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پژوهشکده‌ی آمار

Bivariate Semi-Logistic Distribution and Processes

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Abstract. Bivariate semi-logistic and Marshall-Olkin bivariate semi-logistic distributions are introduced. Some properties of these distributions are studied. First order autoregressive processes with bivariate semi-logistic and Marshall-Olkin bivariate semi-logistic distributions as marginals are introduced and studied.

Keywords. Autoregressive processes; Bivariate semi-logistic distribution; Minification processes; Stationarity.

1 Introduction

Logistic distribution has attracted the attention of many researchers due to the application of this distribution in various fields. Balakrishnan (1992) discussed the application of logistic distribution in population growth, medical diagnosis and public health. A few more interesting use of logistic distribution are in the analysis of survival data. Balakrishnan (1992) discussed the analysis of bioavailability data when successive samples are from logistic distribution.

Univariate logistic distribution in its reduced form is defined by the expressions

$$\bar{F}(x) = \frac{1}{1+\exp\{x\}}, \qquad -\infty < x < \infty \tag{1}$$

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and

$$f(x) = \frac{\exp\{x\}}{(1+\exp\{x\})^2}.$$

This distribution is symmetric about zero and resembles closely to the normal distribution.

Although multivariate data sets with logistic like marginals have always been around, it was not until 1961 that a bivariate logistic model was proposed. Gumbel (1961) actually provided three bivariate logistic models, one of which has cumulative distribution function

$$F(x,y) = \frac{1}{1+\exp\{-x\}+\exp\{-y\}}, \qquad -\infty < x, y < \infty. \tag{2}$$

Location and scale parameters can be introduced to generalize this expression. Gumbel in his paper studied the regression properties and verified that the correlation coefficient is $\frac{1}{2}$. A multivariate extension of the Gumbel's bivariate logistic is proposed by Malik and Abraham (1973). For applications of the logistic distribution see Balakrishnan (1992) and Kotz et al.(2000).

The study on minification processes began with the work of Tavares (1980). In his work, the observations are generated by the equation

$$X_n = k \min(X_{n-1}, \varepsilon_n), \qquad n \geqslant 1 \tag{3}$$

where $k > 1$ is a constant and $\{\varepsilon_n\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a given marginal distribution. Because of the structure of (3), the process $\{X_n\}$ is called minification process. Sim (1986) developed a first order autoregressive Weibull process and studied its properties. Arnold (1993) developed a logistic process involving Markovian minimization.

Giving slight modifications to (3), several other minification models have been constructed so far. Yeh et al. (1988) considered a first order autoregressive minification process having Pareto marginal distribution. Arnold and Robertson (1989) developed a minification process having logistic marginal distribution. Such minification processes in general have the structure given by

$$X_n = \begin{cases} kX_{n-1} & \text{w.p.} \quad p \\ k \min(X_{n-1}, \varepsilon_n) & \text{w.p.} \quad 1-p \end{cases}, \quad 0 < p < 1,$$

where 'w.p.' stands for 'with probability'. Pillai, Jose and Jayakumar (1995) introduced another minification process having the form

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ k \min(X_{n-1}, \varepsilon_n) & \text{w.p.} \quad 1-p \end{cases}, \quad 0 < p < 1.$$

Lewis and McKenzie (1991) obtained necessary and sufficient conditions on the hazard rate of the marginal distributions for a minification process to exist.

Jayakumar and Thomas Mathew (2005) defined semi-logistic distribution and studied first order autoregressive semi-logistic process.

We say that a random variable X defined on $R = (-\infty, \infty)$ has semi logistic distribution and write $X \stackrel{d}{=} L_s(\alpha)$ if its survival function is

$$\bar{F}_X(x) = \frac{1}{1+\eta(x)} \tag{4}$$

where $\eta(x)$ satisfies the functional equation

$$\eta(x) = \frac{1}{p}\eta\left(\frac{1}{\alpha}\ln p + x\right), \qquad \alpha > 0,\ 0 < p < 1. \tag{5}$$

It can be shown that $\eta(x) = \exp\{\alpha x\}h(x)$ where $h(x)$ is periodic in x with period $\frac{1}{\alpha}\ln p$. For proof, see Kagan, Linnik and Rao (1973). For example, if $h(x) = \exp\{\beta\cos(\alpha x)\}$, it satisfies (5) with $p = \exp\{-2\pi\}$ and $\eta(x)$ is monotone increasing with $0 < \beta < 1$.

In Section 2, we introduce bivariate semi-logistic distribution and study properties of it. Characterizations of bivariate semi-logistic distribution are obtained. Marshall-Olkin bivariate semi-logistic distribution is developed and as a special case Marshall-Olkin bivariate logistic distribution is studied in Section 3. In Section 4, first order autoregressive process having bivariate semi-logistic distribution as marginal is constructed and some properties of first order autoregressive bivariate logistic processes are studied. Section 5 is devoted to first order autoregressive Marshall-Olkin bivariate semi-logistic process.

2 Bivariate Semi-logistic Distribution

A random vector (X, Y) defined on R^2 is said to have bivariate semi-logistic distribution with parameters α_1, α_2 and p and we denote it by $(X,Y) \stackrel{d}{=} BSL(\alpha_1, \alpha_2, p)$ if its survival function is of the form

$$\bar{F}(x,y) = P(X > x, Y > y) = \frac{1}{1+\eta(x,y)} \tag{6}$$

where $\eta(x,y)$ satisfies the functional equation

$$\eta(x,y) = \frac{1}{p}\eta\left(x + \frac{1}{\alpha_1}\ln p,\ \ y + \frac{1}{\alpha_2}\ln p\right)$$
$$0 < p < 1, \quad \alpha_1,\ \alpha_2 > 0, \quad -\infty < x, y < \infty. \tag{7}$$

Lemma 1 *The solution of the functional equation* (7) *is given by*

$$\eta(x, y) = \exp\{\alpha_1 x\}h_1(x) + \exp\{\alpha_2 y\}h_2(y) \tag{8}$$

where $h_1(x)$ *and* $h_2(y)$ *are periodic functions in* x *and* y *with period* $\frac{1}{\alpha_1}\ln p$ *and* $\frac{1}{\alpha_2}\ln p$ *respectively.*

Proof: For proof see Kagan Linnik and Rao (1973).

For example if $h_i(x) = \exp\{\beta\cos(\alpha_i x)\}$, $i = 1, 2$, it satisfies (7) with $p = \exp\{-2\pi\}$ and $\eta(x, y)$ is monotone increasing in x and y with $0 < \beta < 1$.

Figure 1 presents the probability density function of bivariate semi-logistic distribution for $h_i(x) = \exp\{\beta\cos(\alpha_i x)\}$ and for various values of α_1, α_2 and β.

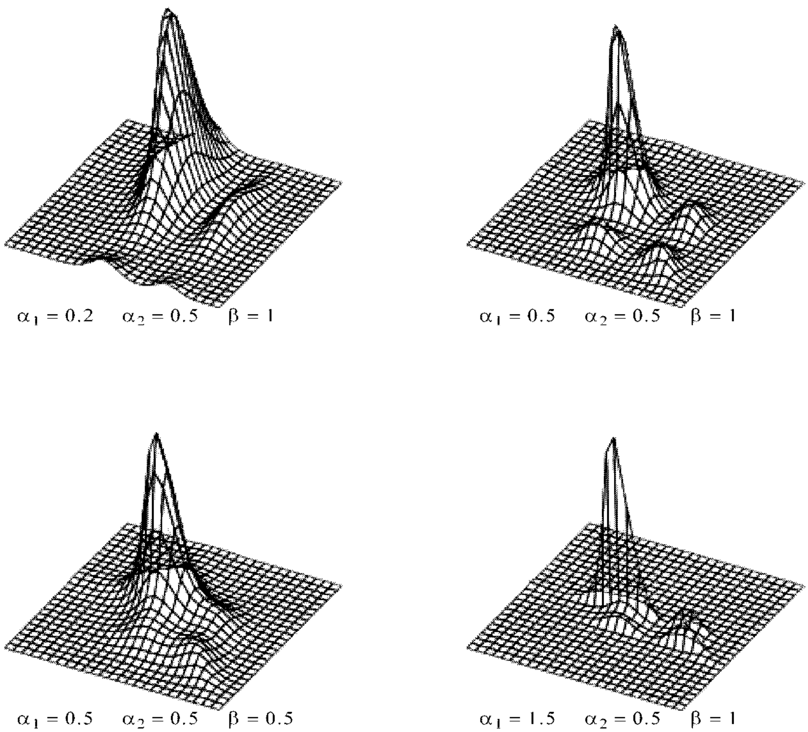

Figure 1. Density plot of bivariate semi-logistic distribution.

In particular if we choose $h_1(x) = h_2(y) = 1$, the $BSL(\alpha_1, \alpha_2, p)$ reduces to a bivariate logistic distribution with survival function

$$\bar{F}(x,y) = \frac{1}{1 + \exp\{\alpha_1 x\} + \exp\{\alpha_2 y\}}, \qquad \alpha_1, \alpha_2 > 0. \tag{9}$$

Gumbel (1961) proved that the bivariate logistic distribution having distribution function (2) is asymmetric. Surface plot of the bivariate logistic distribution with survival function (9) for various values of α_1 and α_2 is presented in Figure 2.

Now we study some characterizing properties of $BSL(\alpha_1, \alpha_2, p)$ distribution via geometric minimization.

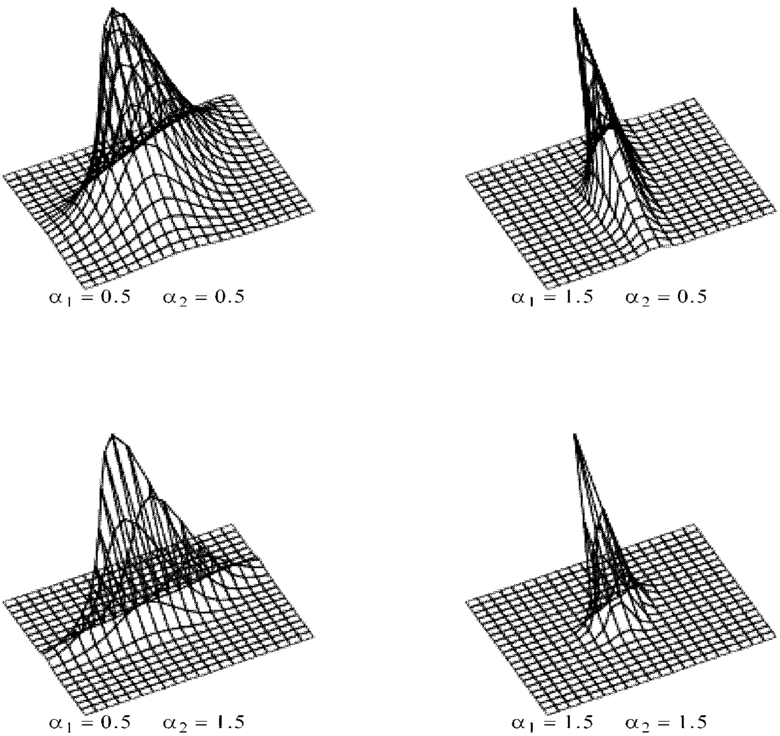

Figure 2. Density plot of bivariate semi-logistic distribution.

Let $\{(X_i, Y_i), i \geqslant 1\}$ be a sequence of independent and identically distributed bivariate random vectors following $BSL(\alpha_1, \alpha_2, p)$ distribution and N be a geometric random variable with parameter p and

$$P(N = n) = pq^{n-1}, \qquad n = 1, 2, ..., \qquad 0 < p < 1,\ q = 1 - p \tag{10}$$

and N is independent of (X_i, Y_i), $i \geqslant 1$.

Define

$$U_N = \min(X_1, X_2, ...X_N) \quad and \quad V_N = \min(Y_1, Y_2, ...Y_N). \tag{11}$$

Theorem 1 *Let $\{(X_i, Y_i), i \geqslant 1\}$ be a sequence of independent and identically distributed bivariate random vectors with common survival function $\bar{F}(x, y)$ and N be a geometric random variable as in (10), which is independent of (X_i, Y_i) for all $i \geqslant 1$. Then the random vectors $\left(U_N - \frac{1}{\alpha_1}\ln p, V_N - \frac{1}{\alpha_2}\ln p\right)$ and (X_1, Y_1) are identically distributed if and only if (X_1, Y_1) have the $BSL(\alpha_1, \alpha_2, p)$ distribution.*

Proof: Let

$$\begin{aligned}\bar{H}(x, y) &= P\left(U_N - \frac{1}{\alpha_1}\ln p > x, V_N - \frac{1}{\alpha_2}\ln p > y\right) \\ &= P\left(U_N > x + \frac{1}{\alpha_1}\ln p, V_N > y + \frac{1}{\alpha_2}\ln p\right) \\ &= \sum_{n=1}^{\infty}\left[\bar{F}\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right)\right]^n pq^{n-1}.\end{aligned}$$

That is,

$$\bar{H}(x, y) = \frac{p\bar{F}\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right)}{1 - q\bar{F}\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right)} \tag{12}$$

Now if $\bar{F}(x, y)$ is as in (6) and (7), the equation (12) becomes

$$\bar{H}(x, y) = \frac{1}{1 + \eta(x, y)} = \bar{F}(x, y)$$

This proves the sufficiency part of the theorem.

Conversely assume that $\bar{H}(x, y) = \bar{F}(x, y)$. Note that any survival function can be represented as

$$\bar{F}(x, y) = \frac{1}{1 + \phi(x, y)}, \tag{13}$$

where $\phi(x,y)$ is a monotonically increasing function in both x and y. Using the representation (13) in (12) with $\bar{H}(x,y)=\bar{F}(x,y)$, we get the equation,

$$\phi(x,y)=\frac{1}{p}\phi\left(x+\frac{1}{\alpha_1}\ln p,\ \ y+\frac{1}{\alpha_2}\ln p\right)$$

This is the functional equation (7) satisfied by $BSL(\alpha_1,\alpha_2,p)$. Hence the proof is complete.

Let $\{N_k, k\geqslant 1\}$ be a sequence of geometric random variables with parameters $p_k, 0<p_k<1$. Define

$$\begin{aligned}\bar{F}_k(x,y) &= P\left(U_{N_{k-1}}>x, V_{N_{k-1}}>y\right), \qquad k=2,3,...\\ &= \frac{p_{k-1}\ \bar{F}_{k-1}(x,y)}{1-(1-p_{k-1})\bar{F}_{k-1}(x,y)}. \qquad (14)\end{aligned}$$

Here we refer $\bar{F}_k$ as the survival function of the geometric (p_{k-1}) minimum of independent and identically distributed random vectors with $\bar{F}_{k-1}$ as the common survival function.

Theorem 2 *Let $\{(X_i,Y_i), i\geqslant 1\}$ be a sequence of independent and identically distributed random vectors with common survival function $\bar{F}(x,y)$. Define $\bar{F}_1=\bar{F}$ and $\bar{F}_k$ as the survival function of the geometric (p_{k-1}) minimum of independent and identically distributed random vectors with common survival function $\bar{F}_{k-1}$, $k=2,3,\ldots$. Then*

$$\bar{F}_k\left(x+\sum_{j=1}^{k-1}\frac{1}{\alpha_1}\ln p_j, y+\sum_{j=1}^{k-1}\frac{1}{\alpha_2}\ln p_j\right)=\bar{F}(x,y) \qquad (15)$$

if and only if (X_1,Y_1) have $BSL(\alpha_1,\alpha_2,p)$ distribution.

Proof: By definition the survival function $\bar{F}_k$ satisfies equation (14). As in (13) we can write

$$\bar{F}_k(x,y)=\frac{1}{1+\phi_k(x,y)}, \qquad k=1,2,3,...$$

Substituting this in (14), we get

$$\phi_k(x,y)=\frac{1}{p_{k-1}}\phi_{k-1}(x,y), \qquad k=2,3,...$$

Recursively using this relation, we have

$$\phi_k(x,y) = \frac{1}{\prod_{j=1}^{k-1} p_j}\phi_1(x,y),$$

since $F_1 = F$ implies $\phi_1 = \phi$. This implies

$$\phi_k\left(x + \sum_{j=1}^{k-1}\frac{1}{\alpha_1}\ln p_j, y + \sum_{j=1}^{k-1}\frac{1}{\alpha_2}\ln p_j\right)$$

$$= \frac{1}{\prod\limits_{j=1}^{k-1} p_j}\phi_1\left(x + \sum_{j=1}^{k-1}\frac{1}{\alpha_1}\ln p_j, y + \sum_{j=1}^{k-1}\frac{1}{\alpha_2}\ln p_j\right) \tag{16}$$

This gives us (15) if we replace ϕ_k by η_k and if we assume that $\eta_1$satisfies (7).

Conversely assume that (15) is true. By the hypothesis of the theorem we have (16). Thus (15) and (16) together leads to the equation,

$$\frac{1}{1 + \frac{1}{\prod_{j=1}^{k-1} p_j}\phi_1\left(x + \sum_{j=1}^{k-1}\frac{1}{\alpha_1}\ln p_j, y + \sum_{j=1}^{k-1}\frac{1}{\alpha_2}\ln p_j\right)} = \bar{F}(x,y) = \frac{1}{1+\eta(x,y)}$$

This implies that

$$\phi(x,y) = \frac{1}{\prod_{j=1}^{k-1} p_j}\phi\left(x + \sum_{j=1}^{k-1}\frac{1}{\alpha_1}\ln p_j, y + \sum_{j=1}^{k-1}\frac{1}{\alpha_2}\ln p_j\right)$$

which is same as (7).

Hence the proof is complete.

Lemma 2 *We say that a random vector (X,Y) on R^2 has bivariate semi extreme value distribution if its survival function is*

$$\bar{F}(x,y) = \exp\{-\eta(x,y)\}$$

where $\eta(x,y)$ satisfies the functional equation (7).

The following theorem establishes the relationship between bivariate semi-logistic distribution and bivariate semi-extreme value distribution.

Theorem 3 *If* $\{(X_i, Y_i), i \geqslant 1\}$ *are independent and identically distributed* $BSL(\alpha_1, \alpha_2, p)$ *random vectors then*

$$(Z_n, W_n) = \left\{ \min\left(X_1 - \frac{1}{\alpha_1}\ln\frac{1}{n}, X_2 - \frac{1}{\alpha_1}\ln\frac{1}{n}, ..., X_n - \frac{1}{\alpha_1}\ln\frac{1}{n}\right), \right.$$
$$\left. \min\left(Y_1 - \frac{1}{\alpha_2}\ln\frac{1}{n}, Y_2 - \frac{1}{\alpha_2}\ln\frac{1}{n}, ..., Y_n - \frac{1}{\alpha_2}\ln\frac{1}{n}\right) \right\} :$$
$$\alpha_1, \alpha_2 > 0, n > 1, n > \alpha_1, n > \alpha_2$$

is asymptotically distributed as bivariate semi-extreme value.

Proof: If the random vector (X, Y) is distributed as $BSL(\alpha_1, \alpha_2, p)$ then $\bar{F}(x, y) = \frac{1}{1+\eta(x,y)}$, where $\eta(x, y)$ satisfies (7)

$$\bar{G}(x, y) = P\left\{ \min\left(X_1 - \frac{1}{\alpha_1}\ln\frac{1}{n}, X_2 - \frac{1}{\alpha_1}\ln\frac{1}{n}, ..., X_n - \frac{1}{\alpha_1}\ln\frac{1}{n}\right) > x, \right.$$
$$\left. \min\left(Y_1 - \frac{1}{\alpha_2}\ln\frac{1}{n}, Y_2 - \frac{1}{\alpha_2}\ln\frac{1}{n}, ..., Y_n - \frac{1}{\alpha_2}\ln\frac{1}{n}\right) > y \right\}$$
$$= \left[\bar{F}\left(x + \frac{1}{\alpha_1}\ln\frac{1}{n}, y + \frac{1}{\alpha_2}\ln\frac{1}{n}\right)\right]^n$$

where the minimum is taken in component wise

$$\bar{G}(x, y) = \left(\frac{1}{1 + \frac{\eta(x,y)}{n}}\right)^n.$$

Taking limit when n$\rightarrow \infty$, we get

$$G(x, y) = \exp\{-\eta(x, y)\}.$$

3 Marshall-Olkin Bivariate Semi Logistic Distribution

By various methods new parameters can be introduced to expand families of distributions. Introduction of a scale parameter leads to accelerate life model and taking powers of a survival function introduces a parameter that leads to proportional hazards model. Marshall and Olkin (1997) introduced a new method of adding a parameter to expand families of distribution. Let (X, Y) be a random vector with joint survival function $\bar{F}(x, y)$ Then

$$\bar{G}(x, y) = \left[\frac{\delta\bar{F}(x, y)}{1 - (1 - \delta)\bar{F}(x, y)}\right], \qquad -\infty < x, y < \infty, 0 < \delta < \infty \tag{17}$$

is a proper bivariate survival function (see Marshall and Olkin, 1997). The family of distributions of the form (17) shall be called Marshall-Olkin bivariate family of distributions.

From (17), We define Marshall-Olkin bivariate semi-logistic ($MOBSL(\alpha_1, \alpha_2, \delta, p)$) distribution with survival function

$$\bar{G}(x, y) = P(X > x, Y > y) = \frac{1}{1 + \frac{1}{\delta}\eta(x, y)} \tag{18}$$

where $\eta(x, y)$ satisfies the functional equation (7). The density plot of $MOBSL(\alpha_1, \alpha_2, \delta, p)$ distribution for $h_i(x) = \exp\{\beta \cos(\alpha_i x)\}$ $i = 1, 2$ and for various values of $\alpha_1, \alpha_2, \beta$ and δ with $p = \exp\{-2\pi\}$ is presented in Figure 3.

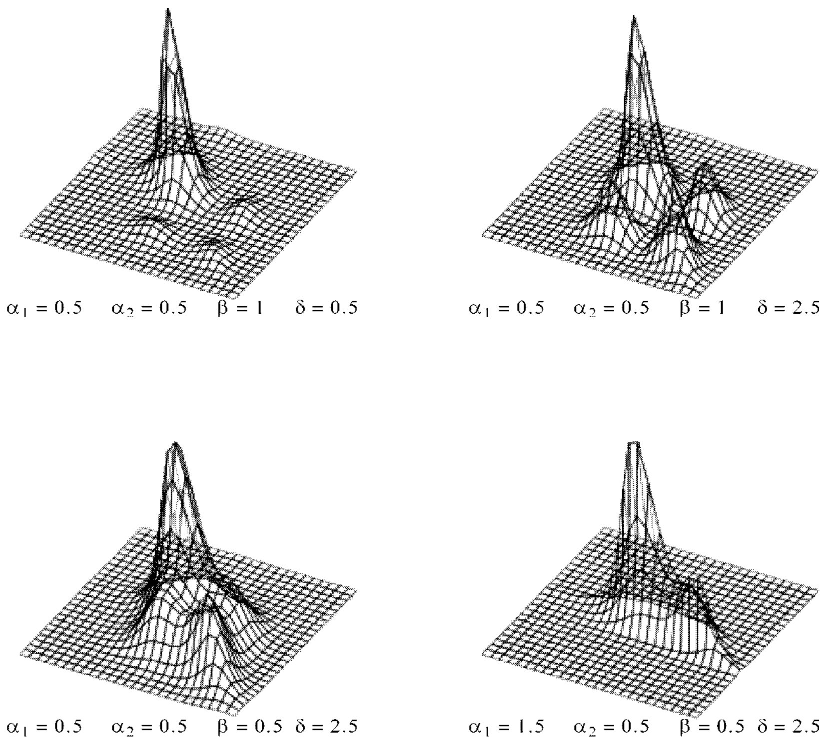

Figure 3. Density plot of Marshall-Olkin bivariate semi-logistic distribution.

Theorem 4 *Let N be a geometric random variable with parameter p such that $P(N = n) = pq^{n-1}$, $n = 1, 2, \ldots$, $0 < p < 1$, $q = 1 - p$. Consider a sequence $\{(X_i, Y_i), i \geqslant 1\}$ of independent and identically distributed random vectors with common survival function $\bar{F}(x, y)$. Assume that N and (X_i, Y_i) are independent for $i \geqslant 1$. Let $U_N = \min(X_1, X_2, ..., X_N)$ and $V_N = \min(Y_1, Y_2, ..., Y_N)$. The random vectors (U_N, V_N) are distributed as $MOBSL(\alpha_1, \alpha_2, \delta, p)$ if and only if (X_i, Y_i) have bivariate semi-logistic distribution.*

Proof: Consider

$$\begin{aligned}\bar{G}(x, y) &= P(U_N > x, V_N > y) \\ &= \sum_{n=1}^{\infty} [\bar{F}(x, y)]^n pq^{n-1}, \\ &= \frac{p\bar{F}(x, y)}{1 - (1-p)\bar{F}(x, y)}.\end{aligned}$$

Let $\bar{F}(x, y) = \frac{1}{1+\eta(x,y)}$, which is the survival function of $BSL(\alpha_1, \alpha_2, p)$.

Substituting this in the above equation, we have $\bar{G}(x, y) = \frac{p}{p+\eta(x,y)}$, which is the survival function of $MOBSL(\alpha_1, \alpha_2, \delta, p)$ with $\delta = p$.

Conversely suppose that

$$\begin{aligned}\bar{G}(x, y) &= \frac{p}{p + \eta(x, y)} \\ &= \frac{p\frac{1}{1+\eta(x,y)}}{1 - (1-p)\frac{1}{1+\eta(x,y)}}, \\ &= \frac{p\bar{F}(x, y)}{1 - (1-p)\bar{F}(x, y)}.\end{aligned}$$

Therefore

$$\bar{F}(x, y) = \frac{1}{1 + \eta(x, y)}.$$

Hence the proof is complete.

As a special case of (17) and (18) we have the Marshall-Olkin bivariate logistic ($MOBL(\alpha_1, \alpha_2, \delta)$) distribution, defined by the survival function

$$\bar{G}(x, y) = \frac{1}{1 + \frac{1}{\delta}\left(\exp\{\alpha_1 x\} + \exp\{\alpha_2 y\}\right)}, \quad \alpha_1, \alpha_2, 0 < \delta < 1$$

The bivariate density function is given by

$$f(x,y) = \frac{2\alpha_1\alpha_2 \exp\{\alpha_1 x\}\exp\{\alpha_2 y\}}{\delta^2\left\{1+\frac{1}{\delta}(\exp\{\alpha_1 x\}+\exp\{\alpha_2 y\})\right\}^3},$$

$$-\infty < x, y < \infty, \alpha_1, \alpha_2, 0 < \delta < 1.$$

A plot of the $MOBL(\alpha_1, \alpha_2, \delta)$ distribution for various values of α_1, α_2 and δ is presented in Figure 4.

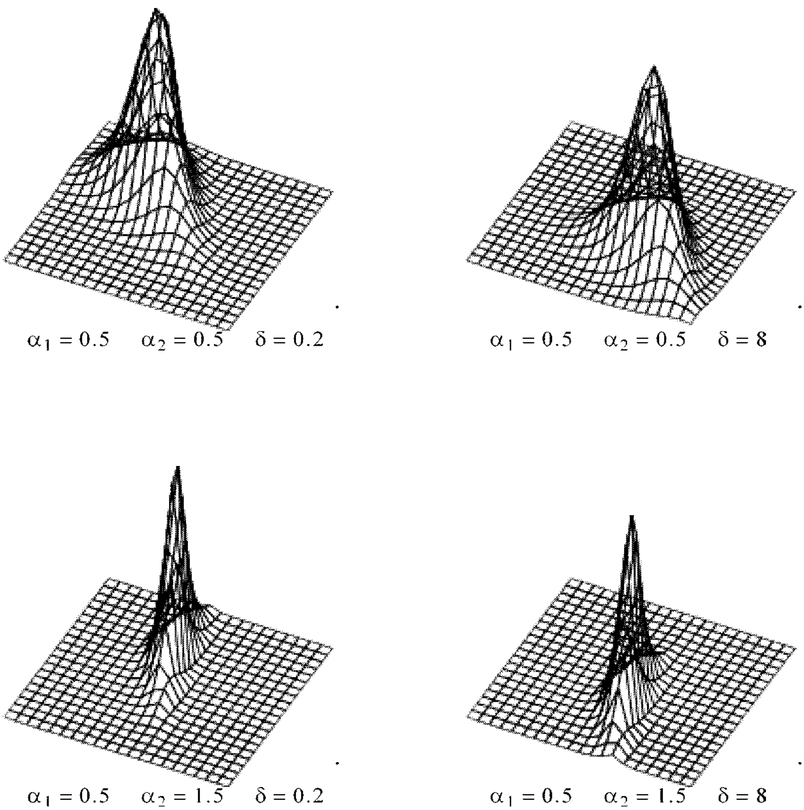

Figure 4. Density plot of the Marshall-Olkin bivariate logistic distribution.

The marginal distributions are

$$f_1(x) = \frac{\alpha_1 \exp\{\alpha_1 x\}}{\delta\left(1+\frac{1}{\delta}\exp\{\alpha_1 x\}\right)^2}, \qquad -\infty < x < \infty, \alpha_1 > 0, 0 < \delta < 1$$

$$f_2(y) = \frac{\alpha_2 \exp\{\alpha_2 y\}}{\delta\left(1+\frac{1}{\delta}\exp\{\alpha_2 y\}\right)^2}, \qquad -\infty < y < \infty, \alpha_2 > 0, 0 < \delta < 1$$

$$f(x|y) = \frac{2\alpha_1 \exp\{\alpha_1 x\}\left(1+\frac{1}{\delta}\exp\{\alpha_2 y\}\right)^2}{\delta\left(1+\frac{1}{\delta}(\exp\{\alpha_1 x\}+\exp\{\alpha_2 y\}\right)^3}, \qquad -\infty < x, y < \infty, \alpha_1, \alpha_2 > 0, 0 < \delta < 1$$

$$f(y|x) = \frac{2\alpha_2 \exp\{\alpha_2 y\}\left(1+\frac{1}{\delta}\exp\{\alpha_1 x\}\right)^2}{\delta\left(1+\frac{1}{\delta}(\exp\{\alpha_1 x\}+\exp\{\alpha_2 y\}\right)^3}, \qquad -\infty < x, y < \infty, \alpha_1, \alpha_2 > 0, 0 < \delta < 1$$

The conditional generating function

$$H(t_1|y) = (\delta + \exp\{\alpha_2 y\})^{\frac{t_1}{\alpha_1}}\,\Gamma\left(2 - \frac{t_1}{\alpha_1}\right)\Gamma\left(1 + \frac{t_1}{\alpha_1}\right)$$

$$H(t_2|x) = (\delta + \exp\{\alpha_1 x\})^{\frac{t_2}{\alpha_2}}\,\Gamma\left(2 - \frac{t_2}{\alpha_2}\right)\Gamma\left(1 + \frac{t_2}{\alpha_2}\right)$$

$$E(X|Y) = \frac{1}{\alpha_1}\ln\left(\delta + \exp\{\alpha_2 y\}\right) - \frac{1}{\alpha_1}$$

$$E(Y|X) = \frac{1}{\alpha_2}\ln\left(\delta + \exp\{\alpha_1 x\}\right) - \frac{1}{\alpha_2}$$

$$E(X) = \frac{\ln(\delta)}{\alpha_1}$$

$$E(Y) = \frac{\ln(\delta)}{\alpha_2}$$

$$E\left(X - E(X)\right)^{2r} = \frac{2\Gamma(2r+1)}{\alpha_1^{2r}}\left(1 - \frac{1}{2^{2r-1}}\right)\zeta(2r), \qquad r = 1, 2, \ldots$$

$$E\left(Y - E(Y)\right)^{2r} = \frac{2\Gamma(2r+1)}{\alpha_2^{2r}}\left(1 - \frac{1}{2^{2r-1}}\right)\zeta(2r), \qquad r = 1, 2, \ldots$$

where $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ is Raiman zeta function. The bivariate moment generating function

$$H(t_1, t_2) = \delta^{\frac{t_1}{\alpha_1}+\frac{t_2}{\alpha_2}-1}\Gamma\left(1 + \frac{t_1}{\alpha_1}\right)\Gamma\left(1 + \frac{t_2}{\alpha_2}\right)\Gamma\left(1 - \frac{t_1}{\alpha_1} - \frac{t_2}{\alpha_2}\right) \qquad (19)$$

For the bivariate logistic distribution defined in Gumbel (1961) having distribution function (2), the coefficient of correlation ρ has a fixed value $\frac{1}{2}$. Using the bivariate moment generating function (19) we can find the correlation coefficient between X and Y for the $MOBSL(\alpha_1, \alpha_2, \delta)$ distribution and it is observed that the value is not restricted to $\frac{1}{2}$.

4 First Order Autoregressive Bivariate Semi-Logistic (BSLAR(1)) Process

The BSLAR(1) minification process $\{(X_n, Y_n), n \geqslant 0\}$ is defined as follows:

Let $\{(\varepsilon_n, \nu_n), n \geqslant 1\}$ be a sequence of independent and identically distributed bivariate real random vectors.

Define

$$\begin{aligned} X_n &= \min\left(X_{n-1} - \frac{1}{\alpha_1}\ln p, \varepsilon_n\right) \\ Y_n &= \min\left(Y_{n-1} - \frac{1}{\alpha_2}\ln p, \nu_n\right), \qquad n \geqslant 1, 0 \leqslant p \leqslant 1, \alpha_1, \alpha_2 > 0. \end{aligned} \tag{20}$$

Assume that (X_0, Y_0) is independent of $\{(\varepsilon_n, \nu_n)\}$. Then it follows that $\{(X_n, Y_n), n \geqslant 0\}$ is a bivariate Markov sequence. As ε_n and ν_n are real random variables assume that either both are ∞ with probability p or both are finite with probability $1-p$ and hence we can represent them as

$$(\varepsilon_n, \nu_n) = \begin{cases} (-\infty, \infty) & \text{with probability } p \\ (\omega_n, \tau_n) & \text{with probability } 1-p. \end{cases}$$

Theorem 5 *Assume that $(X_0, Y_0) \stackrel{d}{=} (\omega_1, \tau_1)$. The process defined by (20) is stationary if and only if has $BSL(\alpha_1, \alpha_2, p)$ distribution.*

Proof: From (20)

$$\begin{aligned} \bar{G}_n(x,y) &= P(X_n > x, Y_n > y) \\ &= \bar{G}_{n-1}\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right)\left[p + (1-p)\bar{F}(x,y)\right] \end{aligned} \tag{21}$$

where $\bar{F}(x,y)$ is the survival function of (ω_1, τ_1).

Assume that $\{(X_n, Y_n), n \geqslant 0\}$ is stationary and $(X_0, Y_0) \stackrel{d}{=} (\omega_1, \tau_1)$. Then for $n = 1$, (21) gives

$$\bar{G}_0\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right) = \frac{\bar{F}(x,y)}{p + (1-p)\bar{F}(x,y)} \tag{22}$$

If we write $\bar{F}(x,y) = \frac{1}{1+\eta(x,y)}$, the equation (22) leads to the relation

$$\eta(x,y) = \frac{1}{p}\eta\left(x + \frac{1}{\alpha_1}\ln p, \ \ y + \frac{1}{\alpha_2}\ln p\right).$$

That is $\bar{F}(x, y)$ is of the form (7) and hence by (21), the random vector (X_1, Y_1) is distributed as $BSL(\alpha_1, \alpha_2, p)$. By induction we can prove that $\{(X_n, Y_n), n \geqslant 0\}$ is a $BSL(\alpha_1, \alpha_2, p)$ Markov sequence.

Conversely assume that $\{(\omega_n, \tau_n), n \geqslant 0\}$ is a sequence of independent and identically distributed $BSL(\alpha_1, \alpha_2, p)$ random vectors and $(X_0, Y_0) \stackrel{d}{=} (\omega_1, \tau_1)$. For $n = 1$ from (21) we get

$$\bar{G}_1(x, y) = \bar{F}\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right)\left[p + (1-p)\bar{F}(x, y)\right]$$

If we write $\bar{F}(x, y) = \frac{1}{1+\eta(x,y)}$ and applying (7), we get

$$\bar{G}_1(x, y) = \frac{1}{1 + \eta(x, y)}.$$

That is, (X_1, Y_1) has $BSL(\alpha_1, \alpha_2)$ distribution. Now by (21) using the method of induction we can prove that $\{(X_n, Y_n)\}$ is a a sequence of $BSL(\alpha_1, \alpha_2, p)$ random vectors. That is $\{(X_n, Y_n), n \geqslant 0\}$ is a stationary $BSL(\alpha_1, \alpha_2, p)$ sequence.

Corrollary 1 *Let be an arbitrary random vector with survival function $\bar{G}_0(u, v)$ and $\{(\omega_n, \tau_n), n \geqslant 0\}$ is a sequence of independent and identically distributed $BSL(\alpha_1, \alpha_2)$ random vectors. Then the bivariate sequence $\{(X_n, Y_n), n \geqslant 0\}$ defined by (20) converges in distribution to $BSL(\alpha_1, \alpha_2)$ as $n \to \infty$.*

Proof: The model defined by (20) and the relations (21), (6) and (7) together imply that

$$\begin{aligned} G_n(x, y) &= G_0\left(x + \frac{1}{\alpha_1}\ln p, y + \frac{1}{\alpha_2}\ln p\right)\left[\frac{1 + p^n\eta(x, y)}{1 + \eta(x, y)}\right] \\ &\to \frac{1}{1 + \eta(x, y)}, \end{aligned}$$

as $n \to \infty$.

Remark 1 *Since the bivariate logistic distribution defined in (9) is is a special case of $BSL(\alpha_1, \alpha_2)$, similar to BSLAR(1) process we can define first order autoregressive bivariate logistic (BLAR(1)) process.*

5 First Order Autoregressive Marshall-Olkin Bivariate Semi-Logistic Model (MOBSLAR(1))

We consider the first order autoregressive time series model with Marshall-Olkin bivariate semi-logistic distribution.

Theorem 6 *Consider the bivariate autoregressive minification process* $\{(X_n, Y_n)\}$ *having the structure*

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad p \\ \min(X_{n-1}, \varepsilon_n) & w.p. \quad 1-p \end{cases}$$

$$Y_n = \begin{cases} \mu_n & w.p. \quad p \\ \min(Y_{n-1}, \mu_n) & w.p. \quad 1-p \end{cases}, \quad 0 < p < 1. \tag{23}$$

where $\{(\varepsilon_n, \mu_n)\}$ *is a sequence of independent and identically distributed innovation random variables. Then* $\{(X_n, Y_n)\}$ *is stationary with marginal distribution Marshall-Olkin bivariate semi-logistic* $MOBSL(\alpha_1, \alpha_2, \delta, p)$ *if and only if* $\{(\varepsilon_n, \mu_n)\}$ *is jointly distributed as bivariate semi-logistic* $(BSL(\alpha_1, \alpha_2, p))$.

Proof: From (23) we have

$$\bar{F}_{X_n, Y_n}(x, y) = p\bar{G}_{\varepsilon_n, \mu_n}(x, y) + (1-p)\bar{F}_{X_{n-1}, Y_{n-1}}(x, y)\bar{G}_{\varepsilon_n, \mu_n}(x, y)$$

Under stationarity, we have

$$\bar{F}_{X,Y}(x, y) = \frac{p\bar{G}_{\varepsilon,\mu}(x, y)}{1 - (1-p)\bar{G}_{\varepsilon,\mu}(x, y)}.$$

If we take $G_{\varepsilon,\mu}(x, y) = \frac{1}{1+\eta(x,y)}$, then we get

$$\bar{F}_{X,Y}(x, y) = \frac{p}{p + \eta(x, y)},$$

which is the survival function of $MOBSL(\alpha_1, \alpha_2, p)$.

Conversely, if we take

$$\bar{F}_{X,Y}(x, y) = \frac{p}{p + \eta(x, y)},$$

it can be shown that

$$G_{\varepsilon,\mu}(x, y) = \frac{1}{1 + \eta(x, y)}.$$

If we assume that $\{(X_{n-1}, Y_{n-1})\}$ is distributed as $MOBSL(\alpha_1, \alpha_2, \delta, p)$ and $\{(\varepsilon_n, \mu_n)\}$ is $BSL(\alpha_1, \alpha_2, p)$ then we can establish that

$$\bar{F}_{X_n, Y_n}(x, y) = \frac{p}{p + \eta(x, y)}.$$

Even if (X_0, Y_0) is arbitrary, it is easy to establish that $\{(X_n, Y_n)\}$ is asymptotically distributed as $MOBSL(\alpha_1, \alpha_2, \delta, p)$.

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