Approximate Analytical Solution to Flow over a Flat Plate by Variational Iteration Method

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ABSTRACT

Variational iteration method is employed to investigate the flow over a flat plate. General Lagrange multipliers are introduced in this method to construct correction functionals for problems. The multipliers in the functionals can be identified optimally via variational theory. Comparison with Adomian decomposition method and Howarth's numerical solution reveals that the approximate solutions obtained by the proposed method are of high accuracy.

Key words: Variational Iteration Method, Flow over a Flat Plate, Blasius equation**.**

حل تحلیلی تقریبی جریان بر روی صفحه تخت به روش وردشی تکراری

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چکیده

در این مقاله مطالعه‌ای تحلیلی برجریان روی صفحه تخت به روش وردشی تکراری انجام شده است. در این روش به منظور بدست آوردن فانکشنال از ضرایب لاگرانژ دریک چرخه تکراری استفاده شده است. به کمک روش محاسبات وردشی این ضرایب بصورت بهینه قابل تعیین می‌باشند. روش وردشی تکراری برای حل مساله بلازیوس بصورت تحلیلی تقریبی بکار گرفته شده و نتایج حاصل با نتایج موجود عددی و تحلیلی واهم‌نهشتی ادومین مقایسه شده است. دقت بالا در تکرارهای کم نشان دهنده قابلیت این روش در حل مسائل مشابه می‌باشد.

واژه‌های کلیدی: روش وردشی تکراری، جریان بر روی صفحه تخت، معادله بلازیوس

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1. Introduction

The Falkner-Skan equation describes a nonlinear, one-dimensional third-order boundary value problem, whose solutions are the similarity solutions of the two-dimensional incompressible laminar boundary layer equations. No closed-form solutions are available for this two-point boundary value problem [1].

In 1908, Blasius [2] gave a solution in the form of a power series. Howarth [3], in 1938, used the Runge-Kutta numerical method and did hand computations to analyze the flat-plate flow. Lock [4, 5], in 1951, studied the laminar boundary layer between parallel streams. Later, Potter [6], in 1957, investigated laminar boundary layer solutions for mass transfer across the plane interface between two-current parallel fluid streams. Blasius solution for a flow past a flat-plate was investigated by Abussita, in 1994, and the existence of a solution was established [7]. Asaithambi [1], in 1998, presented a finite-difference method for the solution of the Falkner-Skan equation. Liao [8, 9], in 1998, gave an analytical solution in a family of power series with parameter $\hbar$, by means of the homotopy analysis method, which is valid in the whole region $x \in [0,+\infty)$. Jianguo Lin [10], in 1999, obtained an analytical solution by use of parameter iteration method. Recently, Abbasbandy [11], Wang [12] and Najafi et. al. [13] obtained an approximate solution to Blasius equation using Adomian decomposition method.

The variational iteration method was first proposed by Ji-Huan He in 1998 [14, 15] who systematically illustrated in 1999 [16]. It was successfully applied to autonomous ordinary differential equations in [17], to nonlinear wave equations [18], to circuit theory [19], to nonlinear polycrystalline solids [20] and to some other fields, as well. A combination of a perturbation method, the variational iteration method, the method of variation of constants and an averaging method were used to establish an approximate solution to a one degree of freedom weakly nonlinear system [21]. The variational iteration method has many merits and has many advantages over the Adomian decomposition method [13, 22].

The present work is motivated by the desire to obtain analytical solution to classical Blasius equation using variational iteration method. The results obtained via variational iteration method (VIM) are compared with the numerical solutions [13, 23, 24] which confirms the validity of the proposed method. The differential equations will be taken in their dimensional form [25]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{3}$$

Here, p is also independent of z. Since a plate placed edgewise will not greatly disturb a uniform stream, as suggested by experience with air and water, an ideal flow of primary interest is the uniform flow [25]:

$$u \equiv U_\infty,\ v \equiv 0 \text{ for all } x, y. \tag{4}$$

The classical Flat-Plate Problem is defined as that of finding a solution of (1-3) differing from (4) only in the neighborhood of the plate the extent of which shrinks to zero as $\nu \to 0$.

Since it is assumed the velocity of potential flow is constant, then $dp/dx \equiv 0$. The boundary-layer equations reduce to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{6}$$

$$\begin{aligned} &y = 0:\ u = v = 0, \\ &y = \infty:\ u = U_\infty. \end{aligned} \tag{7}$$

By reasoning there is no preferred length; one can suppose the velocity profiles at varying distances from the leading edge are similar to each other. The velocity curves, u(y), for varying distances, x, can be made identical by selecting suitable scale factors for u and y, the problem of affinity or similarity of velocity profiles. The scale factors for u and y appear quite naturally as the free-stream velocity, U_∞, and the boundary-layer thickness, δ_x, respectively. The thickness increases with the current distance, x. Hence the principle of similarity of velocity profiles in the boundary-layer can be written as:

$$\frac{u}{U_\infty} = f\left(\frac{y}{\delta}\right), \tag{8}$$

Where, the function f must be the same at all distances, x, from the leading edge. It was found that $\delta \sim \sqrt{\nu t}$, where t denoted the time from the start of the motion. In this case, time is substituted by the time consumed by a fluid particle while traveling from the leading edge to the point x. For a particle outside the boundary layer this is $t=x/U_\infty$. Hence $\delta \sim \sqrt{\nu x/U_\infty}$.

By introducing new dimensionless coordinate, $\eta \sim y/\delta$, both partial differential equations (5,6) will be transformed into an ordinary differential equation for the stream function by the following similarity (stretching) transformations:

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}, \tag{9}$$

$$\psi = \sqrt{\nu x U_\infty}\, f(\eta), \tag{10}$$

$$f(0)=0,\quad f'(0)=0,\quad f'(\infty)=1, \tag{11}$$

Where, $f(\eta)$ denotes the dimensionless stream function. The resulting differential equation is nonlinear and of third order. The classical Blasius equation is:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0. \tag{12}$$

It is a special case of two-dimensional laminar boundary layer flows over a semi-infinite flat plate which is governed by Falkner-Skan equation:

$$f'''(\eta) + \frac{1}{2}(m+1) f(\eta) f''(\eta) + m[1-(f'(\eta))^2] = 0, \tag{13}$$

subject to the boundary conditions:

$$f(0)=0,\quad f'(0)=0,\quad f'(\infty)=1, \tag{14}$$

where m is a constant. As long as m>0, the solutions are known to exist and they are also unique [26].

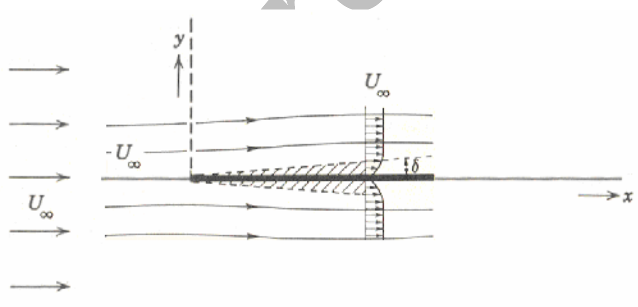

Fig. 1. Boundary layer over a flat plate.

For $m \geq 0$ they are:

- m = 0: Blasius flow over a flat plate with a sharp edge.
- 0<m<1: Flow over a wedge with half angle $\theta_{1/2}=m\pi/(m+1)$ with $0<\theta_{1/2}<\pi/2$.
- m = 1: Heimenz flow toward a plane stagnation point.
- 1<m<2: Flow into a corner with $\theta_{1/2}>\pi/2$.
- 2<m: No corresponding simple ideal flow.

Here, the special case, m=0 as in Fig. 1 is studied.

2. Variational iteration method

The variational iteration method has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. To illustrate the basic concepts of the variational iteration method, consider the following differential equation:

$$Lu + Nu = g(x), \tag{15}$$

Where, L is a linear operator, N a nonlinear operator, and g(x) a known analytic function. According to He's variational iteration method [14-22], A correction functional can be constructed as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n(\tau) + N\hat{u}_n(\tau) - g(\tau)]d\tau, \tag{16}$$

Where, λ is a general Lagrangian multiplier [14-23], which can be identified optimally via the variational theory. The subscript n denotes the nth order approximation, $\hat{u}_n$ is considered as a restricted variation, i.e. $\delta\hat{u}_n = 0$.

Eq. (16) is called a correction functional. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that Lagrange multiplier can be exactly identified.

3. Stationary conditions

The problem of optimization is ubiquitous in nature. The simplest problem of the calculus of variation [16] is to determine a function y = f (x) for which the value of a given functional:

$$J = \int_{x_1}^{x_2} F(y, y'; x)dx + g_1(x)y\big|_{x=x_1} - g_2(x)y\big|_{x=x_2}, \tag{17}$$

is a maximum or a minimum. The extremum condition (stationary condition) of the functional (17) requires that:

$$\delta J = \delta\int_{x_1}^{x_2} F(y,y';x)dx + g_1\delta y\big|_{x=x_1} - g_2\delta y\big|_{x=x_2}$$
$$= \int_{x_1}^{x_2} \delta F(y,y';x)dx + g_1\delta y\big|_{x=x_1} - g_2\delta y\big|_{x=x_2}$$
$$= \int_{x_1}^{x_2}\left\{\frac{dF}{dy}\delta y + \frac{dF}{dy'}\delta y'\right\}dx + g_1\delta y\big|_{x=x_1} - g_2\delta y\big|_{x=x_2}$$
$$= \int_{x_1}^{x_2}\left\{\frac{dF}{dy}\delta y + \frac{dF}{dy'}\frac{d}{dx}(\delta y)\right\}dx + g_1\delta y\big|_{x=x_1} - g_2\delta y\big|_{x=x_2} \tag{18}$$
$$= \int_{x_1}^{x_2}\left\{\left[\frac{dF}{dy}\delta y - \frac{d}{dx}\left(\frac{dF}{dy'}\right)\right]\delta y + \frac{d}{dx}\left(\frac{dF}{dy'}\delta y\right)dx + g_1\delta y\big|_{x=x_1} - g_2\delta y\big|_{x=x_2}\right.$$
$$= \int_{x_1}^{x_2}\left\{\left[\frac{dF}{dy}\delta y - \frac{d}{dx}\left(\frac{dF}{dy'}\right)\right]\delta y\right\}dx + \left[\frac{dF}{dy'}\delta y\right]_{x_1}^{x_2} + g_1\delta y\big|_{x=x_1} - g_2\delta y\big|_{x=x_2}$$
$$= 0.$$

For arbitrary δy, from the above relation, we have:

$$\frac{dF}{dy} - \frac{d}{dx}\left(\frac{dF}{dy'}\right) = 0, \tag{19}$$

and the boundary conditions

$$\frac{dF}{dy'}(x_1) - g_1(x_1) = 0,$$
$$\frac{dF}{dy'}(x_2) - g_2(x_2) = 0. \tag{20}$$

Equation (19) is called Euler–Lagrange's differential equation, or Euler's equation, and Equation (20) is known as the natural boundary conditions.

4. Application to the Blasius equation

The correction variational functional for Blasius equation can be expressed as follows:

$$f_{n+1}(\eta) = f_n(\eta) + \int_0^{+\infty}\lambda(f'''(\tau) + \frac{1}{2}\hat{f}''(\tau)\hat{f}(\tau))d\tau, \tag{21}$$

Where, λ is a general Lagrangian multiplier. Making the correction functional Equation (21), stationary, noticing that:

$$\delta\hat{f}_n = 0, \quad \delta\hat{f}''_n = 0,$$

$$\delta f_{n+1}(\eta) = \delta f_n(\eta) + \delta\int_0^{+\infty}\lambda(f'''(\tau) + \frac{1}{2}\hat{f}''(\tau)\hat{f}(\tau))d\tau, \tag{22}$$

$$\delta f_{n+1}(\eta) = \delta f_n(\eta) + \left(\frac{\partial^2\lambda}{\partial\tau^2}\delta f_n(\tau)\right)\bigg|_{\tau=\eta} - \left(\frac{\partial\lambda}{\partial\tau}\delta f_n'(\tau)\right)\bigg|_{\tau=\eta} + (\lambda\delta f_n''(\tau))\big|_{\tau=\eta} - \int_0^{+\infty}\left(\frac{\partial^3\lambda}{\partial\tau^3}\delta f_n(\tau)\right)d\tau = 0, \tag{23}$$

yields the following stationary conditions:

$$\delta f_n:\ 1 + \frac{\partial^2\lambda}{\partial\tau^2}\bigg|_{\tau=\eta} = 0,$$
$$\delta f_n':\ 1 + \frac{\partial\lambda}{\partial\tau}\bigg|_{\tau=\eta} = 0,$$
$$\delta f_n'':\ \lambda\big|_{\tau=\eta} = 0, \tag{24}$$
$$\delta f_n:\ \frac{\partial^3\lambda}{\partial\tau^3}\bigg|_{\tau=\eta} = 0.$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda(\tau) = -\frac{1}{2}(\tau-\eta)^2. \tag{25}$$

As a result, the following iteration formula is obtained:

$$f_{n+1}(\eta) = f_n(\eta) - \frac{1}{2}\int_0^{+\infty}(\tau-\eta)^2\left(f'''(\tau) + \frac{1}{2}f''(\tau)f(\tau)\right)d\tau. \tag{26}$$

Starting with the initial condition given by Equation (11), f_0 is as follows:

$$f_0(\eta) = A + \eta B + \frac{1}{2}\eta^2 C \tag{27}$$

Where, A, B, C are unknown constants to be further determined by imposing initial conditions. By the iteration formula and using the initial conditions given by Equation (11), and assuming $\sigma = f''(0)$, one can obtain the following results:

$$f_0(\eta) = \frac{1}{2}\sigma\eta^2,$$
$$f_1(\eta) = \frac{1}{2}\sigma\eta^2 - \frac{1}{240}\sigma^2\eta^5,$$
$$f_2(\eta) = \frac{1}{2}\sigma\eta^2 - \frac{1}{240}\sigma^2\eta^5 + \frac{11}{161280}\sigma^3\eta^8 - \frac{1}{5702400}\sigma^4\eta^{11},$$
$$f_3(\eta) = \frac{1}{2}\sigma\eta^2 - \frac{1}{240}\sigma^2\eta^5 + \frac{11}{161280}\sigma^3\eta^8 - \frac{5}{4257792}\sigma^4\eta^{11} + \frac{10033}{1394852659200}\sigma^5\eta^{14} - \frac{5449}{125076897792000}\sigma^6\eta^{17} + \frac{83}{571875655680000}\sigma^7\eta^{20} - \frac{1}{6282355064832000}\sigma^8\eta^{23}, \quad (28)$$

and so on, the rest of the components of the iteration formula (22) can be obtained using symbolic packages such as Maple. In 1938, Howarth [2] evaluated σ by means of a numerical technique. Inserting his value for σ, $\sigma = 0.332057$, into Equation (28) leads to the approximate solution of Blasius equation.

5. Results and discussion

The calculations are conducted up to five iterations and the results are shown in Figure **2** From this figure, it is obvious that the results are in good agreement with those reported in [2]. In table 1, we were compared the results for some iterations with the results obtained by five terms of ADM and also with those given by Howarth [2]. As it was shown there, the results achieved by five iterations of VIM is more accurate than five-term ADM solution. The maximum relative error with respect to numerical solution is 0.018 % for the values of f. Some of the computed results for the variations with η of the functions f′ and f″ are listed in tables 2 and 3, respectively. It is evident that more accuracy will be achieved if more iteration is done, as it is shown in tables.

6. Conclusion

The classical Blasius equation has been analyzed using the variational iteration method. The variational iteration method supplies reliable results in the form of analytical approximation. The results show that variational iteration method is an effective mathematical tool which can play a very important role in nonlinear sciences.

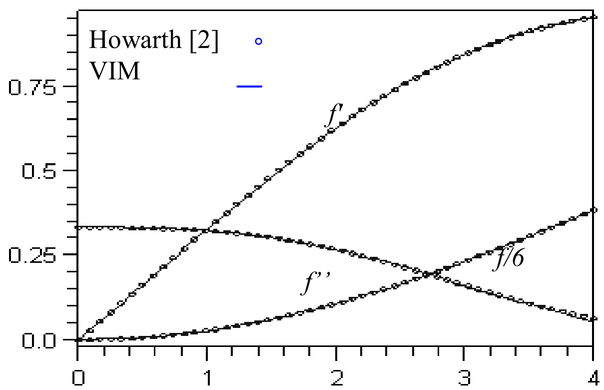

Fig. 2. Variation with η of the functions f, f′ and f″.

Table 1 Comparison between the results of VIM, ADM [13] and NS-numerical solution [1] for $f(\eta)$.

η	Iteration 2	Iteration 3	Iteration 4	Iteration 5	5 terms of ADM	NS
0	0	0	0	0	0	0
0.4	0.0265598571	0.0265598571	0.0265598571	0.0265598571	0.0265598571	0.0265598545
0.8	0.1061081146	0.1061081135	0.1061081135	0.1061081136	0.1061081136	0.1061080991
1.2	0.2379485667	0.2379484768	0.2379484775	0.2379484774	0.2379484774	0.2379484616
1.6	0.4203224242	0.4203203084	0.4203203451	0.4203203446	0.4203203446	0.4203202793
2	0.6500473318	0.6500229272	0.6500237491	0.6500237248	0.6500237230	0.6500236442
2.4	0.9224582617	0.9222793955	0.9222897261	0.9222892012	0.9222891315	0.9222890925
2.8	1.2318523950	1.2308957420	1.2309825960	1.2309756580	1.2309741820	1.2309759131
3.2	1.5726632550	1.5686086130	1.5691512670	1.5690873780	1.5690668780	1.5690931694
3.6	1.9415750270	1.9272140110	1.9299109240	1.9294657590	1.9292598180	1.9295229916
4	2.3407183420	2.2966396300	2.3078123340	2.3053280130	2.3037288970	2.3057439186

Table 2 Comparison between the results of VIM, ADM [13] and NS-numerical solution [1] for $f'(\eta)$.

η	Iteration 2	Iteration 3	Iteration 4	Iteration 5	ADM	NS
0	0	0	0	0	0	0
0.4	0.1327640264	0.1327640264	0.1327640264	0.1327640264	0.1327640264	0.1327640265
0.8	0.2647088860	0.2647088716	0.2647088717	0.2647088717	0.2647088717	0.2647088692
1.2	0.3937765256	0.3937757028	0.3937757104	0.3937757104	0.3937757104	0.3937757132
1.6	0.5167704689	0.5167559613	0.5167562812	0.5167562753	0.5167562750	0.5167562518
2	0.6298931481	0.6297596134	0.6297653307	0.6297651251	0.6297651066	0.6297651170
2.4	0.7297377137	0.7289251308	0.7289847460	0.7289810721	0.7289804954	0.7289812478
2.8	0.8148264064	0.8111199451	0.8111199451	0.8111199451	0.8114952756	0.8115088384
3.2	0.8877132152	0.8740566837	0.8763746377	0.8760440910	0.8759182113	0.8760805951
3.6	0.9575540213	0.9149013723	0.9250542543	0.9230260695	0.9219093057	0.9233288380
4	1.0428838270	0.9261633706	0.9636603096	0.9535805869	0.9458328284	0.9555173538

Table 3 Comparison between the results of VIM, ADM [13] and NS-numerical solution [1] for $f''(\eta)$.

η	Iteration 2	Iteration 3	Iteration 4	Iteration 5	ADM	NS
0	0.332070000	0.3320700000	0.332070000	0.332070000	0.332057000	0.332057000
0.4	0.331469510	0.331469509	0.331469509	0.331469509	0.331469509	0.331469506
0.8	0.327389122	0.327388943	0.327388943	0.327388943	0.327388943	0.327388947
1.2	0.316595650	0.316588804	0.316588887	0.316588886	0.316588886	0.316588885
1.6	0.296751001	0.296660665	0.296663253	0.296663194	0.296663190	0.296663203
2	0.267378922	0.266716051	0.266752906	0.266751276	0.266751101	0.266751293
2.4	0.231140035	0.227795886	0.228114304	0.228090173	0.228085653	0.228091533
2.8	0.195259008	0.182275778	0.184216427	0.183985459	0.183915833	0.184006427
3.2	0.172871465	0.131389991	0.140520745	0.138921057	0.138189373	0.139127959
3.6	0.183946709	0.070045944	0.105217205	0.096590902	0.090864430	0.098086168
4	0.255308458	−0.021656840	0.093887536	0.05578987	0.020336915	0.064234085

References

1. Asaithambi, A., "A Second-order Finte-Difference Method for the Falkner-Skan equation", Applied Mathematics and Computation, Vol. 156, pp. 779-786, 2004.
2. Blasius, H., "Grenzschichten in Flussigkeiten mit kleiner reibung, Z. Math. u. Phys., Vol. 56, No. 1, 1908.
3. Howarth, L., "On the Solution of the Laminar Boundary Layer Equations", The Roy. Soc. London A, Vol. 164, No.919, pp. 547–579, 1938.
4. Lock, R.C., "The Velocity Distribution in the Laminar Boundary Layer Between Parallel Streams", Quart. J. Mech. Math., Vol. 4, No.1, pp. 42–63, 1951.
5. Lock, R.C., "Hydrodynamic Stability of the Flow in the Laminar Boundary Layer Between Parallel Streams", The Cambridge Phil. Soc., Vol. 50, No.1, pp. 105–124, 1954.
6. Potter, O.E., "Mass Transfer Between Co-Current Fluid Streams and Boundary Layer Solutions", Chem. Eng. Sci., Vol. 6, No.4-5, pp. 170–182, 1957.
7. Abussita, A.M.M., "A Note on a Certain Boundary-layer Equation", Applied Mathematics and Computation, Vol. 64, No.1, pp. 73–77, 1994.
8. Liao, SJ., "An Explicit, Totally Analytic Solution of Laminar Viscous Flow Over a Semi-infinite Flat Plate", Commun Nonlinear Sci. Numer. Simul., Vol. 3, No. 2, pp. 53–57, 1998.
9. Liao, SJ., "An Explicit, Totally Analytic Approximate Solution for Blasius' Viscous Flow Problem", Int J. Non-linear Mech., Vol. 34, pp. 759–778, 1999.
10. Jianguo, L., "A New Approximate Iteration Solution of Blasius Equation", Communications in Non-linear Science & Numerical Simulation, Vol. 4, No. 2, 1999.
11. Abbasbandy, S., "A numerical Solution of Blasius equation by Adomian's Decomposition Method and Comparison with Homotopy Perturbation Method", Chaos, Solitons and Fractals, Vol. 31, pp. 257-260, 2005.
12. Wang, L., "A New Algorithm for Solving Classical Blasius Equation", Applied Mathematics and Computation, Vol. 157, pp. 1–9, 2004.
13. Najafi M., Khoramishad, H., Massah H., Moghimi M., Daemi M., "A study of Blasius viscous flow: An ADM Analytical Solution", WSEAS Transactions on Mathematics, No. 7, Vol. 5, pp. 801-805, 2006.
14. He, J.H. "Approximate Analytical Solution for Seepage Fow with Fractional Derivatives in Porous Media", Computer Methods in Applied Mechanics and Eng., Vol. 167, No. 1, pp. 57-68, 1998.
15. He, J.H., "Approximate Solution of Non-linear Differential Equations with Convolution Product Non-linearities", Computer Methods in Applied Mechanics and Eng., Vol. 167, No. 1, pp. 69-73, 1998.
16. He, J.H., "Variational Iteration Method - a Kind of Non-linear Analytical Technique: Some Examples", Int. J. Non-linear Mechanics, Vol. 34, No. 4, pp. 699-708, 1999.
17. He, J.H., "Variational Iteration Method for Autonomous Ordinary Differential Systems", Applied Mathematics and Computation, Vol. 114, No. 2, pp. 115-123, 2000.
18. He, J.H., Wu, X.H., "Construction of Solitary Solution and Compacton-Like Solution by Variational Iteration Method", Chaos, Solitons & Fractals, Vol. 29, No. 1, pp. 108-113, 2006.
19. He, J.H., Y.Q. Wan, Guo, Q., "An Iteration Formulation for Normalized Diode Characteristics", International J. Circuit Theory and Applications, Vol. 32, No. 6, pp. 629-632, 2004.
20. Draganescu, Gh.E., Capalnasan, V. "Nonlinear Relaxation Phenomena in Polycrystalline Solids", International J. Nonlinear Sciences and Numerical Simulation, Vol. 4, No. 3, pp. 219–26, 2003.
21. Marinca, V., "An Approximate Solution for One-Dimensional Weakly Non-linear Oscillations", Int. J. Non-linear Sci. Numer. Simul., Vol. 3, No. 2, pp. 107–120, 2002.
22. Najafi M., Moghimi M., Massah H.R., Khoramishad H., Daemi M., Solving Oscillation Equations via ADM and VIM, WSEAS Transactions on Information Science & Applications, Vol. 3, No. 8, pp. 1429-1436, 2006
23. He, J.H., "Approximate Analytical Solution of Blasius' Equation", Commun Nonlinear Sci Numer Simul, Vol. 4, No. 1, pp. 75–78, 1999.
24. Cortell, R., "Numerical Solutions of the classical Blasius Flat-plate Problem", Applied Mathematics and Computation, Vol. 170, pp. 706–710, 2005.
25. Schlichting, H., "Boundary-Layer Theory", The 7th ed., McGraw-Hill Book Company, pp. 135-150, 1979.
26. Panton, R.L., "Incompressible Flow", John Wiley and Sons, NY, pp. 593-598, 1996.