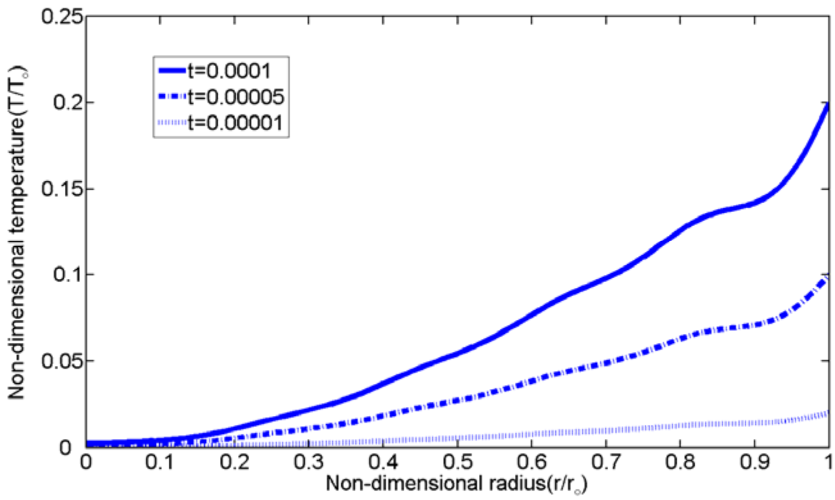

Figure3.b Non-dimensional temperature distribution due to input $T(1,t)=\sin(10^{-3}t)$

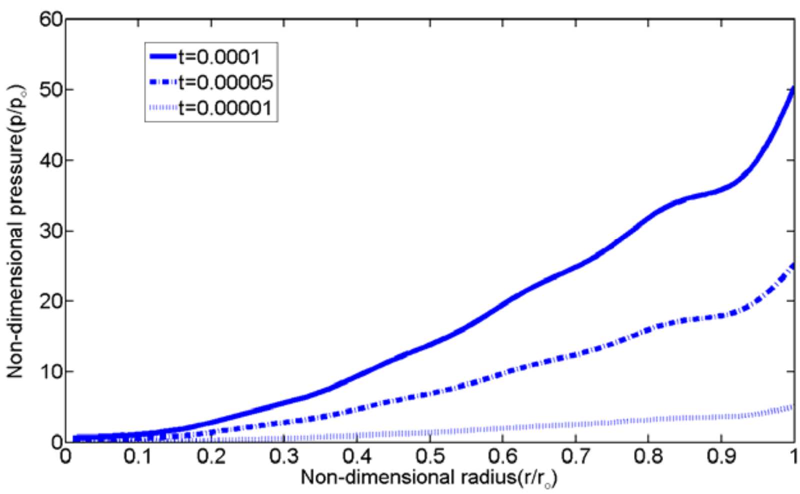

Figure 3.c Non-dimensional pressure distribution due to input $T(1,t)=\sin(10^{-3}t)$

5 Conclusion

In this paper analytical solution for the Quasi-Static porothermoelasticity of thick cylinders under radial temperature is presented. The method is based on the eigenfunctions Fourier expansion, which is a classical and traditional method of solution of the typical initial and boundary value problems. The non-competetive strength of this method is its ability to reveal the fundamental mathematical and physical properties and interpretations of the problem under studying. In the Quasi-Static porothermoelastic problem of radial-symmetric cylinder, the governing equations are a system of partial differential equations with two independent variables, radius (r) and time (t). The traditional procedure to solve this class of problems is to eliminate the time variable by using the Laplace transform. The resulting system is a set of ordinary differential equations in terms of the radius variable, which falls in the Bessel functions family. This method of analysis brings the Laplace parameter (s) in the argument of the Bessel functions, causing hardship or impossibility in carrying out the exact inverse of the Laplace transformation. As a result, the numerical inverse of the Laplace transformation is used in the papers dealing with this type of problems in literature. In the present paper, to prevent this problem, when the Laplace transform is applied to the particular solutions, it is postponed after eliminating the radius variable r by H-Fourier Expansion. Thus, the Laplace parameter (s) appears in polynomial function forms and hence the exact Laplace inversion transformation is possible.

6 Acknowledgments

The present research work is supported by Islamic Azad University, South-Tehran Branch.

References

[1] Bai, B., "Response of Saturated Porous Media Subjected to Local Thermal Loading on the Surface of Semi-infinite Space", Acta Mech. Sinica, Vol. 22, pp. 54-61, (2006).

[2] Bai, B., "Fluctuation Responses of Saturated Porous Media Subjected to Cyclic Thermal Loading", Computers and Geotechnics, Vol. 33, pp. 396-403, (2006).

[3] Droujinine, A., "Generalized Anelastic Asymptotic Ray Theory", Wave Motion, Vol. 43 pp. 357-367, (2006).

[4] Bai, B., and Li, T., "Solution for Cylindrical Cavety in Saturated Thermoporoelastic Medium" Acta Mech. Sinica, Vol. 22, No. 1, pp. 85-92, (2009).

[5] Hetnarski, R.B., "Solution of the Quasi-static Problem of Thermoelasticity in the Form of Series of Functions", Arch. Mech. Stos., Vol. 16, pp. 919-941, (1964).

[6] Hetnarski, R.B., and Ignaczak, J., "Generalized Thermoelasticity: Closed-form Solutions" J. Thermal Stresses, Vol. 16, pp. 473-498, (1993).

[7] Hetnarski, R.B., and Ignaczak, J., "Generalized Thermoelasticity: Response of Semi-space to a Short Laser Pulse", J. Thermal Stresses, Vol. 17, pp. 377-396, (1994).

[8] Georgiadis, H.G., and Lykotrafitis, G., "Rayleigh Waves Generated by a Thermal Source: A Three Dimensional Transient Thermoelasticity Solution", J. Applied Mechanics, Vol. 72, pp. 129-138, (2005).

[9] Wagner, P., "Fundamental Matrix of the System of Dynamic Linear Thermoelasticity" J. Thermal stresses, Vol. 17, pp. 549-565, (1994).

[10] Smith, D.W., and Booker, J.R, " Green's Functions for a Fully Coupled Thermoporoelastic Material", Int. J. Numer Anal. Meth. Geo. Mech. Vol. 17, pp. 139–163, (1993).

[11] Kosheleva, V., and Ghassemi, A., "Complex Variable BEM for Thermo and Poroelasticity" Engineering Analysis with Boundary Elements, Vol. 28, pp. 825–832, (2004).

[12] Zhou, X., and Ghassemi A., "Finite Element Analysis of Coupled Chemo-poro-thermo-Mechanical Effects around a Wellbore in Swelling Shale", International Journal of Rock Mechanics & Mining Sciences, Vol. 46, pp. 769–778, (2009).

Nomenclature

c_w : Heat capacities of pore water
c_s : Heat capacities of solid grains
k : Hydraulic conductivity
n : The porosity
u : Displacement component in the radial direction
P : Pore pressure component
C : Coefficient of volumetric compression of solid skeleton
C_s : Coefficient of volumetric compression of solid grains
C_w : Volumetric compression of pure water
E_s : Elastic modulus of solid grains
K : Coefficient of heat conductivity
Y : Coupling parameter
T : Temperature component
Z : Coupling parameter
E : Elastic modulus of solid skeleton
$T_\circ$: Initial reference temperature

Greek symbols

α_s : Coefficient of linear thermal expansion of solid grains
α_p : Coupling parameter
α_w : Coefficients of linear thermal expansion
β : Thermal expansion factor
γ_w : Unit of pore wate
ν : Poission's ratio of solid skeleton

ν_s : Poission's ratio of solid grains

ρ_w : Densities of pore water

ρ_s : Densities of solid grains

Appendix

$$d_1 = -\alpha \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}, \qquad d_2 = -\beta \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}, \qquad d_3 = -\rho \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}$$

$$d_4 = -Z\frac{T_\circ}{K}, \qquad d_5 = Y\frac{T_\circ}{K}, \qquad d_6 = -\beta\frac{T_\circ}{K}, \qquad d_7 = -\alpha_p \frac{\gamma_w}{k}\frac{1}{M}, \qquad d_8 = Y\frac{\gamma_w}{k}$$

$$d_9 = -\alpha\frac{\eta}{k}, \qquad d_{10} = -\frac{(1+\nu)(1-2\nu)}{(1-\nu)E}, \qquad d_{11} = -\frac{1}{K}, \qquad d_{12} = \frac{\eta}{K}$$

$$d_{13} = \int_0^1 F(r)rdr, \qquad d_{14} = \int_0^1 G(r)rdr, \qquad d_{15} = \int_0^1 W(r)rdr$$

$$d_{16} = \frac{7}{4}\int_0^1 rdr, \qquad d_{17} = \frac{3}{2}d_2\int_0^1 r^2dr, \qquad d_{18} = \frac{3}{2}d_2\int_0^1 r^2dr, \qquad d_{19} = d_3\int_0^1 r^3dr, \qquad d_{20} = \frac{15}{4}\int_0^1 rdr$$

$$d_{21} = \frac{7}{2}d_6\int_0^1 r^2dr, \qquad d_{22} = d_4\int_0^1 r^3dr, \qquad d_{23} = d_5\int_0^1 r^3dr, \qquad d_{24} = \frac{15}{4}\int_0^1 rdr, \qquad d_{25} = \frac{7}{2}d_9\int_0^1 r^2dr$$

$$d_{26} = d_8\int_0^1 r^3dr, \qquad d_{27} = d_7\int_0^1 r^3dr$$

$$\zeta_1 = \zeta_5 = \zeta_{12} = \frac{\left\{m_4 - \frac{m_1 m_6}{m_3}\right\}}{\left\{\frac{m_2 m_6}{m_3} - m_5\right\}}$$

$$\zeta_2 = \zeta_7 = \zeta_{15} = \left\{-\frac{m_1}{m_3} - \frac{m_2}{m_3}\zeta_1\right\}$$

$$\xi_3 = \frac{\left\{\left\{m_8 - m_9\frac{m_2}{m_3}\right\}\frac{\left\{m_{11}\zeta_5 - \frac{\{m_{10} - d_2\zeta_5 - d_1\zeta_7\}m_6}{m_3}\right\}}{\left\{m_5 - \frac{m_2 m_6}{m_3}\right\}} - m_{11}\zeta_7 + m_9\frac{\{m_{10} - d_2\zeta_5 - d_1\zeta_7\}}{m_3}\right\}}{\left\{m_7 - \frac{\left\{m_4 - \frac{m_1 m_6}{m_3}\right\}}{\left\{m_5 - \frac{m_2 m_6}{m_3}\right\}}m_8 - m_9\frac{m_1}{m_3}A_2 + m_9\frac{m_2}{m_3}\frac{\left\{m_4 - \frac{m_1 m_6}{m_3}\right\}}{\left\{m_5 - \frac{m_2 m_6}{m_3}\right\}}\right\}}$$

$$\xi_4 = \left\{ -\frac{\left\{ m_4 - \frac{m_1 m_6}{m_3} \right\}}{\left\{ m_5 - \frac{m_2 m_6}{m_3} \right\}} \xi_3 - \frac{\left\{ m_{11}\zeta_5 - \frac{\{m_{10} - d_2\zeta_5 - d_1\zeta_7\} m_6}{m_3} \right\}}{\left\{ m_5 - \frac{m_2 m_6}{m_3} \right\}} \right\}$$

$$\xi_6 = \left\{ -\frac{m_1}{m_3}\xi_3 - \frac{\{m_{10} - d_2\zeta_5 - d_1\zeta_7\}}{m_3} - \frac{m_2}{m_3}\xi_4 \right\}$$

$$\xi_8 = -\frac{\left\{ \begin{array}{l} \dfrac{\left\{ -m_3\dfrac{m_{11}}{m_6} - d_2 - m_{14}\dfrac{m_8}{m_9} \right\}}{\left\{ m_2 - m_3\dfrac{m_5}{m_6} \right\}\left\{ m_2 - m_3\dfrac{m_8}{m_9} \right\}} \left\{ \left\{ -m_{24} - m_3\dfrac{m_{26}}{m_9} \right\} + \left\{ -m_9\dfrac{m_5}{m_6} + m_8 \right\}\left\{ m_1 - m_3\dfrac{m_7}{m_9} \right\}\zeta_9 \right\} \\ + \dfrac{\left\{ -m_9\dfrac{m_{11}}{m_6} - m_{11}\dfrac{m_8}{m_9} \right\}}{\left\{ m_2 - m_3\dfrac{m_8}{m_9} \right\}} \left\{ -\left\{ -m_{24} - m_3\dfrac{m_{26}}{m_9} \right\} - \left\{ m_1 - m_3\dfrac{m_7}{m_9} \right\}\zeta_9 \right\} - m_{11}\dfrac{m_{26}}{m_9} - m_{11}\dfrac{m_7}{m_9}\zeta_9 \\ + \dfrac{\left\{ -m_9\dfrac{m_5}{m_6} + m_8 \right\}}{\left\{ m_2 - m_3\dfrac{m_5}{m_6} \right\}} \left\{ \begin{array}{l} -\left\{ m_{23} - m_{14}\dfrac{m_{26}}{m_9} \right\} + \dfrac{\left\{ m_{10} - m_{14}\dfrac{m_7}{m_9} \right\}}{\left\{ m_2 - m_3\dfrac{m_8}{m_9} \right\}}\left\{ m_1 - m_3\dfrac{m_7}{m_9} \right\}\zeta_9 \\ + \dfrac{\left\{ m_{10} - m_{14}\dfrac{m_7}{m_9} \right\}}{\left\{ m_2 - m_3\dfrac{m_8}{m_9} \right\}}\left\{ -m_{24} - m_3\dfrac{m_{26}}{m_9} \right\} \end{array} \right\} \end{array} \right\}}{\left\{ m_7 - m_9\dfrac{m_4}{m_6} \right\} - \left\{ -m_9\dfrac{m_5}{m_6} + m_8 \right\}\dfrac{\left\{ m_1 - m_3\dfrac{m_4}{m_6} \right\}}{\left\{ m_2 - m_3\dfrac{m_5}{m_6} \right\}}}$$

$$\zeta_9 = -\frac{\left\{ \left\{ +m_{25} - m_6\dfrac{m_{26}}{m_9} \right\} - \left\{ m_5 - m_6\dfrac{m_8}{m_9} \right\}\dfrac{\left\{ -m_{24} - m_3\dfrac{m_{26}}{m_9} \right\}}{\left\{ m_2 - m_3\dfrac{m_8}{m_9} \right\}} \right\}}{\left\{ \left\{ m_4 - m_6\dfrac{m_7}{m_9} \right\} - \left\{ m_5 - m_6\dfrac{m_8}{m_9} \right\}\dfrac{\left\{ m_1 - m_3\dfrac{m_7}{m_9} \right\}}{\left\{ m_2 - m_3\dfrac{m_8}{m_9} \right\}} \right\}}$$

$$\zeta_{10} = \frac{1}{\left\{m_2 - m_3 \frac{m_5}{m_6}\right\}} \left\{ \begin{array}{l} -\left\{m_1 - m_3 \frac{m_4}{m_6}\right\}\xi_8 + \left\{m_{10} - m_{14} \frac{m_7}{m_9}\right\} \frac{\left\{-m_{24} - m_3 \frac{m_{26}}{m_9}\right\}}{\left\{m_2 - m_3 \frac{m_8}{m_9}\right\}} - \left\{m_{23} - m_{14} \frac{m_{26}}{m_9}\right\} \\ + \frac{\left\{m_1 - m_3 \frac{m_7}{m_9}\right\}}{\left\{m_2 - m_3 \frac{m_8}{m_9}\right\}} \left\{ \left\{m_{10} - m_{14} \frac{m_7}{m_9}\right\}\zeta_9 + \left\{-m_3 \frac{m_{11}}{m_6} - d_2 - m_{14} \frac{m_8}{m_9}\right\}\zeta_9 \right\} \\ + \left\{-m_3 \frac{m_{11}}{m_6} - d_2 - m_{14} \frac{m_8}{m_9}\right\} \frac{\left\{-m_{24} - m_3 \frac{m_{26}}{m_9}\right\}}{\left\{m_2 - m_3 \frac{m_8}{m_9}\right\}} \end{array} \right\}$$

$$\zeta_{11} = \left\{ -\frac{\left\{m_1 - m_3 \frac{m_7}{m_9}\right\}}{\left\{m_2 - m_3 \frac{m_8}{m_9}\right\}}\zeta_9 - \frac{\left\{-m_{24} - m_3 \frac{m_{26}}{m_9}\right\}}{\left\{m_2 - m_3 \frac{m_8}{m_9}\right\}} \right\}$$

$$\zeta_{13} = \left\{ -\frac{m_4}{m_6}\xi_8 - \frac{m_5}{m_6}\zeta_{10} - \frac{m_{11}}{m_6}\zeta_{11} \right\}$$

$$\zeta_{14} = \left\{ -\frac{m_{26}}{m_9} - \frac{m_8}{m_9}\zeta_{11} - \frac{m_7}{m_9}\zeta_9 \right\}$$

$$m_1 = -\beta^2 + \lambda^2 d_3, \qquad m_2 = -d_2\beta, \qquad m_3 = -d_1\beta$$

$$m_4 = \lambda d_6 \beta, \qquad m_5 = -\beta^2 + \lambda d_4, \qquad m_6 = \lambda d_5$$

$$m_7 = \beta\lambda d_9, \qquad m_8 = \lambda d_8, \qquad m_9 = -\beta^2 + \lambda d_7$$

$$m_{10} = d_3 \lambda^2 \frac{3}{\beta} - \beta, \qquad m_{11} = 2\beta, \qquad m_{12} = -3 + d_3 \lambda^2 \frac{15}{\beta^2}$$

$$m_{13} = -3d_2 \frac{1}{\beta}, \qquad m_{14} = -\frac{3}{2} d_1, \qquad m_{15} = -\frac{3}{\beta} d_1$$

$$m_{16} = \beta - \frac{5}{\beta} d_3 \lambda^2, \qquad m_{17} = \lambda d_4 \frac{3}{\beta} + \beta, \qquad , m_{18} = \beta + d_7 \lambda \frac{3}{\beta}$$

$$m_{19} = d_5 \frac{3}{\beta}\lambda, \qquad m_{20} = d_8 \frac{3}{\beta}\lambda, \qquad m_{21} = 3\lambda d_6$$

$$m_{22} = 3d_9\lambda, \qquad m_{23} = m_{12} + m_{13}\zeta_{12} + m_{15}\zeta_{15}$$

$$m_{24} = m_{16} - m_{14}\zeta_{15} - \frac{1}{2} d_1 \zeta_{15} + d_2 \zeta_{12}$$

$$m_{25} = m_{21} + m_{17}\zeta_{12} + m_{19}\zeta_{15}$$

$$m_{26} = m_{18}\zeta_{15} + m_{20}\zeta_{12} + m_{22}$$

$$
\begin{aligned}
C_0 &= 1+\zeta_{16}+\zeta_{17} \\
C_1 &= \zeta_1\zeta_{16}+\zeta_2\zeta_{17} \\
C_2 &= \zeta_3\zeta_{17} \\
E_0 &= \zeta_4+\zeta_5\zeta_{16}+\zeta_7\zeta_{17} \\
E_1 &= \zeta_6\zeta_{16}+\zeta_8\zeta_{17} \\
E_2 &= \zeta_9\zeta_{17} \\
D_0 &= \zeta_{10}+\zeta_{11}\zeta_{16}+\zeta_{13}\zeta_{17} \\
D_1 &= \zeta_{12}\zeta_{16}+\zeta_{14}\zeta_{17} \\
D_2 &= \zeta_{15}\zeta_{17}
\end{aligned}
$$

$$
\begin{aligned}
H_0 &= C_0 J_{\frac{3}{2}}(\beta r)+C_1 r J_{\frac{5}{2}}(\beta r)+C_2 r^2 J_{\frac{7}{2}}(\beta r) \\
H_1 &= E_0 J_{\frac{1}{2}}(\beta r)+E_1 r J_{\frac{3}{2}}(\beta r)+E_2 r^2 J_{\frac{5}{2}}(\beta r) \\
H_2 &= D_0 J_{\frac{1}{2}}(\beta r)+D_1 r J_{\frac{3}{2}}(\beta r)+D_2 r^2 J_{\frac{5}{2}}(\beta r)
\end{aligned}
$$

چکیده

در این پروژه یک کره متخلخل توپر و توخالی شبه استاتیک تحت شوکهای مکانیکی، حرارتی یک بعدی (r,t) مورد بررسی قرار می گیرد.یک روش کاملاَ تحلیلی بکار برده شده و حل یکتائی برای معادلات شبه استاتیک ارائه شده است. شرایط مرزی حرارتی مکانیکی و فشار و نیز منبع تولید گرما و نیروهای حجمی وحجم تزریق شده بر واحد حجم یک منبع توزیع آب در حالت عمومی بررسی شده و این روش هیچ محدودیتی در استفاده ندارد و این عمومیت اجازه می دهد که محدوده وسیعی از مسائل کاربردی را پوشش دهد. در این یک مسئله ترموالاستیک متخلخل کوپله، شامل دیسک توپر تحت شوکهای حرارت و مکانیکی بررسی می شود. ابتدا معادلات کوپله محیط متخلخل با معادلات انتقال حرارت و الاستیسته استخراج می شود. ویژگی بارز این تحقیق ارائه یک روش مستقیم جهت حل دقیق مسئله کوپله است. شیوه ای که دارای مزایای بیشتری نسبت به روشهای عددی مثل اجزا محدود وروش تابع پتانسیل مرسوم برای حل معادلات فوق می باشد ومحدودیت های تابع پتانسیل درمورد شرایط مرزی را ندارد با توجه به مقدمات بالا لازم است فرمولبندی کاملا کوپله بین معادله حاکم بر مساله عرضه شود .