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A One-Stage Two-Machine Replacement Strategy Based on the Bayesian Inference Method

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ABSTRACT

In this research, we consider an application of the Bayesian Inferences in machine replacement problem. The application is concerned with the time to replace two machines producing a specific product; each machine doing a special operation on the product when there are manufacturing defects because of failures. A common practice for this kind of problem is to fit a single distribution to the combined defect data, usually a distribution with an increasing hazard rate. While this may be convenient, it does not adequately capture the fact that there are two different underlying causes of failures. A better approach is to view the defect as arising from a mixture population: one due to the first machine failures and the other due to the second one. This allows one to estimate the various parameters of interest including the mixture proportion and the distribution of time between productions of defective products for each machine, separately. To do this, first we briefly introduce the data augmentation method for Bayesian inferences in the context of the finite mixture models. Then, we discuss the analysis of time-to-failure data and propose an optimal decision-making procedure for machine replacement strategy. In order to demonstrate the application of the proposed method we provide a numerical example.

Keywords: Reliability; Weibull distribution; Preventive maintenance; Bayesian inference

1. INTRODUCTION AND LITERATURE REVIEW

Preventive maintenance (PM) involves the repair, replacement, and maintenance of equipments in order to avoid unexpected failure during use. The objective of any PM program is the minimization of the total cost of inspection, repair, and equipment downtime (measured in terms of lost production capacity or reduced product quality) (Mann et al. 1995).

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In order to perform PM two approaches have evolved in the literature. The traditional approach is based on the use of statistical and reliability analysis of equipment failure. Under statistical-reliability (S-R)-based PM, the objective of achieving the minimum total cost is pursued by establishing fixed and statistically optimal PM intervals, at which to replace or overhaul equipment or components. The second approach involves the use of sensor-based monitoring of equipment condition in order to predict the time of machine failure. Under condition-based (C-B) PM, intervals between PM work are no longer fixed, but are performed only "when needed" (Mann et al. 1995).

The primary disadvantage of (S-R)-based PM is that the results of the calculations are based on the use of the mean value as the measure of central tendency. If the standard deviations of these means are large, then the probability of ascertaining the maintenance interval with accuracy is small. In many of these cases, the plant is over-maintained. Other disadvantages include more emergency maintenance, more overtime, and less equipment utilization (Mann et al. 1995).

With the development of a condition-based maintenance (CBM) technique, a more dynamic preventive maintenance practice could be applied. By integrating prediction tools, CBM can determine the required maintenance action prior to any predicted failure based on the conditions observed prior to a previous failure. From this aspect, this technique can be called condition-based predictive maintenance (CBPM) (Zhou et al. 2006). It has been proven that CBPM is an effective way to minimize maintenance costs, improve operational safety, and reduce the frequency and severity of in-service system failures (Zhou et al. 2006 and Mobley 1989).

CBM is carried out in response to significant deterioration in a unit's condition or performance as indicated by a change in a monitored parameter. PM allows the machine to be taken off-line at a predetermined time, which allows production loss to be minimized by scheduling production around the down time (Saranga 2002).

CBM techniques can be classified according to the type of symptoms they are designed to detect. According to Moubray (1990) the classifications are:

- Dynamic effects, such as vibration and noise levels;
- Particles released into the environment;
- Chemicals released into the environment;
- Physical effects, such as cracks, fractures, wear and deformation;
- Temperature rise in the equipment;
- Electrical effects, such as resistance, conductivity, dielectric strength, etc.

CBM has been widely accepted in practice in the past few years since it enables maintenance decisions to be made based on the current state of the equipment, thus avoiding unnecessary maintenance (replacement) and hence making timely maintenance actions when there is a strong indication of impending failure (Jardine et al. 1997).

The available literature on discrete time maintenance models predominantly treats an equipment deterioration process as a Markov chain. Sherwin and Al-Najjar (1999) presented a Markov model to determine the inspection intervals for a phased deterioration monitored complex components in a system with severe down time costs. An example involved roller bearing in paper mills with three phases; no defect, possible defect and final deterioration towards failure. In the last phase, continuous monitoring was used. The output of the model was an optimum inspection rate for each phase given a switching rule for going over to continuous monitoring. Wang and Hwang (2004) presented a Markov model that could be applied to construct the relationships among maintenance cycle, maintenance personnel allocation, human recovery factor, and system's tolerance time. Zhou et al. (2006) presented a dynamic opportunistic condition-based predictive maintenance policy for a continuously monitored multi-unit series system that was proposed based on short-term optimization with the integration of imperfect effect into maintenance actions. In their research, it was assumed that a unit's hazard rate distribution in the current maintenance cycle could be directly derived through CBPM. Whenever one of the units fails or reaches its reliability threshold, the whole system has to stop and PM opportunities arise for the system units. Jardine et al. (1997) presented an optimal replacement policy based on Markov stochastic process. Gupta and Lawsirirat (2006) presented a simulation based optimization method for strategically optimum maintenance of monitoring-enabled multi-component systems using continuous-time jump deterioration models. Sherwin (1999) with the concept of opportunity maintenance suggests new ways to construct and update preventive schedules for a complex system by making better use of system failure down time to do preventive work. Sinuany-Stern et al. (1997) concentrated on the 2-action version of this preventive schedules problem. They suggested an extremely practicable decision rule in partial observability, and proved empirically that this rule more than satisfactory competes with the state-of-the-art generic algorithm when implemented with its recommended grid usage. Sinuany-Stern (1993) considered a production system (machine) which deteriorates over time and the system deterioration over time was assumed to be Markovian. Moreover, the time scale assumed discrete and the 'true' state of the system (excellent, medium and bad) was not directly observable. What is observed was the performance of the system measured in terms of 'number of defectives' per time period. At the end of each period, a decision was to be made: whether to replace the system or not and the objective was to minimize the total cost in the long run.

Some authors applied Bayesian inference approach in machine replacement strategy. Mazzuchi and Soyer (1996) proposed a Bayesian approach to machine replacement problem in which they tried to minimize the total maintenance costs per unit time, $TC(T)$, including the cost of doing a planned preventive maintenance and the cost of in-service replacement. The objective function of their research was defined as:

$$TC(T) = \frac{C_f \int_0^T m(x)dx + C_p}{T}$$

Where C_p was the cost of performing a planned preventive maintenance, C_f was the cost of in-service replacement, and $m(x)$ was the hazard density function of the time to failure. Merrick et al. (2003) presented a Bayesian semi-parametric proportional hazard model to describe the failure behavior of machine tools. The semi-parametric setup was introduced using a mixture of *Dirichlet* process prior.

The amount of literature on single item maintenance is enormous. In one hand, when the parameters of the failure distribution are unknown but constant, Valdez-Florez and Feldman (1989) performed an extensive review. On the other hand, Wilson and Popova (1998) considered situations in which the parameters are random variables and Bayesian parametric analysis was performed.

While in most of the published research an infinite time horizon is assumed (see Chen and Popova (2000) for Bayesian policies over a finite time horizon), Damien et al. (2007) considered the problem of a finite horizon single item maintenance optimization structured as a combination of preventive and corrective maintenance in a nuclear power plant environment. They presented Bayesian semi-parametric models to estimate the failure time distribution and costs involved. The objective function of the optimization problem was the expected total cost of maintenance over the pre-defined finite time horizon. Typically, the mathematical modeling of failure times was based on parametric models.

Sethi et al. (2000) considered a single machine, multiproduct manufacturing system, operating at finitely many quality levels, in which the quality of the machine deteriorates according to a continuous-time Markov process. They assumed that the only way to improve the quality is to replace the machine by a new one and derived some conditions on the stability of the system under a simple class of scheduling-replacement policy named the recurrence of the total work backlog.

Hamada et al. (2004) presented a fully Bayesian approach that simultaneously combined non-overlapping (in time) basic event and higher-level event failure data in fault tree quantification. Such higher-level data often correspond to train, subsystem, or system failure events. The fully Bayesian approach also automatically propagates the highest-level data to lower levels in the fault tree.

Childress and Durango-Cohen (2005) formulated a stochastic version of the parallel machine replacement problem and analyzed the structure of optimal policies under general classes of replacement cost functions.

Hritonenko et al. (2007) combined known continuous- and discrete-time models of equipment replacement. They showed that the optimal equipment lifetime was shorter when the embodied technological change was more intense.

2. PROBLEM STATEMENT

In this research, we employ the concept of condition monitoring in machine replacement problem. Similar to Sinuany-Stern (1993) research, we observe the true state of the machines indirectly by measuring their performance in terms of ‘number of defective items’ per time period assuming that the machines are close to “wear-out” period such that the hazard rate is increasing. Furthermore, while in machine replacement strategies, a usual practice is to assume that the state space of the decision system is a discrete one, in this paper we relax this assumption and consider continuous state spaces with *Weibull* probability distribution. By this assumption since the complexity of the decision-making problem increases, we will consider only one stage. Moreover, in order to define a continuous state space, we employ the idea of the finite mixture models. Finite mixture models arise in a variety of industrial and scientific applications. In theses applications, we are concerned with the time to replace two machines that produce a specific product and each machine doing a specific operation when there failures due to manufacturing defects. A common practice to model this problem is to fit a single distribution, usually a distribution with an increasing hazard rate, to the combined defect data. Increasing hazard rate is due to the assumption of the machines being close to wear-out period according to “bathtub model.” In the bathtub model, the failure rate, $m(.)$, decreases with age or usage when the equipment is new – the time during which this applies is known as the infant mortality period – and it is usually relatively short. This is followed by a period of constant failure rate (the useful life period), and then a “wear-out” period. The latter is characterized by a failure rate, which is an increasing function of age or usage (Tsang, 1995). While this approach may be convenient, it does not adequately capture the fact that there are two different underlying causes of defect. A better approach is to view the defect as arising from a mixture population: one due to the first machine failures and the other due to the second machine failures.

This allows one to estimate the various parameters of interest including the mixture proportion and the machine reliability distribution, separately.

The main difference between this research and the previous ones is the choice of the state variable. By analyzing the outcome of the machines (products) and based upon the quality of the products, we try to determine the optimal strategy using Bayesian inference. The optimal strategy in previous researches was obtained based on the machine failure data.

In section three, we present the assumptions and some of derivations required in the new approach. We discuss the ways to analyze time-to-failure data in section four. Notations are defined in section five. The application of the Bayesian inference in decision-making method is given in section six. In order to demonstrate the application of the proposed method we provide a numerical example in section seven. A sensitivity analysis comes in section eight, and finally, the conclusion comes in section nine.

3. ASSUMPTION AND DERIVATIONS

The assumptions involved in the proposed methodology are:

1. The horizon of the decision-making process is one stage.

2. The time between producing defective products, (t), follows a *Weibull* probability distribution with parameters of α and β as $m\left(t \mid \alpha, \beta\right) = \alpha\beta t^{\beta-1}$.

3. An appropriate prior for α, $Q(\alpha)$, is assumed to be a *Gamma* distribution with parameters of a and b given by $Q(\alpha)=\frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}$.

4. For the prior distribution of the shape parameter, β, it is convenient to define a discrete distribution by discretization of the *beta* density on (β_L, β_U) (Mazzuchi and Soyer, 1996). This allows for great flexibility in representing prior uncertainty. The *beta* density is given by

$$g(\beta) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \frac{(\beta-\beta_L)^{c-1}(\beta_U-\beta)^{d-1}}{(\beta_U-\beta_L)^{c+d-1}} \quad \text{for } 0 \le \beta_L \le \beta \le \beta_U$$

where $\beta_U, \beta_L, c, d > 0$ are the specified parameters. Then, we define the prior distribution for β as

$$P_L = \Pr\{\beta = \beta_l\} = \int_{\beta_l - \delta/2}^{\beta_l + \delta/2} g(\beta)\, d\beta$$

where $\beta_l = \beta_L + \frac{2l-1}{2}\delta$ and $\delta = \frac{\beta_U - \beta_L}{k}$ for $l = 1, 2, \ldots, k$

5. If t_i represents the time between production of defective products then, using likelihood method, at the current time t_B we obtain the posterior state of variable α and β_l, denoted by $\overline{\alpha}$ and $\overline{\beta_1}$, as (Mazzuchi and Soyer, 1996):

$$f\left(\overline{\alpha}\middle|\beta_l, t_1, t_2, \ldots, t_n\right) = Gamma\left(a^*, b^*\right)$$

Where $a^* = a + n$ and $b^* = b + \sum_{i=1}^{n} t_i^{\beta_l} + \left(t_B - \sum_{i=1}^{n} t_i\right)^{\beta_l}$

And

$$\Pr\left\{\overline{\beta}_1 \middle| t_1, t_2, \ldots, t_n\right\} = \frac{\beta_l^n \left\{\prod_{j=1}^{n} t_j\right\}^{\beta_l - 1} \Big/ \left(b^*\right)^{a^*}}{\sum_{i=1}^{k} P_i \beta_i^n \left\{\prod_{j=1}^{n} t_j\right\}^{\beta_i - 1} \Big/ \left(b^*\right)^{a^*}} P_l$$

6. We assume that when the time between productions of defective products is less than a threshold like T^*, then the machine has failed and should be replaced. This has practical applications in many industries like the press shops when the evaluation of the machines is based on the quality of the products.

7. We consider three types of costs in the model:

a. Replacement cost

b. Cost of defective products in a stage

c. Cost of system failure (production break down)

8. Assuming *k(t)* to be the distribution function of time between production of defective products, then, according to assumption 5, the probability of machine failure using Bayesian inference is:

$$\text{Probability of machine failure} = \int_0^{T^*} f(t)\, dt = \int_0^{T^*} \int_0^{\infty} f(t|\alpha, \beta)\, g(\alpha, \beta)\, d\alpha\, d\beta\, dt$$

where $g(\alpha, \beta) = \sum_{i=1}^{k} Q\left(\alpha \middle| \beta_i\right) \Pr\{\beta_i\}$. Substituting for the terms and using integration we have:

$$\begin{aligned}\int_0^{T^*} f(t)\, dt &= \int_0^{T^*} \sum_{i=1}^{k} \alpha\beta_i t^{\beta_i - 1} e^{-\alpha t^{\beta_i}} \frac{b^{*a^*}}{\Gamma\left(a^*\right)} \alpha^{a^* - 1} e^{-b^*\alpha} \Pr\{\beta_i\}\, d\alpha dt = \sum_{i=1}^{k} \Pr\{\beta_i\} \int_0^{T^*} \alpha\beta_i t^{\beta_i - 1} e^{-\alpha t^{\beta_i}} \frac{b^{*a^*}}{\Gamma\left(a^*\right)} \alpha^{a^* - 1} e^{-b^*\alpha} d\alpha dt \\ &= \sum_{i=1}^{k} \Pr\{\beta_i\} a^* b^{*a^*} \frac{-1}{\left(t^{\beta_i} + b^*\right)^{a^*}} \Bigg|_0^{T^*} = \sum_{i=1}^{k} \Pr\{\beta_i\} a^* b^{*a^*} \left(\frac{1}{b^{*a^*}} - \frac{1}{\left(T^{*\beta_i} + b^*\right)^{a^*}}\right)\end{aligned} \qquad (1)$$

9. Assuming an overall time in decision making stage to be *H*, according to failure function, the expected number of defective products in the stage can be calculated by Bayesian inference as:

The expected number of defective products $=\int_0^H m(t)\,dt = \int_0^H \int_0^\infty m(t|\alpha,\beta)\,g(\alpha,\beta)\,d\alpha\,d\beta dt$

Hence we have

$$\begin{aligned}\int_0^H m(t)\,dt &= \int_0^H \int_0^\infty \sum_{i=1}^{k} \alpha\beta_i t^{\beta_i-1} k(\alpha|\beta_i)\Pr\{\beta_i\}\,d\alpha dt \\ &= \sum_{i=1}^{k}\left(\Pr\{\beta_i\}\int_0^{\mathrm{H}}\int_0^\infty \alpha\beta_i t^{\beta_i-1}\frac{b^{*a^*}}{\Gamma(a^*)}\alpha^{a^*-1}e^{-b^*\alpha}d\alpha dt\right) \\ &= \sum_{i=1}^{k}\int_0^{\mathrm{H}}\beta_i t^{\beta_i-1}\frac{a^*}{b^*}dt \\ &= \sum_{i=1}^{k}\left(\Pr\{\beta_i\}\frac{a^*}{b^*}H^{\beta_i}\right)\end{aligned}$$

10. After machine replacement, the time between productions of defective products for new machine follows a negative exponential with constant failure rate λ, where λ follows *Gamma* $(1,\lambda')$.

4. ANALYSIS OF TIME-TO-FAILURE DATA

Instead of fitting a single distribution to the time to repair or replace of two machines that produce a specific product, it is much more informative to separately estimate the parameters of the mixture model. In other words, we ideally model the time as a mixture of data from the two distributions as in equation (2).

$$f(t,p) = pf_1(t)+(1-p)f_2(t) \tag{2}$$

Where $f_1(t)$ and $f_2(t)$ are the probability distributions of the first and the second machine failures, respectively, and the proportion p measures the extent of the first machine failure. The mixing proportion p and the parameters of $f_1(t)$ and $f_2(t)$ have important physical interpretations in the decision-making process. However, before applying this model, first we employ the Bayesian inference to the mixture model using data augmentation technique (Sinuany-Stern 1993). To do this, let $z_i = 1$ if the defective product belongs to the first machine, and $z_i = 2$ if the second one has produced it. To choose the prior probability distribution of the mixing proportion, $f(p)$, in the absence of specific prior knowledge, it is common to use non-informative priors. Referring to Jeffrey's non-informative prior (Tsang 1995), for the mixing proportion *p*, we take a *Beta* prior distribution with parameters $a=b=\frac{1}{2}$.

It is possible to express the unknown parameter *p*, in terms of a metric $f(p)$, so that the corresponding likelihood is data translated. This means that the likelihood curve for $f(p)$ is completely determined a priori except for its location, which depends on the data yet to be observed. Then to say that we know little a priori relative to what the data is going to tell us, may be expressed by saying that we are almost equally willing to accept one value of $f(p)$ as another. This state of indifference may be expressed by taking $f(p)$ to locally uniform, and the resulting priori distribution is called non-informative for $f(p)$ with respect to the data.

According to Jeffreys' Rule, assuming $\phi(p) = -\underset{x|p}{E}\left[\frac{\partial^2 \log p(x|p)}{\partial p^2}\right]$, the informative for p should be chosen so that locally $f(p) = \phi^{0.5}(p)$. Specifically for the Beta distribution case, the information measure is $\phi(p) \approx p^{-1}(1-p)^{-1}$. Hence $f(p) \approx p^{-0.5}(1-p)^{-0.5}$. In other words, we conclude that $f(p)$ as a non-informative prior distribution is Beta (0.5, 0.5) (See Box and Tiao 1992).

With the aforementioned preliminaries, we are now ready to specify the data augmentation algorithm for the problem at hand. Let $t = (t_1, t_2, ..., t_n)$ be the exact failure times, $z = (z_1, z_2, ..., z_n)$ be the latent variables corresponding to the defect modes, and R to be the number of times z_i's equal to one. Then, we can obtain the distribution of $p|t, z$ as:

$$
\begin{aligned}
f(p|t,z) &= \frac{f(t,z|p)f(p)}{\int f(t,z|p)f(p)dp} \\
&= \frac{p^R(1-p)^{n-R}\frac{\Gamma(a+b)p^a(1-p)^b}{\Gamma(a)\Gamma(b)}}{\int p^R(1-p)^{n-R}\frac{\Gamma(a+b)p^a(1-p)^b}{\Gamma(a)\Gamma(b)}dp} \\
&= \frac{\Gamma(a+R+b+n-R)p^{a+R}(1-p)^{b+n-R}}{\Gamma(a+R)\Gamma(b+n-R)}
\end{aligned}
$$

which is a *Beta* distribution with parameters of $R+a$ and $n-R+b$ (Tanner and Wong 1987; Nair et al. 2001).

5. NOTATIONS

We will use the following notations and definitions in the rest of the paper:

C	cost of producing one unsatisfactory product
C_i	cost of replacing machine *i*
C_{if}	cost of in-service replacement for machine *i*
p	measures the extent of the of the first machine to produce defective products
P_{irf}	Probability of in-service replacement for machine *i* after replacement
P_{if}	Probability of in-service replacement for machine *i* before replacement
$\frac{1}{\lambda_i'}$	Standard expected mean of failure rate for machine *i* after replacement (new machine failure rate)
H	overall time in period
$r(\mathrm{p})$	distribution function of *p*

The probability of an in-service replacement for machine one in the case of replacement, assuming $f\ (t) = \lambda_1^{'} e^{-\lambda_1^{'} t}$ is:

$$\int_0^{T^*} \mathrm{k}(\mathrm{t})\ \mathrm{dt} = \int_0^{T^*}\int_0^{\infty} \mathrm{f}(\mathrm{t}|\lambda) h(\lambda) d\lambda\ \mathrm{dt}$$
$$= \int_0^{T^*}\int_0^{\infty} \lambda e^{(-\lambda t)} \lambda_1^{'} e^{(-\lambda\lambda^{'})} d\lambda\ \mathrm{dt}$$
$$= \int_0^{T^*}\int_0^{\infty} \lambda \lambda_1^{'} e^{-\lambda(\lambda^{'}+t)} d\lambda dt$$
$$= \int_0^{T^*} \lambda^{'} \frac{1}{(\lambda^{'}+t)^2} dt = \frac{T^*}{\lambda^{'}+T^*}$$

In other words,

$$P_{1rf} = \int_0^{T^*} f\ (t)\ dt = \frac{T^*}{\lambda_1^{'}+T^*}$$

The probability of an in-service replacement for machine one in the case of not replacing machine one, assuming increasing failure rate, will be calculated using equation (1) as:

$$P_{1f} = \sum_{i=1}^{k} \Pr\{\overline{\beta}_i\} \left(1 - \left(\frac{b^*}{T^{*\beta_i} + b^*}\right)^{a^*}\right)$$

6. THE DECISION MAKING APPROACH

In order to apply Bayesian inference, assume n defective products has been produced. According to what we state in section 4, for pre-replacement situation we have:

1. $p|t,z$ is distributed as $Beta(R+a, n-R+b)$
2. $\overline{\alpha}_1|t,z$ is distributed as Gamma denoted by $\Gamma\left(a+R, b+\sum_{i:z_i=1}^{n} t_i^{\beta_l} + \left(t_B - \sum_{i:z_i=1}^{n} t_i\right)^{\beta_l}\right)$
3. discrete distribution of $\overline{\beta}_1|t,z$ is

$$\Pr\{\overline{\beta}_1 = \beta_l\,|t,z\} = \frac{\beta_l^n \left\{\prod_{j,z_j=1}^{n} t_j\right\}^{\beta_l - 1} \Big/ (b^*)^{a^*}}{\sum_{i=1}^{k} P_i \beta_i^n \left\{\prod_{j,z_j=1}^{n} t_j\right\}^{\beta_i - 1} \Big/ (b^*)^{a^*}} P_l \tag{3}$$

The distributions of $\overline{\alpha}_2|t,z$ and $\overline{\beta}_2|t,z$ are obtained in a similar manner.

For machines' post-replacement situation, to determine *p*, the extent of a machine to produce defective products, for machine one and two we let the stochastic numbers e_1 and e_2 to denote the numbers of defective products that the new machine one and two produces in the decision-making stage, respectively. According to Bayesian inference, if we decide to replace any machines, we have the following results:

4. $p\,|\,$after machine one replacment is distributed as $Beta(a+e_1, n-R+b)$

5. $p\,|\,$after machine two replacment is distributed as $Beta(R+a, b+e_2)$

6. $p\,|\,$after replacing both machines is distributed as $Beta(a+e_1, b+e_2)$

Then by applying the expected cost for each alternative, we get:

$$V = Min\begin{Bmatrix} C_1 + \mathrm{P}_{1\mathrm{rf}}c_{1f} + \mathrm{P}_{2\mathrm{f}}c_{2f} + E\left(\text{cost of defective products}\right), \\ C_2 + \mathrm{P}_{1\mathrm{f}}c_{1f} + \mathrm{P}_{2\mathrm{rf}}c_{2f} + E\left(\text{cost of defective products}\right), \\ \mathrm{P}_{1\mathrm{f}}c_{1f} + \mathrm{P}_{2\mathrm{f}}c_{2f} + E\left(\text{cost of defective products}\right), \\ C_1 + C_2 + \mathrm{P}_{1\mathrm{rf}}c_{1f} + \mathrm{P}_{2\mathrm{rf}}c_{2f} + E\left(\text{cost of defective products}\right) \end{Bmatrix} \quad (4)$$

The first and the second term in the minimization program given in (4) indicate the expected cost when we replace the first and the second machine, respectively. The third term refers to the expected cost when none of the machines are replaced, and the fourth term is the expected cost when both machines are replaced.

An important feature of this approach is that we use the historical data of defective products to update the distribution function of failure rates and with considering appropriate assumptions, we determine the optimal policy at the start of the current stage.

Applying equation (4) and conditional expectation, we have the following result:

$$E\left(\text{cost of defective products}\right) = C\int_0^1\left(p\int_0^\infty m_1(t)\,dt + (1-p)\int_0^\infty m_2(t)\,dt\right)r(p)\,d \quad (5)$$

In order to calculate the expected costs of defective products in equation (4) we define the following events:

FMR: The event of the First Machine Replacement
SMR: The event of the Second Machine Replacement
NMR: The event of No Machine Replacement
BMR: The event of Both Machine Replacements

Then, the costs associated with different decisions are:

$$\text{Cost of } FMR: C_1+\frac{T^*}{\lambda_1^{'}+T^*}C_{1f}+\left(\sum_{i=1}^{k}\Pr\left\{\overline{\beta}_i\right\}\left(1-\left(\frac{b_2^*}{T^{*\beta_i}+b_2^*}\right)^{a_2^*}\right)\right)C_{2f}+E\left(p\middle|FMR\right)C\left(\frac{H}{\lambda_1^{'}}\right)+\left(\left(1-E\left(p\middle|FMR\right)\right)C\left(\sum_{i=1}^{k}\left(\Pr\left\{\overline{\beta}_i\right\}\frac{a_2^*}{b_2^*}H^{\beta_i}\right)\right)\right)$$

$$\text{Cost of } SMR: C_2+\left(\frac{T^*}{\lambda_2^{'}+T^*}\right)C_{2f}+\left(\sum_{i=1}^{k}\Pr\left\{\overline{\beta}_i\right\}\left(1-\left(\frac{b_1^*}{T^{*\beta_i}+b_1^*}\right)^{a_1^*}\right)\right)C_{1f}+E\left(p\middle|SMR\right)C\left(\sum_{i=1}^{k}\left(\Pr\left\{\overline{\beta}_i\right\}\frac{a_1^*}{b_1^*}H^{\beta_i}\right)\right)+\left(\left(1-E\left(p\middle|SMR\right)\right)C\left(\frac{H}{\lambda_2^{'}}\right)\right)$$

$$\text{Cost of } NMR: \left(\sum_{i=1}^{k}\Pr\left\{\overline{\beta}_i\right\}\left(1-\left(\frac{b_2^*}{T^{*\beta_i}+b_2^*}\right)^{a_2^*}\right)\right)C_{2f}+\left(\sum_{i=1}^{k}\Pr\left\{\overline{\beta}_i\right\}\left(1-\left(\frac{b_1^*}{T^{*\beta_i}+b_1^*}\right)^{a_1^*}\right)\right)C_{1f}+E\left(p\middle|NMR\right)C\left(\sum_{i=1}^{k}\left(\Pr\left\{\overline{\beta}_i\right\}\frac{a_1^*}{b_1^*}H^{\beta_i}\right)\right)+\left(\left(1-E\left(p\middle|NMR\right)\right)C\left(\sum_{i=1}^{k}\left(\Pr\left\{\overline{\beta}_i\right\}\frac{a_2^*}{b_2^*}H^{\beta_i}\right)\right)\right)$$

$$\text{Cost of } BMR: C_1+C_2+\frac{T^*}{\lambda_1^{'}+T^*}C_{1f}+\left(\frac{T^*}{\lambda_2^{'}+T^*}\right)C_{2f}+E\left(p\middle|BMR\right)C\left(\frac{H}{\lambda_1^{'}}\right)+\left(\left(1-E\left(p\middle|BMR\right)\right)C\left(\frac{H}{\lambda_2^{'}}\right)\right)$$

Since $p\,|\,\text{machine one replacment}$ is distributed as $Beta(a+e_1, n-R+b)$, we have:

$$\begin{aligned}
E\left(p\middle|FMR\right) &= E\left(E\left(p\middle|FMR\middle|e_1\right)\right)\\
&= E\left(\frac{a+e_1}{n-R+b+a+e_1}\right)\\
&= \sum_{i=1}^{\infty}\left(\int_0^{\infty}\frac{\exp\left(-\lambda_1 H\right)\left(\lambda_1 H\right)^i\left(a+i\right)}{\Gamma\left(i+1\right)\left(n-R+b+a+i\right)}f\left(\lambda_1\right)d\lambda_1\right)\\
&= \sum_{i=1}^{\infty}\left(\int_0^{\infty}\frac{\exp\left(-\lambda_1 H\right)\left(\lambda_1 H\right)^i\left(a+i\right)}{\Gamma\left(i+1\right)\left(n-R+b+a+i\right)}\lambda_1^{'}e^{-\lambda_1\lambda_1^{'}}d\lambda_1\right)\\
&= \sum_{i=1}^{\infty}\frac{H^i\left(a+i\right)\lambda_1^{'}}{\left(\lambda_1^{'}+H\right)^{i+1}\left(n-R+b+a+i\right)}
\end{aligned}$$

We note that in deriving the last equation we assumed a Poisson distribution for the number of defective products in stage (with mean of $\lambda_1 H$).
By similar reasoning, we have:

$$E\left(p\middle|SMR\right)=\sum_{i=1}^{\infty}\frac{H^i\left(a+R\right)\lambda_2^{'}}{\left(\lambda_2^{'}+H\right)^{i+1}\left(R+a+b+i\right)}$$

Based on the expectation of *beta* distribution we have:

$$E\left(p\middle|NMR\right)=\frac{a+R}{n+a+b}$$

$$E(p|BMR)=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left(\int_0^{\infty}\int_0^{\infty}\frac{\exp\left(-(\lambda_1+\lambda_2)H\right)(\lambda_1 H)^i(\lambda_2 H)^j(a+i)}{\Gamma(i+1)\Gamma(j+1)(j+b+a+i)}\lambda_1^{'}e^{-\lambda_1\lambda_1^{'}}\lambda_2^{'}e^{-\lambda_2\lambda_2^{'}}d\lambda_1 d\lambda_2\right)$$
$$=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{H^{i+j}(a+i)\lambda_1^{'}\lambda_2^{'}}{(\lambda_1^{'}+H)^{i+1}(\lambda_2^{'}+H)^{j+1}(b+a+i+j)}$$

which can be approximated by

$$E(p|BMR)=\frac{\frac{H}{\lambda_1^{'}}}{\frac{H}{\lambda_1^{'}}+\frac{H}{\lambda_2^{'}}}=\frac{\lambda_2^{'}}{\lambda_2^{'}+\lambda_1^{'}}$$

7. A NUMERICAL EXAMPLE

Consider a manufacturing process in which there are two serial machines producing a specific product. For increasing hazard rate assumption, we have generated values of β from standard *beta* distribution with parameter *c=2* and *d=1* and values of α have been generated from an exponential distribution with mean 1. Generated data are shown in Table 1.

Assuming $C_1=1000,\ C_2=1300,\ T^*=0.6,\ \lambda_1^{'}=50,\ \lambda_2^{'}=50, C_{1f}=2000, C_{2f}=2500, H=100, C=5$, we have:

Cost of FMR : 2667.61
Cost of SMR : 2098.15
Cost of BMR : 2357.43
Cost of NMR : 2242.53

Table 1: Time between productions of defective products

Machine Two	*Machine One*
0.63308	0.527716
0.822558	0.025934
0.140781	0.402137
0.21674	0.072486
0.023657	2.215276
0.6156	11.37043
0.280686	1.93114
0.091386	4.389833
0.741723	1.209809
0.34208	0.016308

Hence, the decision regarding the replacement of machine two is the best.

8. A SENSITIVITY ANALYSIS

To analyze the effect of different values of the model parameters, in this section we present a sensitivity study of the numerical example given in section 7. This study is based on different values of the in-service replacement costs and the defective product cost.

If the in-service replacement cost is unknown, then based on different combinations of C_{1f} and C_{2f} we can estimate the cost associated with each decision and determine the best strategy. Table 2 shows the results of this study.

The results of Table 2 show that the decision *FMR* is never the best. For large values of both C_{1f} and C_{2f} replacing both machines is the best decision. Small values of C_{1f} and C_{2f} leads *NMR* to be the best and moderate values of C_{1f} and C_{2f} results in replacing machine two.

The results of the sensitivity analysis based on different defective product costs are shown in Table 3. From the results of Table 3, we see that small values of *C* leads to replacing machine two and for the large values of *C*, replacing both machines is the best decision. The other two decisions are not

Table2: The estimated cost of each decision for different combinations of C_{1f} and C_{2f}

C_{1f}	C_{2f}	*NMR*	*BMR*	*SMR*	*FMR*
1000	1000	1114.773	2331.344	1756.604	1855.283
1200	1200	1286.962	2335.613	1823.966	1964.38
1400	1400	1459.151	2339.881	1891.327	2073.476
1600	1600	1631.341	2344.15	1958.688	2182.573
1800	1800	1803.53	2348.419	2026.05	2291.67
2000	2000	1975.719	2352.688	2093.411	2400.766
2200	2200	2147.908	2356.957	2160.773	2509.863
2400	2400	2320.098	2361.225	2228.134	2618.959
2600	2600	2492.287	2365.494	2295.495	2728.056
2800	2800	2664.476	2369.763	2362.857	2837.153
3000	3000	2836.665	2374.032	2430.218	2946.249
3200	3200	3008.855	2378.3	2497.58	3055.346
3400	3400	3181.044	2382.569	2564.941	3164.442
3600	3600	3353.233	2386.838	2632.302	3273.539
3800	3800	3525.422	2391.107	2699.664	3382.636
4000	4000	3697.612	2395.375	2767.025	3491.732
4200	4200	3869.801	2399.644	2834.387	3600.829
4400	4400	4041.99	2403.439	2901.748	3709.926
4600	4600	4214.179	2408.182	2969.109	3819.022
4800	4800	4386.369	2412.451	3036.471	3928.119
5000	5000	4558.558	2416.719	3103.832	4037.215
5200	5200	4730.747	2420.988	3171.193	4146.312
5400	5400	4902.936	2425.257	3238.555	4255.409

the best in none of the cases. The pictorial representation of the results of Table 3 is also given in Figure 1.

9. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

In this paper, first we briefly introduced the data augmentation method for Bayesian inferences in the context of the finite mixture models. Then, we discussed the analysis of time-to-failure data and the application of the methods of decision-making for the machine replacement strategy. In order to demonstrate the application of the proposed method we provided a numerical example.

The proposed approach may be applied to any decision-making problems in which we need to update the probability distribution function of its states by Bayesian inference approach.

For further researches in the subject of this research, we recommend the following:

1. In general, we can consider the case in which there are *n* machines instead of two.

2. We can consider other distribution functions for the time between producing of defective products.

3. Failure rates, costs, and the probabilities in the mixture model all may be considered as fuzzy parameters.

Table3: The estimated cost of each decision for different values of *C*

C	*FMR*	*SMR*	*BMR*	*NMR*
1	2418.892	2001.958	2349	2038.72
2	2480.784	2025.915	2351	2089.44
3	2542.676	2049.873	2353	2140.16
4	2604.568	2073.83	2355	2190.88
5	2666.46	2097.788	2357	2241.6
6	2728.352	2121.746	2359	2292.32
7	2790.244	2145.703	2361	2343.04
8	2852.136	2169.661	2363	2393.76
9	2914.028	2193.618	2365	2444.48
10	2975.92	2217.576	2367	2495.2
11	3037.812	2241.534	2369	2545.92
12	3099.704	2265.491	2371	2596.64
13	3161.596	2289.449	2373	2647.36
14	3223.488	2313.406	2375	2698.08
15	3285.38	2337.364	2377	2748.8
16	3347.272	2361.322	2379	2799.52
17	3409.164	2385.279	2381	2850.24
18	3471.056	2409.237	2383	2900.96
19	3532.948	2433.194	2385	2951.68
20	3594.84	2457.152	2387	3002.4

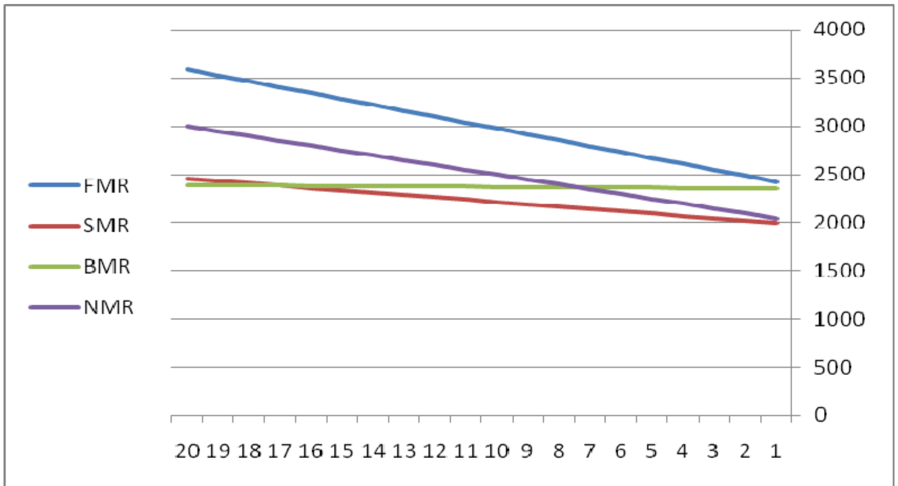

Figure 1: The cost diagrams of different decisions

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